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Weyl type theorems for p-hyponormal and M-hyponormal operators

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Abstract. "Generalized Weyl's theorem holds" for an operator when the complement in the spectrum of the B-Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues; and "generalized a-Weyl's theorem holds" for an operator when the complement in the approximate point spectrum of the semi-B-essential approximate point spectrum coincides with the isolated points of the approximate point spectrum which are eigenvalues. If T or T^* is p-hyponormal or M-hyponormal then for every $f \in H(\sigma(T))$, generalized Weyl's theorem holds for $f(T)$, so Weyl's theorem holds for $f(T)$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$. Moreover, if T^* is p-hyponormal or M-hyponormal then for every $f \in H(\sigma(T))$, generalized a-Weyl's theorem holds for $f(T)$ and hence a-Weyl's theorem holds for $f(T)$.

1. Introduction. Throughout this paper, $B(H)$ and $K(H)$ denote respectively the algebra of bounded linear operators and the ideal of compact operators acting on an infinite-dimensional separable Hilbert space H. If $T \in B(H)$, we write $N(T)$ and $R(T)$ for the null space and range of T, with $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T)$; $\sigma(T)$ for the spectrum of T; $\sigma_{\mathrm{a}}(T)$ for the approximate point spectrum of T; $E_0(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity; and $E_0^{\mathrm{a}}(T)$ for the isolated points of $\sigma_{\mathrm{a}}(T)$ which are eigenvalues of finite multiplicity. Let $\varrho_{\mathrm{a}}(T) = \mathbb{C} \setminus \sigma_{\mathrm{a}}(T)$.

An operator $T \in B(H)$ is called *Fredholm* if it has closed finite-codimensional range and finite-dimensional null space. The *index* of a Fredholm operator $T \in B(H)$ is given by $\operatorname{ind}(T) = n(T) - d(T)$.

An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent, or equivalently, if T is Fredholm and $T - \lambda I$ is invertible for all sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. For $T \in B(H)$, we write $\alpha(T)$ for the ascent of T and $\beta(T)$ for the descent of T.

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The *essential spectrum* $\sigma_{\mathrm{e}}(T)$, *Weyl spectrum* $\sigma_{\mathrm{w}}(T)$, and *Browder spectrum* $\sigma_{\mathrm{b}}(T)$ of $T \in B(H)$ are defined by

$$\begin{aligned}\sigma_{\mathrm{e}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},\\ \sigma_{\mathrm{w}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},\\ \sigma_{\mathrm{b}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.\end{aligned}$$

We say that *Weyl's theorem holds for* $T \in B(H)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = E_0(T),$$

and that *Browder's theorem holds for* $T \in B(H)$ if

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T).$$

Let

$$\begin{aligned}\mathrm{SF}_+(H) &= \{T \in B(H) : R(T) \text{ is closed and } n(T) < \infty\},\\ \mathrm{SF}_+^-(H) &= \{T \in B(H) : T \in \mathrm{SF}_+(H) \text{ and } \mathrm{ind}(T) \le 0\},\\ \sigma_{\mathrm{SF}_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } \mathrm{SF}_+(H)\}.\end{aligned}$$

For $T \in B(H)$, the *essential approximate point spectrum* and the *Browder essential approximate point spectrum* are defined by

$$\begin{aligned}\sigma_{\mathrm{SF}_+^-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } \mathrm{SF}_+^-(H)\},\\ \sigma_{\mathrm{ab}}(T) &= \bigcap\{\sigma_{\mathrm{a}}(T+K) : TK = KT \text{ and } K \in K(H)\}.\end{aligned}$$

Recall that by [12, Theorem 2.1] a complex number λ is not in $\sigma_{\mathrm{ab}}(T)$ if and only if $T - \lambda I \in \mathrm{SF}_+(H)$ and $\alpha(T - \lambda I) < \infty$.

We say that *a-Weyl's theorem holds for* $T \in B(H)$ if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E_0^{\mathrm{a}}(T),$$

and that *a-Browder's theorem holds for* $T \in B(H)$ if $\sigma_{\mathrm{SF}_+^-}(T) = \sigma_{\mathrm{ab}}(T)$.

It is known [7] that for any $T \in B(H)$ we have the implications:

$$\text{a-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem},$$
$$\text{a-Weyl's theorem} \Rightarrow \text{a-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

For a bounded linear operator T and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ to $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n, the range space $R(T^n)$ is closed and $T_{[n]}$ is upper (resp. lower) semi-Fredholm, then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. Moreover if $T_{[n]}$ is a Fredholm (resp. Weyl, Browder) operator, then T is called a *B-Fredholm* (resp. *B-Weyl*, *B-Browder*) operator. Similarly, we can define the B-Fredholm spectrum $\sigma_{\mathrm{BF}}(T)$, B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ and B-Browder spectrum $\sigma_{\mathrm{BB}}(T)$. A *semi-B-Fredholm operator* is an upper or a lower semi-B-Fredholm operator.

