On Nikodym-type sets in high dimensions

by

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Abstract. We prove that the complement of a higher-dimensional Nikodym set must have full Hausdorff dimension.

1. Introduction. In [4] Nikodym constructed a subset $F$ of the unit square in $\mathbb{R}^2$ such that $F$ has planar measure 1, and for every point $x \in F$ there exists a line passing through $x$ intersecting $F$ in that single point. Such paradoxical sets are called Nikodym sets.

Falconer [3] extended Nikodym’s result to higher dimensions. He proved that for every $n > 2$ there exists a set $F \subset \mathbb{R}^n$ such that the complement of $F$ has Lebesgue measure zero, and for every $x \in F$ there is a hyperplane $H$ so that $x \in H$ and $F \cap H = \{x\}$. We call such a set an $n$-Nikodym set.

The purpose of this paper is to show that the complement of an $n$-Nikodym set, even though small in terms of Lebesgue measure, must be large in terms of Hausdorff dimension. Namely, we use ideas from [1] and [2] to prove the following.

Theorem. The Hausdorff dimension of the complement of an $n$-Nikodym set is equal to $n$.

A few remarks about our notation. $\mathcal{L}^k(\cdot)$ denotes $k$-dimensional Lebesgue measure and $\text{card}(\cdot)$ cardinality; $B(x, r)$ is the ball with center $x$ and radius $r$; $\chi_A$ is the characteristic function of the set $A$; finally, $x \preceq y$ means $x \leq Cy$, where $C$ is some positive constant not necessarily the same at each occurrence.

2. Proof of the Theorem. Let $E$ be the complement of an $n$-Nikodym set in $\mathbb{R}^n$. Without loss of generality we may assume that there is a subset $A$ of the unit cube with $\mathcal{L}^n(A) > 0$ such that for every $x \in A$ there exists a set $H_x$ with the following properties:

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(P1) $H_x$ is a rotated translation of $[0, 1] \times \cdots \times [0, 1] \times \{0\}$. 

(P2) The center of $H_x$ is the point $x$.

(P3) The normal vector to $H_x$ makes an angle less than $\pi/100$ with the unit vector $e_n = (0, \ldots, 0, 1)$.

(P4) $H_x \cap E = H_x \setminus \{x\}$, so in particular $\mathcal{L}^{n-1}(E \cap H_x) = 1$.

We will show that for every $\varepsilon > 0$ the $(n - \varepsilon)$-dimensional Hausdorff measure of $E$ is not zero. Therefore, the Hausdorff dimension of $E$ must equal $n$. To this end, fix a countable covering $\{B(x_j, r_j)\}$ of $E$, and for every integer $k$ let

$$J_k = \{j : 2^{-k} \leq r_j \leq 2^{-(k-1)}\},$$
$$E_k = E \cap \bigcup_{j \in J_k} B(x_j, r_j), \quad \tilde{E}_k = \bigcup_{j \in J_k} B(x_j, 2r_j).$$

We will bound $\sum_j r_j^{n-\varepsilon}$ from below by a constant depending only on $\varepsilon$.

Notice that for every $x \in A$ there exists an integer $k_x$ such that

$$\mathcal{L}^{n-1}(E_{k_x} \cap H_x) \geq \frac{1}{4k_x^2}.$$ 

Indeed, if this were not the case for some $x \in A$, we would have

$$1 = \mathcal{L}^{n-1}(E \cap H_x) \leq \sum_k \mathcal{L}^{n-1}(E_k \cap H_x) \leq \sum_k \frac{1}{4k^2} < \frac{1}{2}.$$ 

Now let

$$A_k = \left\{ x \in A : \mathcal{L}^{n-1}(E_k \cap H_x) \geq \frac{1}{4k^2} \right\}.$$ 

Then

$$A = \bigcup_k A_k.$$ 

Therefore, there must be an integer $N$ such that

$$\mathcal{L}^n(A_N) \geq \frac{\mathcal{L}^n(A)}{2N^2},$$

because otherwise we would have

$$\mathcal{L}^n(A) \leq \sum_k \mathcal{L}^n(A_k) \leq \sum_k \frac{\mathcal{L}^n(A)}{2k^2} < \mathcal{L}^n(A).$$

Next, we decompose the unit cube into a grid of small cubes, each of side $2^{-N}$:

$$[0, 1]^n = \bigcup_{i_1, \ldots, i_n = 1}^{2^N} \prod_{k=1}^{n} [(i_k - 1)2^{-N}, i_k 2^{-N}] = \bigcup_{i_1, \ldots, i_n = 1}^{2^N} Q_{i_1 \ldots i_n}.$$
Let

\[ I = \{(i_1, \ldots, i_n) : Q_{i_1 \ldots i_n} \cap A_N \neq \emptyset\}. \]

