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Hilbert C^*-modules from group actions: beyond the finite orbits case

by

Michael Frank (Leipzig), Vladimir Manuilov (Moscow and Harbin) and Evgenij Troitsky (Moscow)

Abstract. Continuous actions of topological groups on compact Hausdorff spaces X are investigated which induce almost periodic functions in the corresponding commutative C^*-algebra. The unique invariant mean on the group resulting from averaging allows one to derive a C^*-valued inner product and a Hilbert C^*-module which serve as an environment to describe characteristics of the group action. For Lyapunov stable actions the derived invariant mean $M(\phi_x)$ is continuous on X for any $\phi \in C(X)$, and the induced C^*-valued inner product corresponds to a conditional expectation from $C(X)$ onto the fixed-point algebra of the action defined by averaging on orbits. In the case of self-duality of the Hilbert C^*-module all orbits are shown to have the same cardinality. Stable actions on compact metric spaces give rise to C^*-reflexive Hilbert C^*-modules. The same is true if the cardinality of finite orbits is uniformly bounded and the number of closures of infinite orbits is finite. A number of examples illustrate typical situations appearing beyond the classified cases.

1. Introduction. To investigate continuous actions of groups on topological spaces, several approaches may be applied. In the present paper the authors continue their work started in [5, 18] which relies on the Gelfand duality of locally compact Hausdorff spaces and commutative C^*-algebras. In the dual picture some well-known results from functional analysis and non-commutative geometry can be applied to get new insights, often also for related non-commutative situations of group actions on general C^*-algebras.

Consider a continuous action of a topological group Γ on a compact Hausdorff space X. In view of the Gelfand duality it can be seen as a continuous action of Γ on the commutative C^*-algebra $C(X)$ of all continuous complex-valued functions on X. Let us denote the subalgebra of Γ-invariant functions on X by $C_\Gamma(X) \subset C(X)$.

We wish to introduce on $C(X)$ the structure of a pre-Hilbert C^*-module over $C_\Gamma(X)$ which expresses significant properties of the action of Γ on X.

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One way to find suitable C^*-valued inner products on $C(X)$ is the search for conditional expectations $E : C(X) \to C_\Gamma(X)$ which are a kind of mean over the group action of Γ on $C(X)$ and canonically give rise to the Hilbert $C_\Gamma(X)$-module structures on $C(X)$ we are looking for. We followed that approach in [5, 18] (see [15] for a related discussion).

Here we want to consider a more general approach closer to the topological background. For $\phi, \psi \in C(X)$ and for the derived group maps

$$\phi_x : \Gamma \to \mathbb{C}, \qquad \phi_x(g) = \phi(gx) \qquad (x \in X), \tag{1.1}$$

we want to select a suitable normalized invariant mean m_Γ on Γ such that a $C_\Gamma(X)$-valued inner product on $C(X)$ could be defined by

$$\langle \phi, \psi \rangle(x) := m_\Gamma(\phi_x \overline{\psi_x}) \qquad (x \in X). \tag{1.2}$$

Of course, we would have to suppose Γ to be amenable at this point to warrant the existence of the (left) invariant mean m_Γ. The product (1.2) has to satisfy at least the following two properties (with Γ-invariance following from the definition):

1) The resulting functions $m_\Gamma(\phi_x \overline{\psi_x})$ are continuous in $x \in X$.
2) The value $\langle \phi, \phi \rangle(x)$ is always positive if $\phi(gx) \neq 0$ for some $g \in \Gamma$ and some $x \in X$.

One can observe that property 2) would follow e.g. from the following supposition:

2′) For any $x \in X$ and any $\phi \in C(X)$ the map ϕ_x is almost periodic on Γ.

The supposition 2′) would allow us:

- to avoid the restriction on Γ to be amenable,
- to overcome the dependence on the particular choice of m_Γ,

by passing from (1.2) to

$$\langle \phi, \psi \rangle(x) := M(\phi_x \overline{\psi_x}), \tag{1.3}$$

where the map M is the unique invariant mean on almost periodic functions with respect to the given action of Γ, when 1) and 2′) are supposed to hold. The link to results in [5, 18] is given by constructing a suitable conditional expectation $E_\Gamma : C(X) \to C_\Gamma(X)$ by the rule

1′) For any $\varphi \in C(X)$ the function $E_\Gamma(\phi)(x) := M(\phi_x)$ is continuous in x.

Properties 1′) and 2′) ensure that the formula (1.2) makes $C(X)$ a pre-Hilbert C^*-module over $C_\Gamma(X)$. Let us denote its completion by $L_\Gamma(X)$.

We are interested in two questions here:

- For which actions do conditions 1′) and 2′) hold?
- If they hold, what properties does $L_\Gamma(X)$ have?

Our reference on almost periodic functions is [3]. Hilbert C^*-modules were introduced in [14] and [10]. For facts on Hilbert C^*-modules we refer the reader to [8, 7, 13]. Recall that for a Hilbert C^*-module L over a C^*-algebra A the A-dual module L' is the module of all bounded A-linear maps from L to A. The C^*-module L is called *self-dual* (resp. *C^*-reflexive*) if $L = L'$ (resp. if $L = L''$).

Our paper is organized as follows: In Section 2 we give some sufficient conditions for conditions $1'$) and $2'$) to hold, hence, for the existence of $C_\Gamma(X)$-valued inner products on the C^*-algebra $C(X)$. We also show that our type of averaging is the same as averaging over orbits. Section 3 deals with more restrictive situations in which the resulting Hilbert C^*-module turns out to be self-dual. In Section 4 we revisit the situation of C^*-reflexive Hilbert C^*-modules and obtain an important restriction on X. In Section 5 we give some examples showing different possible behaviors of an averaging.

2. Lyapunov stability and continuity of averaging. We want to find conditions under which a well defined averaging over the group action on orbits exists in the case of infinite orbits. For this purpose we introduce, in addition to the uniform continuity discussed in [5, 18], the condition of Lyapunov stability. The latter condition ensures uniform continuity, the well definedness of averaging and the existence of a conditional expectation onto the fixed-point algebra, which gives rise to a C^*-valued inner product and a resulting Hilbert C^*-module structure. In subsequent sections we apply this tool to characterize those group actions on compact Hausdorff spaces with infinite orbits.

Definition 2.1. We say that an action of a group G on a locally compact Hausdorff space X is *uniformly continuous* if for every point $x \in X$ and every neighborhood U_x of x there exists a neighborhood V_x of x such that $g(V_x) \subseteq U_x$ for every $g \in G_x$, where G_x denotes the stabilizer of x.

Theorem 2.2. *Let an action of a topological group Γ on a compact Hausdorff space X be uniformly continuous. If all orbits are finite and if their size is uniformly bounded then the average $M(\varphi_x)$ is continuous with respect to $x \in X$ for any $\varphi \in C(X)$.*

Proof. If an orbit Γx is finite then the function φ_x on Γ is exactly periodic, hence

$$M(\varphi_x) = \frac{1}{\#\Gamma x} \sum_{gx \in \Gamma x} \varphi(gx),$$

so the average on Γ is the same as the average over orbits. The continuity of the latter is provided by Lemma 2.11 from [5]. ∎

Example 5.2 below demonstrates that if infinite orbits are present, the uniform continuity is not sufficient for the continuity of the average.