Let $T \in B(H)$ and let

$$\triangle(T) = \{n \in \mathbb{N} : R(T^n) \cap N(T) \subseteq R(T^m) \cap N(T) \text{ for all } m \geq n\}.$$

Then the *degree of stable iteration* $\mathrm{dis}(T)$ of T is defined as $\mathrm{dis}(T) = \inf \triangle(T)$. Let T be a semi-B-Fredholm operator and let $d = \mathrm{dis}(T)$. It follows from [5, Proposition 2.1] that $T_{[m]}$ is a semi-Fredholm operator and $\mathrm{ind}(T_{[m]}) = \mathrm{ind}(T_{[d]})$ for each $m \geq d$. This enables us to define the index of a semi-B-Fredholm operator T as the index of $T_{[d]}$.

In the case of a normal operator T acting on a Hilbert space, Berkani [2, Theorem 4.5] showed that $\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T)$, where $E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T. This generalizes the classical Weyl's theorem. So we say that T *obeys generalized Weyl's theorem* ([4, Definition 2.13]) if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T).$$

Similarly, let $\mathrm{SBF}_+(H)$ be the class of all upper semi-B-Fredholm operators, and $\mathrm{SBF}_+^-(H)$ the class of all $T \in \mathrm{SBF}_+(H)$ such that $\mathrm{ind}(T) \leq 0$. Also let

$$\sigma_{\mathrm{SBF}_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } \mathrm{SBF}_+^-(H)\},$$

called the *semi-B-essential approximate point spectrum*. We say that T *obeys generalized a-Weyl's theorem* if

$$\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_{\mathrm{a}}(T) \setminus E^{\mathrm{a}}(T),$$

where $E^{\mathrm{a}}(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_{\mathrm{a}}(T)$ ([4, Definition 2.13]). From [4, Theorem 3.11], we know that each T satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem and hence Weyl's theorem, but the converse is not true (see [4, Example 3.12]).

An operator $T \in B(H)$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, and *M-hyponormal* if there exists a positive number M such that

$$M\|(T - \lambda I)x\| \geq \|(T - \lambda I)^*x\| \quad \text{for all } \lambda \in \mathbb{C} \text{ and all } x \in H.$$

In this paper we show that if T^* is p-hyponormal or M-hypomormal, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, hence a-Weyl's theorem holds for $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

2. Preliminary results. Let

$$A_1(H) = \{S \in B(H) : \mathrm{ind}(S - \lambda I)\,\mathrm{ind}(S - \mu I) \geq 0 \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{e}}(S)\},$$

$$A_2(H) = \{S \in B(H) : \mathrm{ind}(S - \lambda I)\,\mathrm{ind}(S - \mu I) \geq 0 \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{SF}_+}(S)\}.$$

An operator $T \in B(H)$ is called *approximate-isoloid* (abbr. *a-isoloid*) if every isolated point of $\sigma_{\mathrm{a}}(T)$ is an eigenvalue of T, and *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. Clearly, if T is a-isoloid then it is isoloid. However, the converse is not true.

THEOREM 2.1. *If $T \in B(H)$ obeys generalized Weyl's theorem and it is isoloid, then the following statements are equivalent*:

(1) $T \in A_1(H)$;
(2) $\sigma_{\mathrm{w}}(f(T)) = f(\sigma_{\mathrm{w}}(T))$ *for every* $f \in H(\sigma(T))$;
(3) $\sigma_{\mathrm{BW}}(f(T)) = f(\sigma_{\mathrm{BW}}(T))$ *for every* $f \in H(\sigma(T))$;
(4) *Weyl's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$;
(5) *generalized Weyl's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$.

Proof. (1)⇔(3). By [3, Remark iii], $\operatorname{ind}(T-\lambda I)\operatorname{ind}(T-\mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{e}}(T)$ if and only if $\operatorname{ind}(T-\lambda I)\operatorname{ind}(T-\mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{BF}}(T)$. From [3, Corollary 3.3] and [2, Theorem 3.2], the spectral mapping theorem for the B-Weyl spectrum may be rewritten as the implication, for arbitrary $n \in \mathbb{N}$ and $\lambda_i \in \mathbb{C}$,

$$f(T) = (T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I)g(T) \text{ B-Weyl} \\ \Rightarrow\ T-\lambda_j I \text{ B-Weyl for each } j = 1, \dots, n,$$

where $g(T)$ is invertible. Now if $\operatorname{ind}(T-\lambda I) \geq 0$ on $\mathbb{C} \setminus \sigma_{\mathrm{BF}}(T)$, then

$$\sum_{j=1}^{n} \operatorname{ind}(T-\lambda_j I) = \operatorname{ind} \prod_{j=1}^{n} (T-\lambda_j I) = 0 \\ \Rightarrow\ \operatorname{ind}(T-\lambda_j I) = 0\ (j = 1, \dots, n)$$

and similarly if $\operatorname{ind}(T-\lambda I) \leq 0$ off $\sigma_{\mathrm{BF}}(T)$. If conversely there exist $\lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{e}}(T)$ for which $\operatorname{ind}(T-\lambda I) = -m < 0 < k = \operatorname{ind}(T-\mu I)$, then $f(T) = (T-\lambda I)^k (T-\mu I)^m$ is a Weyl operator whose factors are not B-Weyl. This is a contradiction.