Notice that for each \((i_1, \ldots, i_n) \in I\), property (P2) and (1) imply that there exists a rectangle \(R_{i_1 \ldots i_n}\) such that

- \(R_{i_1 \ldots i_n}\) has dimensions \(1 \times \cdots \times 1 \times 2^{-N}\).
- \(R_{i_1 \ldots i_n}\) is parallel to \(H_x\) for some \(x \in Q_{i_1 \ldots i_n}\).
- \(R_{i_1 \ldots i_n} \cap Q_{i_1 \ldots i_n} \neq \emptyset\).
- \(\mathcal{L}^n(\tilde{E}_N \cap R_{i_1 \ldots i_n}) \geq N^{-2}2^{-N}\).

Now let

\[ R'_{i_1 \ldots i_n} = \begin{cases} R_{i_1 \ldots i_n} & \text{if } (i_1, \ldots, i_n) \in I, \\ \emptyset & \text{otherwise}. \end{cases} \]

Then

\[ N^{-2}\mathcal{L}^n(A) \leq \mathcal{L}^n(A_N) \leq \sum_{(i_1, \ldots, i_n) \in I} 2^{-nN} = 2^{-(n-1)N} N^2 \sum_{(i_1, \ldots, i_n) \in I} N^{-2}2^{-N} \]

\[ \leq 2^{-(n-1)N} N^2 \sum_{i_1, \ldots, i_{n-1} = 1}^{2^N} \mathcal{L}^n(\tilde{E}_N \cap R'_{i_1 \ldots i_n}) \]

\[ = 2^{-(n-1)N} N^2 \sum_{i_1, \ldots, i_{n-1} = 1}^{2^N} \left( \int_{\tilde{E}_N} \sum_{i_n = 1}^{2^N} \chi_{R'_{i_1 \ldots i_n}} \right)^{1/2} \]

\[ \leq 2^{-(n-1)N} N^2 \mathcal{L}^n(\tilde{E}_N)^{1/2} \sum_{i_1, \ldots, i_{n-1} = 1}^{2^N} \left( \sum_{i_n = 1}^{2^N} \chi_{R'_{i_1 \ldots i_n}} \right)^{1/2} \]

\[ = 2^{-(n-1)N} N^2 \mathcal{L}^n(\tilde{E}_N)^{1/2} \sum_{i_1, \ldots, i_{n-1} = 1}^{2^N} \left( \sum_{l,m=1}^{2^N} \chi_{R'_{i_1 \ldots i_{n-1}l} \cap R'_{i_1 \ldots i_{n-1}m}} \right)^{1/2} \]

Now using property (P3), it is easy to show that for fixed \(i_1, \ldots, i_{n-1}\) we have

\[ \mathcal{L}^n(R'_{i_1 \ldots i_{n-1}l} \cap R'_{i_1 \ldots i_{n-1}m}) \leq \frac{2^{-N}}{1 + |m - l|}. \]

Consequently,

\[ \sum_{l,m=1}^{2^N} \mathcal{L}^n(R'_{i_1 \ldots i_{n-1}l} \cap R'_{i_1 \ldots i_{n-1}m}) \leq \log 2^N = N \log 2. \]
Therefore
\[ N^{-2} \mathcal{L}^n(A) \lesssim 2^{-(n-1)N} N^2 \mathcal{L}^n(\tilde{E}_N)^{1/2} 2^{(n-1)N} N^{1/2} \]
and so
\[ \mathcal{L}^n(\tilde{E}_N) \gtrsim N^{-9} \mathcal{L}^n(A)^2. \]
On the other hand, by the definition of \( \tilde{E}_N \) we have
\[ \mathcal{L}^n(\tilde{E}_N) \lesssim \text{card}(J_N) 2^{-nN}. \]
Hence
\[ \text{card}(J_N) \gtrsim 2^{nN} N^{-9} \mathcal{L}^n(A)^2. \]
We conclude that
\[ \sum_j r_j^{n-\varepsilon} \gtrsim \text{card}(J_N)(2^{-N})^{n-\varepsilon} \gtrsim 2^{N\varepsilon} N^{-9} \mathcal{L}^n(A)^2 \gtrsim C_\varepsilon. \]
The proof is complete.

References


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