Now we generalize the approach of [18] and introduce a condition which is sufficient to overcome these difficulties.

Recall from [1] that, on a compact space, there is a unique uniform structure Φ compatible with its topology. It consists of *all* neighborhoods of the diagonal in $X \times X$ [1, Ch. II, Sect. 4, Theorem 1]. If X is a metric space with a metric d then a base for the uniform structure is the set of neighborhoods of the diagonal $\Delta \subset X \times X$ of the form $\{(x, y) : x, y \in X, d(x, y) < \varepsilon\}$, $\varepsilon \in (0, \infty)$.

Definition 2.3. An action of a group Γ on a topological space X with a uniform structure Φ compatible with its topology is called *Lyapunov stable* (or uniformly equicontinuous ([1])) if for any $\mathbb{U} \in \Phi$ and any $x \in X$ there is $\mathbb{V} \in \Phi$ such that $(gx, gy) \in \mathbb{U}$ for any $g \in \Gamma$ whenever $(x, y) \in \mathbb{V}$.

Note that in the case of a *metric* space, this definition takes the following form:

Definition 2.4. An action of a group Γ on a metric space X is called *Lyapunov stable* if for any $\varepsilon > 0$ and any $x \in X$ there exists $\delta > 0$ such that

$$\rho(gx, gy) < \varepsilon \quad \text{for any } g \in \Gamma \quad \text{whenever } \rho(x, y) < \delta.$$

Lemma 2.5. *If an action of a discrete group Γ on a topological space X with a uniform structure is Lyapunov stable then it is uniformly continuous.*

Proof. For $x \in X$ and $\mathbb{U} \in \Phi$ set $\mathbb{U}(x) := \{y \in X : (x, y) \in \mathbb{U}\}$. If W is a neighborhood of x then there is $\mathbb{U} \in \Phi$ such that $\mathbb{U}(x) \subset W$. By stability, there is $\mathbb{V} \in \Phi$ such that $(gx, gy) \in \mathbb{U}$ for any $g \in \Gamma$ whenever $(x, y) \in \mathbb{V}$. Now let $g \in \Gamma_x$. Take any $y \in \mathbb{V}(x)$. Then $(x, gy) \in \mathbb{U}$, hence $gy \in \mathbb{U}(x) \subset W$, i.e. $g(\mathbb{V}(x)) \subset W$ for any $g \in \Gamma_x$. ■

In the case when all orbits are finite, uniform continuity is equivalent to Lyapunov stability:

Proposition 2.6. *Let a discrete group act uniformly continuously on a compact Hausdorff space X and suppose all the orbits are finite. Then the action is Lyapunov stable.*

Proof. Take a neighborhood $\mathbb{W}$ of the diagonal in $X \times X$ and pick a point $x \in X$. Since its orbit is finite, we can select a finite set $\{g_1, \dots, g_s\} \subset \Gamma$ such that $\{g_1 x, \dots, g_s x\}$ is the orbit Γx. Now find a neighborhood U^x of x such that $g_i(U^x) \times g_i(U^x) \subset \mathbb{W}$ for each $i = 1, \dots, s$.

Uniform continuity implies that there exists a neighborhood V^x of x such that $hy \in U^x$ for any $y \in V^x$ and any $h \in \Gamma_x$. Since any $g \in \Gamma$ can

([1]) As was kindly pointed out by the referee.

be written as $g = g_i h$ for some $i = 1, \dots, s$ and some $h \in \Gamma_x$, we have $g(V^x) = g_i(h(V^x)) \subset g_i(U^x)$.

It follows from the compactness of X that there are a finite number of points $x_1, \dots, x_r$ in X such that the sets $V^{x_1}, \dots, V^{x_r}$ form a finite covering for X. Then $\mathbb{W}_0 = V^{x_1} \times V^{x_1} \cup \dots \cup V^{x_r} \times V^{x_r}$ is a neighborhood of the diagonal in $X \times X$.

Take $(y,z) \in \mathbb{W}_0$. Then there is some $1 \le j \le r$ such that $(y,z) \in V^{x_j} \times V^{x_j}$. Hence $(gy, gz) \in g_i(U^{x_j}) \times g_i(U^{x_j})$ for some i. By construction, $g_i(U^{x_j}) \times g_i(U^{x_j}) \subset \mathbb{W}$, so we conclude that $(gy, gz) \in \mathbb{W}$ for any $g \in \Gamma$ whenever $(y,z) \in \mathbb{W}_0$. ∎

PROPOSITION 2.7. *Let a discrete group Γ act* Lyapunov stably *on a compact Hausdorff space X and let $\varphi : X \to \mathbb{C}$ be a continuous function. Then, for any $x \in X$, the function $\varphi_x : \Gamma \to \mathbb{C}$, $\varphi_x(g) := \varphi(gx)$, is almost periodic.*

Proof. The proof repeats the proof of [18, Proposition 29] with obvious modifications to pass from the metric case to the uniform case. ∎

So, under the conditions of Proposition 2.7 the invariant mean $M(\varphi_x)$ is well defined on $C(X)$.

THEOREM 2.8. *Let a discrete group Γ act on a compact Hausdorff space X. If the action is Lyapunov stable, then the conditional expectation $E_\Gamma : C(X) \to C_\Gamma(X)$ defined by $E_\Gamma(\phi)(x) = M(\phi_x)$ is well defined, i.e. conditions* 1′) *and* 2′) *hold.*

Proof. Repeat the proof of [18, Theorem 30] with obvious modifications. ∎

For $x \in X$ let us denote its orbit Γx by γ and the closure of γ in X by $\overline{\gamma}$.

THEOREM 2.9. *Let a discrete group Γ act on a compact Hausdorff space X.*

(i) *If the action is Lyapunov stable, then for the unique invariant mean M on the set of almost periodic functions on Γ we have the equality*

$$M(\varphi_x) = \int_{\overline{\gamma}} \varphi|_{\overline{\gamma}} \, d\mu_{\overline{\gamma}}, \tag{2.1}$$

where $\gamma = \Gamma x$ and $\mu_{\overline{\gamma}}$ is a (unique) invariant measure on $\overline{\gamma}$ of total mass 1.

(ii) *If, in addition, γ is finite, then $M(\varphi_x)$, $x \in \gamma$, can be taken as the standard average, as considered in* [5, 18].

Proof. Evidently, (ii) follows from (i).

Let us show that for $\varphi \in C(X)$ the left-hand side of (2.1) does not depend on $x \in \overline{\gamma}$. First, evidently, it does not depend on the choice of x inside the same orbit. Hence, it is sufficient to verify that the value is the same for gx_2 sufficiently close to any x_1 for $x_1, x_2 \in \overline{\gamma}$ to demonstrate the invariance with respect to the action of Γ. By Lyapunov stability, for any $\varepsilon > 0$ we can find $g_\varepsilon \in \Gamma$ such that $g_\varepsilon x_2$ is so close to x_1 that $|\varphi(gx_1) - \varphi(gg_\varepsilon x_2)| < \varepsilon$ for any $g \in \Gamma$. Then

$$|M(\varphi_{x_1}) - M(\varphi_{x_2})| = |M(\varphi_{x_1}) - M(\varphi_{g_\varepsilon x_2})| = |M(\varphi_{x_1} - \varphi_{g_\varepsilon x_2})| \\ \leq \sup_{g\in\Gamma} |\varphi(gx_1) - \varphi(gg_\varepsilon x_2)| < \varepsilon.$$

Since ε can be arbitrarily small, $M(\varphi_{x_1}) = M(\varphi_{x_2})$ and the value is constant on the closure of each orbit.