(1)⇔(2). See [9, Theorem 5].

(2)⇔(4). Generalized Weyl's theorem implies Weyl's theorem for T. Moreover, [11, Lemma] tells us that if T is isoloid, then

$$f(\sigma(T) \setminus E_0(T)) = \sigma(f(T)) \setminus E_0(f(T)) \quad \text{for every } f \in H(\sigma(T)),$$

thus Weyl's theorem holds for $f(T)$ if and only if $\sigma_{\mathrm{w}}(f(T)) = f(\sigma_{\mathrm{w}}(T))$.

(3)⇒(5). For every $f \in H(\sigma(T))$, we need to prove $\sigma(f(T)) \setminus \sigma_{\mathrm{BW}}(f(T)) = E(f(T))$. Let $\mu \in \sigma(f(T)) \setminus \sigma_{\mathrm{BW}}(f(T))$, that is, $f(T)-\mu I$ is B-Weyl and μ is not in $f(\sigma_{\mathrm{BW}}(T))$. Let

$$\text{(a)} \qquad f(T) - \mu I = (T-\lambda_1 I)\cdots(T-\lambda_m I)g(T),$$

where $g(T)$ is invertible. Then $T-\lambda_i I$ is B-Weyl, and, in particular, an operator of topological uniform descent. Since generalized Weyl's theorem holds for T, it follows that $T-\lambda_i I$ has finite ascent and descent for every

$i \in \{1, \ldots, m\}$ ([8, Corollary 4.9]). Then $f(T) - \mu I$ has finite ascent and descent. Suppose $\alpha(f(T) - \mu I) = \beta(f(T) - \mu I) = p$. Then μ is a pole of $f(T)$ and hence $\mu \in E(f(T))$.

Conversely, let $\mu \in E(f(T))$. Using (a), without loss of generality, we suppose $\lambda_i \in \sigma(T)$. Thus $\lambda_i \in \operatorname{iso} \sigma(T)$. Since T is isoloid, we see that $\lambda_i \in E(T)$. Since generalized Weyl's theorem holds for T, it follows that $T - \lambda_i I$ is B-Weyl and hence $f(T) - \mu I$ is B-Weyl ([3, Corollary 3.3] and [2, Theorem 3.2]). Thus $\mu \in \sigma(f(T)) \setminus \sigma_{\mathrm{BW}}(f(T))$. So we have proved that generalized Weyl's theorem holds for $f(T)$.

(5)⇒(4). See [4, Theorem 3.9]. ■

Recall that if T is an upper semi-Fredholm operator, then $T - \lambda I$ is upper semi-Fredholm and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $|\lambda|$ is sufficiently small. The same holds if T is upper semi-B-Fredholm. In fact, [5, Corollary 3.2] tells us that if $|\lambda|$ is sufficiently small, then $T - \lambda I$ is upper semi-Fredholm. Let $\operatorname{dis}(T) = d$. Then $T_{[d]}$ is upper semi-Fredholm. Hence if $|\lambda|$ is sufficiently small, then

$$N(T_{[d]} - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T_{[d]} - \lambda I)^n] \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n].$$

Since $N(T - \lambda I) \subseteq R(T^d)$, it follows that

$$N(T - \lambda I) = N(T - \lambda I) \cap R(T^d) = N(T_{[d]} - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n].$$

THEOREM 2.2. *If $T \in B(H)$ obeys generalized a-Weyl's theorem and it is a-isoloid, then the following statements are equivalent*:

(1) $T \in A_2(H)$;
(2) $\sigma_{\mathrm{SF}_+^-}(f(T)) = f(\sigma_{\mathrm{SF}_+^-}(T))$ *for every* $f \in H(\sigma(T))$;
(3) $\sigma_{\mathrm{SBF}_+^-}(f(T)) = f(\sigma_{\mathrm{SBF}_+^-}(T))$ *for every* $f \in H(\sigma(T))$;
(4) *a-Weyl's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$;
(5) *generalized a-Weyl's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$.