Thus, we have a continuous functional $m : C(\overline{\gamma}) \to \mathbb{C}$, $m(\phi) = M(\phi_x)$ for $x \in \overline{\gamma}$. By the Riesz–Markov–Kakutani theorem [4, Theorem 3, Sect. IV.6], m has the form

$$m(f) = \int_{\overline{\gamma}} f\, d\mu,$$

where μ is some regular countably additive complex measure on $\overline{\gamma}$. Evidently, μ is invariant. It remains to explain why μ is unique. In fact, this follows from [2, Ch. VII, Sect. 1, Problem 14]. ■

3. Self-duality. After a characterization of the inner structure of Hilbert C^*-modules that arise from Lyapunov stable actions we are going to describe the interrelation between certain properties of the action and self-duality of the resulting Hilbert C^*-module.

Since, by Lyapunov stability, the closure $\overline{\gamma}$ of an orbit is a minimal closed invariant subset containing x, where $\gamma = \Gamma x$, it follows that the closures of two orbits $\gamma = \Gamma x$ and $\gamma' = \Gamma y$ either coincide or do not intersect (cf. [6, Theorem 2.38]). Thus, the space of closures of orbits is well defined as a quotient space of X. Denote it by $\widetilde{X/\Gamma}$.

Lemma 3.1. *Let an action of a discrete group Γ on a compact Hausdorff space X be Lyapunov stable. Then the quotient space of closures of orbits, $\widetilde{X/\Gamma}$, is Hausdorff, and hence it coincides with the Gelfand spectrum of $C_\Gamma(X)$.*

Proof. For $\overline{\gamma_1} \neq \overline{\gamma}$ we can choose neighborhoods $U_1 \supset \overline{\gamma_1}$ and $U \supset \overline{\gamma}$ such that $\overline{U} \cap \overline{U_1} = \emptyset$, i.e. $(\overline{U} \times \overline{U_1}) \cap \Delta = \emptyset$, where $\Delta \subset X \times X$ is the diagonal. Take a neighborhood $\mathbb{U}_0$ of Δ such that $\mathbb{U}_0 \cap (\overline{U} \times \overline{U_1}) = \emptyset$. By the axioms of uniform structure, there exists a neighborhood $\mathbb{U}$ of Δ such that $\mathbb{U} \circ \mathbb{U} \subset \mathbb{U}_0$, where

$$\mathbb{U} \circ \mathbb{V} := \{(u, v) \mid \exists z \in X \text{ such that } (u, z) \in \mathbb{U},\ (z, v) \in \mathbb{V}\}.$$

Thus for this $\mathbb{U}$ there is no $u \in X$ such that $(z, u) \in \mathbb{U}$ and $(u, z_1) \in \mathbb{U}$ for some $z \in \overline{\gamma}$, $z_1 \in \overline{\gamma_1}$, i.e. $(z, z_1) \notin \mathbb{U} \circ \mathbb{U}$ for any $z \in \overline{\gamma}$, $z_1 \in \overline{\gamma_1}$.

By Lyapunov stability we can choose $\mathbb{V} \subset \mathbb{U}$ such that $g\mathbb{V} \subset \mathbb{U}$ for any $g \in \Gamma$. Thus $\mathbb{V}' := \Gamma\mathbb{V} \subset \mathbb{U}$ is an invariant neighborhood of Δ. Set

$$V := \mathbb{V}'[\overline{\gamma}] := \{u \in X \mid \exists z \in \overline{\gamma} \text{ such that } (z, u) \in \mathbb{V}'\} \supset \overline{\gamma},$$
$$V_1 := (\mathbb{V}')^{-1}[\overline{\gamma_1}] := \{u \in X \mid \exists z_1 \in \overline{\gamma_1} \text{ such that } (u, z_1) \in \mathbb{V}'\} \supset \overline{\gamma_1}.$$

Since $\overline{\gamma}$, $\overline{\gamma_1}$ and $\mathbb{V}'$ are invariant, V and V_1 are invariant neighborhoods of $\overline{\gamma}$ and $\overline{\gamma_1}$ respectively. Suppose that they intersect at some v. Then

$$(z, z_1) \in \mathbb{V}' \circ \mathbb{V}' \subset \mathbb{U} \circ \mathbb{U} \quad \text{for some } z \in \overline{\gamma},\ z_1 \in \overline{\gamma_1},$$

which contradicts the properties of $\mathbb{U}$. ∎

THEOREM 3.2. *Let a discrete group Γ act on a compact Hausdorff space X. If the action is Lyapunov stable, then the module $L_\Gamma(X)$ (defined on p.* 132 *as the completion of $C(X)$ with respect to the inner product $\langle \cdot, \cdot \rangle_L$) consists of all functions $\psi : X \to \mathbb{C}$ such that*

(i) *$\psi|_{\overline{\gamma}} \in L^2(\overline{\gamma}, \mu_{\overline{\gamma}})$, where $\mu_{\overline{\gamma}}$ is a unique normalized invariant measure on $\overline{\gamma}$ for any orbit γ,*
(ii) *for any $\varphi \in C(X)$ the function $\langle \psi, \varphi \rangle_L$ is continuous.*

In particular, the average $\langle \psi, \mathbf{1} \rangle_L$ of such a function ψ is continuous on X/Γ. (We do not distinguish functions which differ on $\mu_{\overline{\gamma}}$-negligible sets.)

Proof. Denote the set of all functions satisfying (i) and (ii) by S. We should prove the following facts:

(a) continuous functions on X satisfy these conditions, i.e. $C(X) \subset S$,
(b) $C(X)$ is dense in S with respect to the C^*-valued inner product on the module $L_\Gamma(X)$,
(c) S is a (complete) Hilbert module.

Condition (i) of the assertion above should be interpreted via the equality

$$\int_{\overline{\gamma}} \psi|_{\overline{\gamma}}\, d\mu_{\overline{\gamma}} = M(\psi_x), \qquad \psi \in C(X),\ x \in \gamma, \tag{3.1}$$

by Theorem 2.9(i). Thus, by Proposition 2.7 and Theorem 2.8, condition (a) is fulfilled.

Now take an arbitrary function $\psi(x)$ satisfying conditions (i) and (ii), an arbitrary function $\varphi \in C(X)$ with $\|\varphi\|_L \le 1$, and an arbitrarily small $\varepsilon > 0$. Consider the closure $\overline{\gamma}$ of an orbit. Choose a continuous function $f_{\overline{\gamma}} : \overline{\gamma} \to \mathbb{C}$ such that

$$\int_{\overline{\gamma}} \left|\psi|_{\overline{\gamma}} - f_{\overline{\gamma}}\right|^2 d\mu_{\overline{\gamma}} < \varepsilon^2. \tag{3.2}$$

By normality of X, $f_{\overline{\gamma}}$ can be extended to a continuous function $\widehat{f_{\overline{\gamma}}} : X \to \mathbb{C}$. There exists a neighborhood $U_{\overline{\gamma}}$ of $\overline{\gamma}$ in the Gelfand spectrum $\widetilde{X/\Gamma}$ of $C_\Gamma(X)$ such that

$$\left|\int\limits_{\overline{\gamma'}} (\psi|_{\overline{\gamma'}} - \widehat{f_{\overline{\gamma}}}|_{\overline{\gamma'}})\overline{\varphi}|_{\overline{\gamma'}}\, d\mu_{\overline{\gamma'}}\right| < 2\varepsilon, \qquad \overline{\gamma'} \in U_{\overline{\gamma}}. \tag{3.3}$$

This follows from (3.2) and (ii). Choose a finite subcovering $U_{\overline{\gamma_i}}$ of $\widetilde{X/\Gamma}$ and a subordinated partition of unity ω_i, $i = 1, \ldots, I$. This can be done by Lemma 3.1. We will show that $\sup |\langle \psi - f, \varphi\rangle| \le 2\varepsilon$, where

$$f := \sum_{i=1}^{I} \omega_i^* \widehat{f_{\overline{\gamma_i}}},$$

and ω_i^* is the pullback of ω_i to X. Indeed, take an arbitrary γ_0. Consider neighborhoods $U_{\overline{\gamma_i}}$ such that $\overline{\gamma}_0 \subset U_{\overline{\gamma_i}}$. For simplicity of notation we can assume that this is true for all these neighborhoods. Then

$$\begin{aligned}\left|\int\limits_{\overline{\gamma_0}} (\psi|_{\overline{\gamma_0}} - f|_{\overline{\gamma_0}})\overline{\varphi}|_{\overline{\gamma_0}}\, d\mu_{\overline{\gamma_0}}\right| &= \left|\sum_{i=1}^{I} \omega_i(\overline{\gamma_0}) \int\limits_{\overline{\gamma_0}} (\psi|_{\overline{\gamma_0}} - \widehat{f_{\overline{\gamma_i}}}|_{\overline{\gamma_0}})\overline{\varphi}|_{\overline{\gamma_0}}\, d\mu_{\overline{\gamma_0}}\right| \\ &\le \sum_{i=1}^{I} \omega_i(\overline{\gamma_0}) \cdot \sup_{i=1,\ldots,I}\left|\int\limits_{\overline{\gamma_0}} (\psi|_{\overline{\gamma_0}} - \widehat{f_{\overline{\gamma_i}}}|_{\overline{\gamma_0}})\overline{\varphi}|_{\overline{\gamma_0}}\, d\mu_{\overline{\gamma_0}}\right| \le 2\varepsilon,\end{aligned}$$

because for each i we have the estimate (3.3). Thus $f \in C(X)$ is 2ε-close to ψ. The property (b) is proved.

To prove (c) consider a Cauchy sequence $\psi_n \in S$. Evidently $\psi_n|_{\overline{\gamma}}$ is an L^2-Cauchy sequence with a limit function $\psi_{\overline{\gamma}} \in L^2(\overline{\gamma}, \mu_{\overline{\gamma}})$. Define ψ to be equal to $\psi_{\overline{\gamma}}$ on $\overline{\gamma}$. Clearly ψ is the limit of ψ_n with respect to $\langle \cdot, \cdot\rangle_L$. It remains to verify (ii) for ψ. Consider an arbitrary continuous function φ. Take arbitrary $\overline{\gamma_0}$ and $\varepsilon > 0$. Since $\psi_n \to \psi$ with respect to $\langle \cdot, \cdot\rangle_L$, there exists N such that

$$\sup_{\overline{\gamma}}\left|\int\limits_{\overline{\gamma}} \langle \psi|_{\overline{\gamma}} - \psi_n|_{\overline{\gamma}}, \varphi|_{\overline{\gamma}}\rangle\, d\mu_{\overline{\gamma}}\right| < \varepsilon/3$$

for $n \ge N$. Choose a neighborhood U of $\overline{\gamma_0}$ in $\widetilde{X/\Gamma}$ such that

$$\left|\int\limits_{\overline{\gamma}} \langle \psi_n|_{\overline{\gamma}}, \varphi|_{\overline{\gamma}}\rangle\, d\mu_{\overline{\gamma}} - \int\limits_{\overline{\gamma_0}} \langle \psi_n|_{\overline{\gamma_0}}, \varphi|_{\overline{\gamma_0}}\rangle\, d\mu_{\overline{\gamma_0}}\right| < \varepsilon/3$$

for any $\overline{\gamma} \in U$. Then for any $\overline{\gamma} \in U$,

$$\left|\int\limits_{\overline{\gamma}} \langle \psi|_{\overline{\gamma}}, \varphi|_{\overline{\gamma}}\rangle\, d\mu_{\overline{\gamma}} - \int\limits_{\overline{\gamma_0}} \langle \psi|_{\overline{\gamma_0}}, \varphi|_{\overline{\gamma_0}}\rangle\, d\mu_{\overline{\gamma_0}}\right| < 3 \cdot \varepsilon/3 = \varepsilon. \quad \blacksquare$$

THEOREM 3.3. *Let a discrete group Γ act on a compact Hausdorff space X.*

(i) *Suppose the module $L_\Gamma(X)$ is self-dual and the Gelfand spectrum $\widetilde{X/\Gamma}$ of the algebra $C_\Gamma(X)$ of continuous invariant functions has no isolated points. Then there are only finitely many $\overline{\gamma}$ with γ infinite, and all finite orbits have the same cardinality.*

(ii) *If there are only finitely many $\overline{\gamma}$ with γ infinite, and all finite orbits have the same cardinality, then the module $L_\Gamma(X)$ is self-dual.*

Proof. (i) By [12], the restriction on the Gelfand spectrum implies that $L_\Gamma(X)$ is finitely generated and projective. Let N be the cardinality of one of its generator systems. Thus, the number of points of each finite orbit is $\le N$. This follows from the epimorphy of the restriction map $L_\Gamma(X) \to L_\Gamma(Y)$, where $Y \subset X$ is a closed Γ-invariant set. Indeed, Y is a closed set in a normal space, hence continuous functions on it are extendable by the Tietze theorem.

In this situation of uniform boundedness of the cardinality of finite orbits, the subset X_f, formed by all finite orbits, is a closed (invariant) subset of X. Indeed, suppose an infinite orbit γ is in the closure of X_f. Choose a cover of X by (a finite number of) open sets U_i none of which is covered by the others, and γ is covered by more than N of these U_i's. Let $\mathbb{U} = \bigcup_i (U_i \times U_i)$ be an element of the uniform structure on X. Then there exists another neighborhood $\mathbb{V}$ of the diagonal in $X \times X$ such that $\Gamma(\mathbb{V}) \subset \mathbb{U}$ under the diagonal action. Choosing a finite orbit (of cardinality $\le N$) $\mathbb{V}$-close to γ we obtain a contradiction to the properties of $\mathbb{U}$.