Proof. (1)⇔(3). By [5, Corollary 3.2], $\operatorname{ind}(T - \lambda I) \operatorname{ind}(T - \mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{SF}_+}(T)$ if and only if $\operatorname{ind}(T - \lambda I) \operatorname{ind}(T - \mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{\mathrm{SBF}_+}(T)$.

⇒ For every $f \in H(\sigma(T))$, let $\mu_0 \in \sigma_{\mathrm{SBF}_+^-}(f(T))$ and suppose μ_0 is not in $f(\sigma_{\mathrm{SBF}_+^-}(T))$. We also suppose that

$$\text{(b)} \qquad f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} \cdots (T - \lambda_m I)^{n_m} g(T),$$

where $g(T)$ is invertible, λ_i is not in $\sigma_{\mathrm{SBF}_+^-}(T)$ and $\lambda_i \neq \lambda_j$ if $i, j = 1, \ldots, m$. Then $f(T) - \mu_0 I$ is upper semi-B-Fredholm ([5, Proposition 4.3]) and $\text{ind}(f(T)-\mu_0 I) = \sum_{i=1}^m \text{ind}(T-\lambda_i I)^{n_i} \leq 0$. That is, μ_0 is not in $\sigma_{\text{SBF}_+^-}(f(T))$. This is a contradiction. Thus $\sigma_{\text{SBF}_+^-}(f(T)) \subseteq f(\sigma_{\text{SBF}_+^-}(T))$.

Conversely, suppose $\mu_0 \in f(\sigma_{\text{SBF}_+^-}(T))$ but μ_0 is not in $\sigma_{\text{SBF}_+^-}(f(T))$, that is, $f(T)-\mu_0 I$ is upper semi-B-Fredholm and $\text{ind}[f(T)-\mu_0 I] \leq 0$. Then $T-\lambda_i I$ is upper semi-B-Fredholm ([5, Corollary 4.4]). Since $\text{ind}(f(T)-\mu_0 I) = \sum_{i=1}^m \text{ind}(T-\lambda_i I)^{n_i} \leq 0$, we know that $\text{ind}(T-\lambda_i I) \leq 0$. Thus $T-\lambda_i I \in \text{SBF}_+^-(X)$ and therefore μ_0 is not in $f(\sigma_{\text{SBF}_+^-}(T))$. This is a contradiction.

$\Leftarrow$ Assume the contrary; then there exist λ_0, μ_0 such that $T-\lambda_0 I$, $T-\mu_0 I \in \text{SF}_+(X)$ and $\text{ind}(T-\lambda_0 I) < 0$, $\text{ind}(T-\mu_0 I) > 0$. Let $m = -\text{ind}(T-\lambda_0 I)$ and $n = \text{ind}(T-\mu_0 I)$. Thus n is finite. If m is finite, let $f(T) = (T-\lambda_0 I)^n (T-\mu_0 I)^m$ or else let $f(T) = (T-\lambda_0 I)(T-\mu_0 I)$. Then $f(T) \in \text{SF}_+^-(X)$. So 0 is not in $\sigma_{\text{SBF}_+^-}(f(T))$. But $\mu_0 \in \sigma_{\text{SBF}_+^-}(T)$, hence $0 = f(\mu_0) \in f(\sigma_{\text{SBF}_+^-}(T)) = \sigma_{\text{SBF}_+^-}(f(T))$. This is a contradiction. Hence the result is true.

(1)$\Leftrightarrow$(2). Argue as for (1)$\Leftrightarrow$(3).

(2)$\Leftrightarrow$(4). Suppose that $\sigma_{\text{SF}_+^-}(f(T)) = f(\sigma_{\text{SF}_+^-}(T))$ for every $f \in H(\sigma(T))$. Since a-Weyl's theorem holds for T, it follows that $\sigma_{\text{SF}_+^-}(T) = \sigma_{\text{ab}}(T)$. We know that Browder essential approximate point spectrum satisfies the spectral mapping theorem, so $\sigma_{\text{SF}_+^-}(f(T)) = f(\sigma_{\text{SF}_+^-}(T)) = f(\sigma_{\text{ab}}(T)) = \sigma_{\text{ab}}(f(T))$. Therefore a-Browder's theorem holds for $f(T)$. Now let $\mu \in E_0^{\text{a}}(f(T))$. Using (b), without loss of generality, we suppose that $\lambda_i \in \sigma_{\text{a}}(T)$; then $\lambda_i \in \text{iso}\,\sigma_{\text{a}}(T)$. Since T is a-isoloid, we infer that $\lambda_i \in E_0^{\text{a}}(T)$. Generalized a-Weyl's theorem implies a-Weyl's theorem, so $T-\lambda_i I \in \text{SF}_+^-(H)$ and hence $f(T)-\mu I \in \text{SF}_+^-(H)$. Thus a-Weyl's theorem holds for $f(T)$. Conversely, suppose a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Then $\sigma_{\text{SF}_+^-}(f(T)) = \sigma_{\text{ab}}(f(T)) = f(\sigma_{\text{ab}}(T)) = f(\sigma_{\text{SF}_+^-}(T))$.

(3)$\Rightarrow$(5). For every $f \in H(\sigma(T))$, we need to prove that

$$\sigma_{\text{a}}(f(T)) \setminus \sigma_{\text{SBF}_+^-}(f(T)) = E^{\text{a}}(f(T)).$$

Let $\mu \in \sigma_{\text{a}}(f(T)) \setminus \sigma_{\text{SBF}_+^-}(f(T))$, that is, $f(T)-\mu I \in \text{SBF}_+^-(H)$ and μ is not in $f(\sigma_{\text{SBF}_+^-}(T))$. By (b), then $T-\lambda_i I \in \text{SBF}_+^-(H)$. We can suppose that $\lambda_i \in \sigma_{\text{a}}(T)$. Since generalized a-Weyl's theorem holds for T and T is a-isoloid, it follows that $T-\lambda_i I$ has finite ascent for every $i \in \{1,\ldots,m\}$ ([4, Theorem 2.8]). Thus $f(T)-\mu I$ has finite ascent. Let $\text{dis}(f(T)) = d$. Then $R[(f(T)-\mu I)^d]$ is closed. Now [10, Lemma 2.5] and the perturbation theory of upper semi-B-Fredholm operators imply that $\mu \in \text{iso}\,\sigma_{\text{a}}(f(T))$, so $\mu \in E^{\text{a}}(f(T))$.

Conversely, suppose $\mu \in E^{\text{a}}(f(T))$. Then $\lambda_i \in \text{iso}\,\sigma_{\text{a}}(T)$. Since T is a-isoloid, we find that $T-\lambda_i I \in E^{\text{a}}(T)$. Generalized a-Weyl's theorem holds for T, so $T-\lambda_i I \in \mathrm{SBF}_+^-(H)$ and hence $f(T)-\mu I \in \mathrm{SBF}_+^-(H)$. Thus we have proved that generalized a-Weyl's theorem holds for $f(T)$.

(5)⇒(4). See [4, Theorem 3.11]. ■

3. p-hyponormal and M-hyponormal operators. We start with some elementary results about p-hyponormal and M-hyponormal operators. We call $T \in B(H)$ *paranormal* if for any $x \in H$, $\|Tx\|^2 \le \|T^2x\|\,\|x\|$. If T is paranormal, then $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

LEMMA 3.1 ([14], [16]). *If T is p-hyponormal for some p such that $0 < p \le 1$ or T is M-hyponormal, then the restriction $T|_F$ to any invariant subspace F is also p-hyponormal or M-hyponormal.*

LEMMA 3.2 ([15]). (1) *If T is p-hyponormal, then T is paranormal.*

(2) *If T is invertible and p-hyponormal, then T^{-1} is also p-hyponormal.*

(3) *If T is M-hyponormal and $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

THEOREM 3.3. *If T^* is p-hyponormal or M-hyponormal, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Hence a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is p-hyponormal. By Theorem 2.2, we need to prove that T is a-isoloid, $T \in A_2(H)$ and generalized a-Weyl's theorem holds for T. First we prove that T is a-isoloid. Since T^* is p-hyponormal, a-Weyl's theorem holds for T ([6]) and $\sigma(T) = \sigma_{\mathrm{a}}(T)$. Let $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T) = \operatorname{iso}\sigma(T)$; then $\overline{\lambda} \in \operatorname{iso}\sigma(T^*)$. Since T^* is p-hyponormal, T^* is isoloid ([15, Theorem 1]), so $N(T^*-\overline{\lambda}I) \ne \{0\}$. As $N(T^*-\overline{\lambda}I) \subseteq N(T-\lambda I)$ ([15, Corollary 3]), we have $N(T-\lambda I) \ne \{0\}$, which means that T is a-isoloid.

Since $N(T^*-\overline{\lambda}I) \subseteq N(T-\lambda I)$, it follows that if $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{SF}_+}(T)$, then $\operatorname{ind}(T-\lambda I) \ge 0$. Therefore $T \in A_2(H)$.

Next we prove that generalized a-Weyl's theorem holds for T, that is, $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E^{\mathrm{a}}(T)$. Let $\lambda_0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. Then there exists $\varepsilon > 0$ such that $T-\lambda I \in \mathrm{SF}_+^-(H)$ and

$$N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n] \quad \text{if } 0 < |\lambda-\lambda_0| < \varepsilon.$$

Since a-Weyl's theorem holds for T, it follows that $\alpha(T-\lambda I) < \infty$ and hence

$$N(T-\lambda I) = N(T-\lambda I) \cap \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n] = \{0\} \quad \text{if } 0 < |\lambda-\lambda_0| < \varepsilon$$

([13, Lemma 3.4]). Thus $T-\lambda I$ is bounded from below, which means that $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$. Thus $\lambda_0 \in E^{\mathrm{a}}(T)$.