Thus, as above, $L_\Gamma(X_\mathrm{f})$ is finitely generated. Moreover, it is projective, because the projection associated with a canonical isometric embedding of the finitely generated projective $C(\widetilde{X/\Gamma})$-module $L_\Gamma(X)$ into a standard finitely generated $C(\widetilde{X/\Gamma})$-module $C(\widetilde{X/\Gamma})^N$, say $\pi : C(\widetilde{X/\Gamma})^N \to L_\Gamma(X)$, restricted to X_f gives an epimorphic idempotent mapping

$$\pi' : C(\widetilde{X_\mathrm{f}/\Gamma})^N \to L_\Gamma(X_\mathrm{f}),$$

defined by the restriction of matrix elements of the projection π. Surjectivity follows from the above argument which relied on the Tietze theorem.

We arrive at the case considered in [5] and [18]. As explained in Theorem 2.9, the average over finite orbits is the same as in those papers, and the inner product is the same. Thus, $L_\Gamma(X_\mathrm{f}) = C(X_\mathrm{f})$. By the results of [5] and [18], under our assumptions this module is finitely generated projective if and only if all (finite) orbits have the same cardinality.

Now we turn to the statement about infinite orbits. Suppose there exist an infinite number of closures of infinite orbits: $\overline{\gamma}_i$, $i = 1, 2, \ldots$. We need to construct a $C_\Gamma(X)$-functional on $L_\Gamma(X)$ which is not an element of $L_\Gamma(X)$.

Passing to a subsequence if necessary, we can assume that for each point $z_i \in \widetilde{X/\Gamma}$ representing $\overline{\gamma}_i$, we can choose an open neighborhood U_i of z_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Indeed, suppose the opposite. Then for one of these points, say z_1, and any of its neighborhoods only finitely many points from the set $\{z_2, z_3, \dots\}$ are off this neighborhood (i.e., $z_2, z_3, \dots \to z_1$). We choose a neighborhood $U_1' \ni z_1$ such that there is $z_{i_1} \notin U_1'$, and (by normality) a neighborhood U_1'' of z_1 such that $\overline{U_1''} \subset U_1'$ and there is a neighborhood U_1 of z_{i_1} such that $U_1 \cap \overline{U_1''} = \emptyset$. Take $U_2' \subset U_1''$ such that there exists $z_{i_2} \in U_1'' \setminus U_2'$. Choose, by normality, $U_2'' \ni z_{i_0}$ such that $U_2'' \subset \overline{U_2''} \subset U_2'$ and there exists a neighborhood U_2 of z_{i_2} such that $U_2 \subset U_1''$ and $U_2 \cap \overline{U_2''} = \emptyset$. And so on. Finally, collect the points $\{z_{i_n}\}$ which have the required property.

Let us define $f_i : \overline{\gamma}_i \to \{0, \sqrt{i}\}$ to be the indicator function of the subset of $\overline{\gamma}_i$ with $\mu_i(\operatorname{supp} f_i) = 1/i$ (where μ_i is the invariant measure on $\overline{\gamma}_i$ of total mass 1). Thus $f_i \in L^2(\overline{\gamma}_i, \mu_i)$, $\langle f_i, f_i \rangle_{\overline{\gamma}_i} = \int_{\overline{\gamma}_i} |f_i|^2 \, d\mu_i = 1$ and $\int_{\overline{\gamma}_i} |f_i| \, d\mu_i = 1/\sqrt{i}$.

Choose $\alpha_i \in C(X)$ $(i = 1, 2, \dots)$ such that

(1) $\operatorname{supp} \alpha_i \subset p^{-1}(U_i)$, where $p : X \to \widetilde{X/\Gamma}$ is the canonical projection;
(2) $\alpha_i(X) \subset [0, 1/\sqrt{i}]$, $\alpha_i(\overline{\gamma}_i) = [0, 1/\sqrt{i}]$;
(3) $\|\alpha_i|_{\overline{\gamma}_i} - f_i\|_{L^2} < 1/2^i$ and $\|\alpha_i|_{\overline{\gamma}_i}\|^2_{L^2} - \|f_i\|^2_{L^2} < 1/2^i$;
(4) $\int_{\overline{\gamma}} \left|\alpha_i|_{\overline{\gamma}}\right| d\mu_{\overline{\gamma}} \leq 1/\sqrt{i} + 1/2^{i-1}$ for any $\overline{\gamma}$;
(5) $\int_{\overline{\gamma}} \left|\alpha_i|_{\overline{\gamma}}\right|^2 d\mu_{\overline{\gamma}} \leq 1 + 1/2^{i-1}$ for any $\overline{\gamma}$.

To construct such functions we first approximate f_i by an appropriate continuous function, then extend by the Tietze theorem, and finally multiply by an appropriate partition of unity function. More precisely, we first choose a continuous function $\alpha_i' : \overline{\gamma}_i \to [0, 1/\sqrt{i}]$ with properties (2) and (3). Then we extend it by the Tietze theorem to a continuous function $\alpha_i'' : X \to [0, 1/\sqrt{i}]$. By Theorem 3.2 the functions

$$\langle \alpha_i'', 1 \rangle_L : \widetilde{X/\Gamma} \to [0, +\infty), \qquad \langle \alpha_i'', \alpha_i'' \rangle_L : \widetilde{X/\Gamma} \to [0, +\infty)$$

are continuous and

$$\langle \alpha_i'', 1 \rangle_L(z_i) \in \left(\frac{1}{\sqrt{i}} - \frac{1}{2^i}, \frac{1}{\sqrt{i}} + \frac{1}{2^i} \right), \qquad \langle \alpha_i'', \alpha_i'' \rangle_L(z_i) \in \left(1 - \frac{1}{2^i}, 1 + \frac{1}{2^i} \right).$$

Choose a neighborhood $U_i' \subset U_i$ of z_i such that

$$\langle \alpha_i'', 1 \rangle_L(z_i) \in \left(\frac{1}{\sqrt{i}} - \frac{1}{2^{i-1}}, \frac{1}{\sqrt{i}} + \frac{1}{2^{i-1}} \right),$$
$$\langle \alpha_i'', \alpha_i'' \rangle_L(z_i) \in \left(1 - \frac{1}{2^{i-1}}, 1 + \frac{1}{2^{i-1}} \right).$$

Let $\omega_i : \widetilde{X/\Gamma} \to [0,1]$ be a continuous function with $\omega_i(z_i) = 1$ and $\operatorname{supp} \omega_i \subset U_i'$. Put $\widehat{\omega}_i := p^*\omega_i : X \to [0,1]$ and $\alpha_i := \widehat{\omega}_i \alpha_i''$. These are as required.

Define a function $h : X \to [0, +\infty)$ to be equal to α_i on $p^{-1}(U_i)$ ($i = 1, 2, \ldots$) and 0 otherwise. First, we wish to show that $h \notin L_\Gamma(X)$. Indeed, $\langle h, h\rangle_L$ is greater than $1 - 1/2^i > 1/2$ at each z_i and vanishes at any accumulation point of $\{z_i\}$.