Conversely, if $\lambda_0 \in E^{\mathrm{a}}(T)$, then λ_0 is an isolated point of $\sigma_{\mathrm{a}}(T) = \sigma(T)$, so $\overline{\lambda}_0 \in \operatorname{iso}\sigma(T^*)$. Now using the spectral projection

$$P = \frac{1}{2\pi i} \int_{\partial B_0} (T^* - \lambda I)^{-1} d\lambda,$$

where B_0 is an open disk of center $\overline{\lambda}_0$ which contains no other points of $\sigma(T^*)$, we can represent T^* as the direct sum

$$T^* = T_1 \oplus T_2, \quad \text{where} \quad \sigma(T_1) = \{\overline{\lambda}_0\} \text{ and } \sigma(T_2) = \sigma(T^*) \setminus \{\overline{\lambda}_0\}.$$

Then $T_2 - \overline{\lambda}_0 I$ is invertible.

Case 1: $\lambda_0 = 0$. Then $\sigma(T_1) = \{0\}$. Since T_1 is p-hyponormal, it follows that $T_1 = 0$ ([15, Corollary 2]). Thus $T^* - \overline{\lambda}_0 I = 0 \oplus T_2 - \overline{\lambda}_0 I$.

Case 2: $\lambda_0 \neq 0$. Since T_1 is invertible and paranormal, it follows that T_1^{-1} is paranormal. Then $\|T_1\| = |\lambda_0|$ and $\|T_1^{-1}\| = 1/|\lambda_0|$. For any $x \in R(P)$,

$$\|x\| \le \|T_1^{-1}\| \, \|T_1 x\| = \frac{1}{|\lambda_0|} \|T_1 x\| \le \frac{1}{|\lambda_0|} |\lambda_0| \, \|x\| = \|x\|,$$

which implies that $(1/\overline{\lambda}_0)T_1$ is unitary. Thus T_1 is normal and hence so is $T_1 - \overline{\lambda}_0 I$. Since $T_1 - \overline{\lambda}_0 I$ is quasinilpotent and the only normal quasinilpotent operator is zero, it follows that $T^* - \overline{\lambda}_0 I = 0 \oplus T_2 - \overline{\lambda}_0 I$.

Since $T_2 - \overline{\lambda}_0 I$ is invertible, we know that $T^* - \overline{\lambda}_0 I$ has finite ascent and descent. Then $T - \lambda_0 I$ has finite ascent and descent, and therefore $\lambda_0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$.

From the above proof, we see that generalized a-Weyl's theorem holds for T. Using Theorem 2.2, we conclude that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

If T^* is M-hyponormal, then since M-hyponormality is translation-invariant, it suffices to show that $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) \Leftrightarrow 0 \in E^{\mathrm{a}}(T)$. If $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$, then $T - \lambda I \in \mathrm{SF}_+^-(H)$ and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $|\lambda|$ is sufficiently small. Since T^* is M-hyponormal, it follows that $\alpha(T^* - \overline{\lambda} I) = \beta(T - \lambda I) < \infty$, hence $T - \lambda I$ is Browder. Then $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$ and therefore 0 is an isolated point in $\sigma_{\mathrm{a}}(T) = \sigma(T)$. We have thus proved that $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) \subseteq E^{\mathrm{a}}(T)$. Conversely, suppose $0 \in E^{\mathrm{a}}(T)$. Then 0 is an isolated point of $\sigma_{\mathrm{a}}(T) = \sigma(T)$. Applying the projection P defined above, we have a direct sum $T^* = T_1 \oplus T_2$, where $\sigma(T_1) = \{0\}$ and $\sigma(T_2) = \sigma(T^*) \setminus \{0\}$. Then $T_1 = 0$ and T_2 is invertible. Thus T^* has finite ascent and descent and so T has finite ascent and descent. This implies that T is B-Weyl and $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. Thus T obeys generalized a-Weyl's theorem. Theorem 2.2 tells us now that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. ∎

COROLLARY 3.4. *If T or T^* is p-hyponormal or M-hyponormal, then generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Hence Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. If T^* is p-hyponormal or M-hyponormal, then the result is true by Theorem 3.3. If T is p-hyponormal or M-hyponormal, then Weyl's theorem holds for T, $T \in A_1(T)$ and T is isoloid. By Theorem 2.1, we need to prove that generalized Weyl's theorem holds for T, so we need to prove $\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = E(T)$.

Suppose $\lambda_0 \in E(T)$. Using the same method as for Theorem 3.3, we can prove that $T - \lambda_0 I$ has finite ascent and descent. Suppose $\alpha(T - \lambda_0 I) = \beta(T - \lambda_0 I) = p$. Then $H = N[(T - \lambda_0 I)^p] \oplus R[(T - \lambda_0 I)^p]$, therefore $(T - \lambda_0 I)_{[p]}$ is Weyl and so $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$.

Conversely, if $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$, then $T - \lambda I$ is Weyl and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $|\lambda - \lambda_0|$ is sufficiently small. The fact that Weyl's theorem holds for T implies that $T - \lambda I$ is Browder and hence $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$, so $T - \lambda I$ is invertible if $|\lambda - \lambda_0|$ is sufficiently small. Therefore $\lambda_0 \in \operatorname{iso} \sigma(T)$. If $N(T - \lambda_0 I) = \{0\}$, suppose $\operatorname{dis}(T - \lambda_0 I) = d$. Then $\alpha(T - \lambda_0 I) = 0$. Since $T - \lambda_0 I$ is B-Weyl, we know that $(T - \lambda_0 I)_{[d]}$ is invertible and hence $R[(T - \lambda_0 I)^{d+1}] = R[(T - \lambda_0 I)^d]$. Thus $\beta(T - \lambda_0 I) = \alpha(T - \lambda_0 I) = 0$, that is, $T - \lambda_0 I$ is invertible. This contradicts the fact that $\lambda_0 \in \sigma(T)$. Thus $\lambda_0 \in E(T)$. Therefore generalized Weyl's theorem holds for T. ∎

COROLLARY 3.5. *If T is p-hyponormal or M-hyponormal and if $\sigma_{\mathrm{BW}}(T) = \{0\}$, then T is normal.*

Proof. Since generalized Weyl's theorem holds for T, by assumption, every nonzero point of $\sigma(T)$ is an isolated point of $\sigma(T)$ and an eigenvalue. Hence $\sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$ is a finite set or a countably infinite set whose only cluster point is 0. Let $\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = \{\lambda_n\}$, with $|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$, and let E_n be the orthogonal projection onto $N(T - \lambda_n I)$. Then $TE_n = E_n T = \lambda_n E_n$ and $E_n E_m = 0$ if $n \neq m$. Put $E = \bigoplus_n E_n$. Then $T = \bigoplus_n \lambda_n E_n \oplus T_{(I-E)H}$ with $\sigma(T_{(I-E)H}) = \{0\}$. Since $T_{(I-E)H}$ is also p-hyponormal or M-hyponormal, it follows that $T_{(I-E)H} = 0$. Hence $T = \bigoplus_n \lambda_n E_n$ is normal. ∎

4. Berberian spectra. Suppose that $T \in B(H)$ is reduced by each of its eigenspaces. We write $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{0\mathrm{f}}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0\mathrm{i}}(T)$ for the eigenvalues of infinite multiplicity. If M is the closed linear span of the eigenspaces $N(T - \lambda I)$ ($\lambda \in \pi_0(T)$), then M reduces T. Let $T_1 = T|_M$ and $T_2 = T|_{M^\perp}$. Then ([1, Proposition 4.1]):

(1) T_1 is a normal operator with pure point spectrum;
(2) $\pi_0(T_1) = \pi_0(T)$;

(3) $\sigma(T_1) = \operatorname{cl} \pi_0(T_1)$;
(4) $\pi_0(T_2) = \emptyset$.

In this case, Berberian [1, Definition] defined

$$\tau(T) = \sigma(T_2) \cup \operatorname{acc} \pi_0(T) \cup \pi_{0\mathrm{i}}(T).$$

We shall call $\tau(T)$ the *Berberian spectrum* of T. Berberian has shown that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$.

Let

$$\mathrm{LD}(H) = \{T \in B(H) : \alpha(T) < \infty \text{ and } R(T^{\alpha(T)+1}) \text{ is closed}\},$$
$$\sigma_{\mathrm{LD}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } \mathrm{LD}(H)\}.$$

The following theorem shows the relation of $\sigma_{\mathrm{SBF}_+^-}(T)$, $\sigma_{\mathrm{LD}}(T)$, and Berberian spectra. We also give a relation of $\sigma_{\mathrm{BW}}(T)$, $\sigma_{\mathrm{BB}}(T)$ and Berberian spectra.

THEOREM 4.1. *If $T \in B(H)$ is reduced by each of its eigenspaces, then*

(1) $\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_{\mathrm{LD}}(T) \subseteq \tau(T)$;
(2) $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{BB}}(T) \subseteq \tau(T)$.

Proof. Let M be the closed linear span of the eigenspaces $N(T - \lambda I)$ ($\lambda \in \pi_0(T)$) and write $T_1 = T|_M$ and $T_2 = T|_{M^\perp}$. From the preceding arguments it follows that T_1 is normal, $\pi_0(T_1) = \pi_0(T)$ and $\pi_{0\mathrm{f}}(T_2) = \emptyset$.