Now let us show that $h \in L_\Gamma(X)'$. Let φ be a continuous function on Y such that $\|\langle\varphi, \varphi\rangle\|_L \le 1$. Then for any $\overline{\gamma}$ in some $p^{-1}(U_i)$ we have (using property (5))

$$|\langle h, \varphi\rangle|_{\overline{\gamma}}| = |\langle \alpha_i|_{\overline{\gamma}}, \varphi|_{\overline{\gamma}}\rangle| \le \langle \alpha_i|_{\overline{\gamma}}, \alpha_i|_{\overline{\gamma}}\rangle^{1/2} \cdot \langle\varphi, \varphi\rangle^{1/2} \le 2.$$

For the remaining $\overline{\gamma}$'s this product vanishes. It remains to show that $\langle h, \varphi\rangle$ is a continuous (invariant) function, i.e. for any $\varepsilon > 0$ and any $\overline{\gamma}_0$ in the closure of $\bigcup_i p^{-1}(U_i)$ there is an invariant neighborhood W of $\overline{\gamma}_0$ such that

$$\int_{\overline{\gamma}} \overline{h|_{\overline{\gamma}}} \cdot \varphi|_{\overline{\gamma}}\, d\mu_{\overline{\gamma}} < \varepsilon$$

for any $\overline{\gamma} \in W$. Choose W not intersecting $p^{-1}(U_i)$ for $i = 1, \ldots, k$, where $k > \max(2, (2\sup_{x\in X} |\varphi(x)|/\varepsilon)^2)$. Then (except in the trivial cases) $\overline{\gamma} \in p^{-1}(U_i)$ for some $i > k$. Let us estimate using property (4):

$$\begin{aligned}
\Big|\int_{\overline{\gamma}} \overline{h_{\overline{\gamma}}} \cdot \varphi|_{\overline{\gamma}}\, d\mu_{\overline{\gamma}}\Big| &\le \sup_{x\in X} |\varphi(x)| \cdot \int_{\overline{\gamma}} |\alpha_i|_{\overline{\gamma}}|\, d\mu_{\overline{\gamma}} \\
&= \sup_{x\in X} |\varphi(x)| \cdot \left(\frac{1}{\sqrt{i}} - \frac{1}{2^{i-1}}, \frac{1}{\sqrt{i}} + \frac{1}{2^{i-1}}\right) \\
&< \sup_{x\in X} |\varphi(x)| \cdot \frac{2}{\sqrt{i}} < \varepsilon
\end{aligned}$$

for $i > k$. Hence, the module is not self-dual.

(ii) As explained in the first part of the proof, in this case

$$\begin{aligned}
X &= X_{\mathrm{f}} \sqcup \overline{\gamma}_1 \sqcup \cdots \sqcup \overline{\gamma}_n, \\
L_\Gamma(X) &= L_\Gamma(X_{\mathrm{f}}) \oplus L^2(\overline{\gamma}_1, \mu_{\overline{\gamma}_1}) \oplus \cdots \oplus L^2(\overline{\gamma}_n, \mu_{\overline{\gamma}_n}), \\
(L_\Gamma(X))'_{C_\Gamma(X)} &= (L_\Gamma(X_{\mathrm{f}}))'_{C_\Gamma(X_{\mathrm{f}})} \oplus (L^2(\overline{\gamma}_1, \mu_{\overline{\gamma}_1}))'_{C_\Gamma(\overline{\gamma}_1)} \\
&\qquad \oplus \cdots \oplus (L^2(\overline{\gamma}_n, \mu_{\overline{\gamma}_n}))'_{C_\Gamma(\overline{\gamma}_n)} \\
&= (L_\Gamma(X_{\mathrm{f}}))'_{C_\Gamma(X_{\mathrm{f}})} \oplus (l^2(\mathbb{C}))'_{\mathbb{C}} \oplus \cdots \oplus (l^2(\mathbb{C}))'_{\mathbb{C}} \\
&= (L_\Gamma(X_{\mathrm{f}}))'_{C_\Gamma(X_{\mathrm{f}})} \oplus L^2(\overline{\gamma}_1, \mu_{\overline{\gamma}_1}) \oplus \cdots \oplus L^2(\overline{\gamma}_n, \mu_{\overline{\gamma}_n}).
\end{aligned}$$

As already explained, $L_\Gamma(X_{\mathrm{f}}) = C(X_{\mathrm{f}})$ in this case, and $(C(X_{\mathrm{f}}))'_{C_\Gamma(X_{\mathrm{f}})} = C(X_{\mathrm{f}})$. ∎

EXAMPLE 3.4. Let Y be the cone given by the equation $x^2+y^2 = z^2$, and $Z \subset Y$ be the subset of all points with $z \in J = \{0, 1, 1/2, 1/3, \ldots\}$. Then Z

is an infinite collection of circles with one limit point $(0,0,0)$ added. Let X be the union of three distinct copies of Z. To describe an action of $\mathbb{Z}$ on Z number the circles in the double cone consecutively by numbers of $\mathbb{Z}$ where the number zero is assigned to the point $(0,0,0)$. Consider the discrete group $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_3$, where $\mathbb{Z}$ acts on each circle by the irrational rotation of angle α_i $(i = 1, 2, \dots)$, where $\alpha_i \to 0$, and where $\mathbb{Z}_3$ transposes the cones. Then the module $L_\Gamma(X)$ is not self-dual since the orbits are all infinite except for the fixed point.

4. C^*-reflexivity

4.1. The metric case. In this section we would like to clarify in which situations the Hilbert C^*-module $L_\Gamma(X)$ is C^*-reflexive over $C_\Gamma(X)$. Our previous partial results [5, 18] made us believe that the Hilbert C^*-module $L_\Gamma(X)$ is C^*-reflexive in much more general situations beyond the finite orbit case. It turns out that any countably generated module over a wide class of commutative C^*-algebras is C^*-reflexive.

Theorem 4.1. *Let X be a compact metric space. Then any countably generated module over $C(X)$ is C^*-reflexive.*

Proof. A first version of the proof appeared in [9]. Then Trofimov [17] realized that the formulation in [9] was too general and provided a proof for any compact X with a certain property L. While preparing this paper, we realized that the property L of Trofimov is the same as the property of being a compact Baire space. So, any compact Hausdorff space has property L, and C^*-reflexivity would hold for any countably generated module over any unital commutative C^*-algebra, which is obviously not true, e.g. for von Neumann algebras [11]. Nevertheless, the main part of Trofimov's proof is correct. It was just overlooked that the proof implicitly used the assumption that, for any subset $E \subset X$ and for any point t_0 in the closure of E, there exists a sequence of points $t_n \in E$ which converges to t_0. In other words, the topology on X is supposed to possess a countable base of neighborhoods at any point of X. This is not true in general, but if we restrict ourselves to the case of compact *metric* spaces then this is obviously the case. Under this additional assumption, Trofimov's proof is correct. ∎

Corollary 4.2. *Let X be a compact metric space, and let an action of Γ on X be Lyapunov stable. Then the module $L_\Gamma(X)$ is C^*-reflexive.*

Proof. Since X is metric, the module $L_\Gamma(X)$ is countably generated and the C^*-algebra $C_\Gamma(X)$ is separable, hence its Gelfand spectrum is metrizable. ∎

Example 4.3. Let $D = \prod_{k=1}^{\infty} D_k$, where each D_k is the two-point space with the distance between the two points equal to 2^{-k}, and let $X = J \times D$.