(1) First we will prove that $\sigma_{\mathrm{SBF}_+^-}(T) \subseteq \tau(T)$ and $\sigma_{\mathrm{BW}}(T) \subseteq \tau(T)$.

Suppose $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda I$ is invertible and λ is an isolated point of $\pi_0(T_1)$. Since also $\pi_{0\mathrm{i}}(T) \subseteq \tau(T)$, we have $\lambda \in \pi_{00}(T_1)$. But T_1 is normal, hence Weyl's theorem holds for T_1. Therefore $T_1 - \lambda I$ is Weyl. Thus $T - \lambda I$ is Weyl. Now we conclude that λ is not in $\sigma_{\mathrm{SBF}_+^-}(T) \cup \sigma_{\mathrm{BW}}(T)$.

(2) Second we will prove that $\sigma_{\mathrm{LD}}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$ and $\sigma_{\mathrm{BB}}(T) = \sigma_{\mathrm{BW}}(T)$.

By definition, $\sigma_{\mathrm{SBF}_+^-}(T) \subseteq \sigma_{\mathrm{LD}}(T)$. Let $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. Then $T-\lambda I \in \mathrm{SF}_+^-(H)$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^\infty R[(T-\lambda I)^n]$ if $|\lambda-\lambda_0|$ is sufficiently small. Suppose that there exists λ such that $|\lambda - \lambda_0|$ is sufficiently small and $\lambda \in \sigma_{\mathrm{a}}(T)$. Since $\pi_0(T_2) = \emptyset$, it follows that $T_1 - \lambda I \in \mathrm{SF}_+^-(M)$ and $T_2 - \lambda I$ is bounded from below. The fact that T_1 is normal implies that $\alpha(T_1 - \lambda I) < \infty$, and hence $\alpha(T - \lambda I) < \infty$. Thus $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap_{n=1}^\infty R[(T - \lambda I)^n] = \{0\}$, so $T - \lambda I$ is bounded from below. This is a contradiction. Thus $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$ or $\lambda_0 \in \varrho_{\mathrm{a}}(T)$. Now [4, Theorem 2.8] shows that λ_0 is not in $\sigma_{\mathrm{LD}}(T)$. Similarly, we can prove that $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{BB}}(T)$. ■

COROLLARY 4.2. *If $T \in B(H)$ is reduced by each of its eigenspaces, then*

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) \subseteq E^{\mathrm{a}}(T), \qquad \sigma(T) \setminus \sigma_{\mathrm{BW}}(T) \subseteq E(T).$$

An operator T is called *reduction-isoloid* (resp. *reduction a-isoloid*) if the restriction of T to any reducing subspace is isoloid (resp. a-isoloid). It is well known that every hyponormal operator is reduction-isoloid.

THEOREM 4.3. *If $T \in B(H)$ is reduced by each of its eigenspaces and is reduction a-isoloid* (*resp. reduction-isoloid*), *then $f(T)$ obeys generalized a-Weyl's theorem* (*resp. generalized Weyl's theorem*) *for every $f \in H(\sigma(T))$. Hence in this case*, *$f(T)$ obeys a-Weyl's theorem* (*resp. Weyl's theorem*) *for every $f \in H(\sigma(T))$.*

Proof. By Theorem 2.2 and Corollary 4.2, we only need to prove $E^{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. Let $\lambda \in E^{\mathrm{a}}(T)$. Since $H = N(T-\lambda I) \oplus N(T-\lambda I)^{\perp}$, we have $T-\lambda I = 0 \oplus S$, where $S = (T-\lambda I)|_{N(T-\lambda I)^{\perp}}$. If $0 \in \sigma_{\mathrm{a}}(S)$, then 0 is an isolated point of $\sigma_{\mathrm{a}}(S)$. But T is reduction a-isoloid, hence $0 \in \pi_0(S)$. This is a contradiction. Therefore 0 is not in $\sigma_{\mathrm{a}}(S)$, which means that S is bounded from below. Then $R[(T-\lambda I)^k] = 0 \oplus R(S^k)$ is closed for every $k \in \mathbb{N}$ and $\alpha(T-\lambda I) < \infty$. Suppose $\alpha(T-\lambda I) = p$. Then $N(T-\lambda I) \cap R[(T-\lambda I)^p] = \{0\}$, and hence $(T-\lambda I)_{[p]}$ is upper semi-B-Fredholm. Thus $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. Now we conclude that generalized a-Weyl's theorem holds for T. ∎

COROLLARY 4.4. *If T is p-hyponormal or M-hyponormal and T is reduction a-isoloid* (*resp. reduction-isoloid*), *then $f(T)$ obeys generalized a-Weyl's theorem* (*resp. generalized Weyl's theorem*) *for every $f \in H(\sigma(T))$.*

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