Let $G = \bigoplus_{k=1}^{\infty} \mathbb{Z}_2$, $G_n = \bigoplus_{k=1}^{n} \mathbb{Z}_2$ and $\pi_n : G \to G_n$, $i_n : G_n \to G$ be the standard projection and inclusion homomorphisms. Denote their composition by $p_n = i_n \circ \pi_n : G \to G$. Let α denote the standard action of G on D. Define an action β of G on X by the formula

$$\beta_g\left(\frac{1}{n}, d\right) = \left(\frac{1}{n}, \alpha_{p_n(g)}(d)\right), \quad n \in \mathbb{N} \setminus \{0\}, \quad \beta_g(0, d) = (0, \alpha_g(d)),$$

where $d \in D$.

It is easy to see that this action has the following properties:

- The orbit of any point of the form $(1/n, d)$ is finite and consists of 2^n elements.
- The orbit of any point of the form $(0, d)$ is infinite.
- The action is continuous.
- The action is Lyapunov stable.

It follows from Corollary 4.2 that the module $L_\Gamma(X)$ is C^*-reflexive in this example.

4.2. The non-metric case. Having clarified how C^*-reflexivity arises in the metric case, let us pass to the case when X is non-metric. To begin, we give an example of a non-C^*-reflexive module $L_\Gamma(X)$.

Example 4.4. Let K be a (non-metrizable) compact space such that $l^2(A)$ is not C^*-reflexive, where $A = C(K)$. That is the case for A being a von Neumann algebra, and one of the most important cases is that of $K = \beta\mathbb{N}$, the Stone–Čech compactification of the positive integers. Consider the compact space $X = K \times S^1$ equipped with the action of $\mathbb{Z}$ by an irrational rotation in the second argument:

$$m(y, s) = (y, e^{\alpha\pi m} s), \quad m \in \mathbb{Z},\ \alpha \in \mathbb{R} \setminus \mathbb{Q},\ s \in S^1 \subset \mathbb{C}.$$

This is an isometric action on S^1 and a trivial one on K, hence it is Lyapunov stable. Evidently, the algebra $C_\Gamma(X)$ of continuous invariant functions is $A = C(K)$. By Theorem 3.2, the module $L_\Gamma(X)$ is the set of all functions $\psi : X \to \mathbb{C}$ such that

- $\psi|_{\overline{\gamma}} \in L^2(\overline{\gamma}, \mu_{\overline{\gamma}})$ for each orbit γ, i.e. $\psi_y(s) = \psi(y, s) \in L^2(S^1)$;
- for any $\varphi \in C(X)$ the function $\langle \psi, \varphi \rangle_L$ is continuous.

Let $\{e_j\}$ be a countable system of functions forming an orthonormal basis of $L^2(S^1)$ (e.g. exponents). Then $\{1_A \cdot e_j\}$ is an orthonormal system in $L_\Gamma(X)$:

$$\langle 1_A \cdot e_j, 1_A \cdot e_k \rangle_L(y) = \int_{S^1} 1_A(y) e_j(s) \overline{e_k(s) 1_A(y)}\, ds = \delta_{jk}.$$

Let us show that the $C(K)$-linear span of $\{1_A \cdot e_j\}$ is dense in $C(X)$ (hence, in $L_\Gamma(X)$) with respect to the Hilbert module distance. Let $\varphi \in C(X)$.

Then for any $\varepsilon > 0$ we can choose a partition $\Delta_1, \dots, \Delta_d$ of S^1 such that $\sup_{\Delta_i}(\varphi - f_i) < \varepsilon/d$, where f_i is independent of $s \in S^1$, i.e. actually $f_i \in A$, and $\sup_{\Delta_i} |f_i| \le 2 \sup_X |\varphi|$. Let χ_i be the indicator function of Δ_i, $i = 1, \dots, d$. Take $\hat{\chi}_i$ to be a $\mathbb{C}$-linear combination of $\{e_j\}$ such that

$$\|\chi_i - \hat{\chi}_i\|_{L^2(S^1)} < \varepsilon/d, \qquad i = 1, \dots, d.$$

Then

$$\hat{\varphi}(y, s) := \sum_{i=1}^{d} f_i(y) \cdot \hat{\chi}_i(s) \in \operatorname{span}_{C(K)}\{1_A \cdot e_j\}.$$

Let $\psi \in C(X)$ with $\|\psi\|_L \le 1$. Then

$$\begin{aligned}
\|\langle \varphi - \hat{\varphi}, \psi\rangle\|_L &= \sup_{y\in K}\Big|\int_{S^1} (\varphi(y,s) - \hat{\varphi}(y,s))\overline{\psi}(y,s)\,ds\Big| \\
&\le \sup_{y\in K}\Big|\int_{S^1}\Big(\varphi(y,s) - \sum_{j=1}^{d} f_j(y)\chi_j(s)\Big)\overline{\psi}(y,s)\,ds\Big| \\
&\quad + \sup_{y\in K}\Big|\sum_{j=1}^{d}\int_{S^1} f_j(y)(\chi_j(s) - \hat{\chi}_j(s))\overline{\psi}(y,s)\,ds\Big| \\
&\le \Big[\sup_{y\in K}\Big(\sup_{s\in S^1}\Big|\varphi(y,s) - \sum_{j=1}^{d} f_j(y)\chi_j(s)\Big|\Big) \\
&\quad + \sup_{y\in K} d \cdot \sup_{j=1,\dots,d} |f_j(y)| \cdot \Big(\int_{S^1} |\chi_j(s) - \hat{\chi}_j(s)|^2\,ds\Big)^{1/2}\Big] \\
&\quad \cdot \Big(\int_{S^1} |\psi(y,s)|^2\,ds\Big)^{1/2} \\
&\le \sup_{i=1,\dots,d}\ \sup_{K\times\Delta_i} |\varphi(y,s) - f_i(y)| \cdot \|\psi\|_L \\
&\quad + (2\sup_{x\in X}\|\varphi(x)\|) \cdot d \cdot \frac{\varepsilon}{d} \cdot \|\psi\|_L \\
&< \varepsilon(1 + 2\sup_{x\in X}\|\varphi(x)\|).
\end{aligned}$$

Thus, $L_\Gamma(X) = l^2(A)$ and is not C^*-reflexive.

Although we are far from obtaining a criterion for C^*-reflexivity, we can give a sufficient condition even in the non-metric case.

THEOREM 4.5. *Consider a Lyapunov stable action of Γ on a compact Hausdorff space X, where X is not necessarily metrizable. Suppose the cardinality of finite orbits is uniformly bounded and the number of closures of infinite orbits is finite. Then $L_\Gamma(X)$ is C^*-reflexive.*

Proof. By the argument in the proof of the first part of Theorem 3.3 (see page 139) the finite orbits form a closed invariant subset $X_{\mathrm{f}} \subset X$. The Gelfand spectrum consists of a closed subspace X_{f}/Γ and a finite number of isolated points corresponding to the closures of infinite orbits. Arguing as in the second part of Theorem 3.3 (see page 141), we reduce this case to the case of pure finite orbits [16]. ■

5. Further examples. We want to show by examples that there are other situations beyond the described above in which a well defined averaging can be found leading to admissible C^*-valued inner products and derived Hilbert C^*-module structures on the corresponding commutative C^*-algebras.

The following example shows that we can have a non-Lyapunov stable action with a good average.

EXAMPLE 5.1. Let $\Gamma = \mathbb{Z}$. Let $X = J \times S^1$ with

$$J = \{0, 1, 1/2, 1/3, \dots\} \subset \mathbb{R}.$$

Let $\alpha_i \to \alpha$ be a sequence of irrational numbers such that α is irrational and α/α_i is irrational for every i. Let the generator of $\mathbb{Z}$ rotate $\{1/i\} \times S^1$ by α_i, and the limit circle $\{0\} \times S^1$ by α. Clearly we have 1′) and 2′) in this case.

The next example demonstrates that if infinite orbits are present, uniform continuity is not sufficient for the continuity of the average.

EXAMPLE 5.2 ([18, Example 25]). Let $X \subset \mathbb{R}^3$ consist of two circles

$$S_\pm : \quad \begin{cases} x = \cos 2\pi t, \\ y = \sin 2\pi t, \qquad t \in (-\infty, +\infty), \\ z = \pm 1, \end{cases}$$

and of a non-uniform spiral

$$\Sigma : \quad \begin{cases} x = \cos 2\pi\tau, \\ y = \sin 2\pi\tau, \qquad \tau \in (-\infty, +\infty). \\ z = (2/\pi) \cdot \arctan \tau, \end{cases}$$

Let the generator g of $\Gamma = \mathbb{Z}$ act on all three components by

$$t \mapsto t + \alpha, \qquad \tau \mapsto \tau + \alpha,$$

where α is a positive irrational number. Then the isotropy group of each point of X is trivial. Hence, the condition of uniform continuity holds automatically.

Let $\varphi : X \to \mathbb{R}$ be the restriction of the function $\mathbb{R}^3 \ni (x, y, z) \mapsto z$ onto X. Then the function φ_x on $\mathbb{Z}$ has the following form: if $x \in S_\pm$ then

$\varphi_x = \pm 1$; if $x \in \Sigma$ then φ_x is a function on $\mathbb{Z}$ such that $\varphi_x(n) \in [-1,1]$ for any $n \in \mathbb{Z}$ and $\lim_{n\to\pm\infty} \varphi_x(n) = \pm 1$. So, φ_x is in general not almost periodic and we cannot average it using our definition. Nevertheless, we can average it using the amenability of the group $\mathbb{Z}$. In this case we get

$$E_\Gamma(\varphi_x) = \begin{cases} \pm 1 & \text{for } x \in S_\pm, \\ 0 & \text{for } x \in \Sigma. \end{cases}$$

Thus we see that $E_\Gamma(\varphi_x)$ is not continuous with respect to $x \in X$.

Example 5.3. In the previous example, let us identify the two circles S_+ and S_-. Then X consists of the spiral Σ and of the circle S. Still, the function φ_x on $\Gamma = \mathbb{Z}$ need not be almost periodic, but there is an almost periodic function ρ on $\mathbb{Z}$ such that for any $\varepsilon > 0$ there is a finite subset $F \subset \mathbb{Z}$ such that $\|\varphi_x - \rho\| < \varepsilon$ on $\mathbb{Z} \setminus F$. This makes it possible to define an average $E_\Gamma(\varphi)$ by $E_\Gamma(\varphi_x) = M(\rho)$. And it is easy to see that, this time, $E_\Gamma(\varphi_x)$ is continuous with respect to $x \in X$.

Example 5.4. Our next example is a modification of Example 4.4. Let $Y = \mathbb{N} \times S^1$, $X = \beta Y$ its Stone–Čech compactification. Let $\Gamma = \mathbb{Z}$ act on Y by rotating each circle by an irrational angle α. This action canonically extends to an action on X.

Let $s \in S^1$. Then the inclusion $\mathbb{N} \to \mathbb{N} \times S^1$, $n \mapsto (n,s)$, canonically extends to a map $s_* : \beta\mathbb{N} \to X$ and $m(s_*(x)) = (s \cdot e^{im\alpha})_*(x)$ for any $x \in \beta\mathbb{N}$ and any $m \in \Gamma = \mathbb{Z}$.

Let $\varphi \in C(X)$. Since $C(X) = C_b(Y)$ (continuous functions on X are canonically identified with bounded continuous functions on Y), φ can be identified with a uniformly bounded sequence $(\varphi^{(1)}, \varphi^{(2)}, \ldots)$ of continuous functions on S^1. Let $x \in \beta\mathbb{N}$. Then

$$\varphi_{s_*(x)}(m) = \varphi((s \cdot e^{im\alpha})_*(x)) \quad \text{for } m \in \mathbb{Z}.$$

Let $\mathcal{U}_x$ be the ultrafilter on $\mathbb{N}$ that corresponds to the point $x \in \beta\mathbb{N}$. Then

$$\varphi_{s_*(x)} = \lim_{\mathcal{U}_x} (\varphi^{(n)}(s))_{n=1}^\infty,$$

where the limit of the sequence $(\varphi^{(n)}(s))_{n=1}^\infty$ is taken over $\mathcal{U}_x$, hence

$$\varphi_{s_*(x)}(m) = \lim_{\mathcal{U}_x} (\varphi^{(n)}((s \cdot e^{im\alpha})_*(x)))_{n=1}^\infty.$$

Take $\varphi^{(n)}(s) = e^{ins}$. Then $\varphi \in C_b(Y) = C(X)$ and

$$\varphi_{1_*(x)}(m) = \lim_{\mathcal{U}_x} (e^{inm\alpha})_{n=1}^\infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that $\lim_{\mathcal{U}} (e^{in\lambda})_{n=1}^\infty = 0$ for any $\lambda \in (0, 2\pi)$, and let $x_0 \in \beta\mathbb{N}$ be the point that corresponds to $\mathcal{U}$. Then

$$\varphi_{1_*(x_0)}(m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

Thus, for the point $y = 1_*(x_0) \in X$ and the function $\varphi \in C(X)$ we see that the function φ_y is not almost periodic on $\mathbb{Z}$.

Nevertheless, there is a "good" averaging in this example. Since any continuous function φ on X is a uniformly bounded sequence of functions $\varphi^{(n)}$, $n \in \mathbb{N}$, on S^1, it is easy to see that $C_\Gamma(X) \cong C_b(\mathbb{N})$, and one can define $E_\Gamma(\varphi)$ by the formula $(E_\Gamma(\varphi))_n = \int_{S^1} \varphi^{(n)}(s)\, ds$.

These examples show that a good averaging (and an inner product with values in $C_\Gamma(X)$) can be defined in a wider class than the Lyapunov stable actions. On the other hand, as the last two examples show, a good averaging, when it exists, may give rise to a degenerate inner product.

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Michael Frank
FB IMN
HTWK Leipzig
Postfach 301166
D-04251 Leipzig, Germany
E-mail: mfrank@imn.htwk-leipzig.de
http://www.imn.htwk-leipzig.de/~mfrank

Evgenij Troitsky
Department of Mechanics and Mathematics
Moscow State University
119991 GSP-1 Moscow, Russia
E-mail: troitsky@mech.math.msu.su
http://mech.math.msu.su/~troitsky

Vladimir Manuilov
Department of Mechanics and Mathematics
Moscow State University
119991 GSP-1 Moscow, Russia
and
Harbin Institute of Technology
Harbin, P.R. China
E-mail: manuilov@mech.math.msu.su
http://mech.math.msu.su/~manuilov

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