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Invertibility of the commutator of an element in a C^*-algebra and its Moore–Penrose inverse

by

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Abstract. We study the subset in a unital C^*-algebra composed of elements a such that $aa^\dagger - aa^\dagger$ is invertible, where $a^\dagger$ denotes the Moore–Penrose inverse of a. A distinguished subset of this set is also investigated. Furthermore we study sequences of elements belonging to the aforementioned subsets.

1. Introduction. Throughout this paper, $\mathcal{A}$ will be a C^*-algebra with unit 1 and we will denote by $\mathcal{A}^{-1}$ the subset of invertible elements in $\mathcal{A}$. An element $a \in \mathcal{A}$ is said to be *idempotent* when $a^2 = a$. The term *projection* will be reserved for an element p of $\mathcal{A}$ which is self-adjoint and idempotent, that is, $p^* = p = p^2$.

An element $a \in \mathcal{A}$ is said to have a *Moore–Penrose inverse* if there exists $x \in \mathcal{A}$ such that

$$axa = a, \qquad xax = x, \qquad (ax)^* = ax, \qquad (xa)^* = xa. \tag{1.1}$$

It can be proved that if $a \in \mathcal{A}$ has a Moore–Penrose inverse, then the element x satisfying (1.1) is unique (see, for example, [Pen]), and we write $x = a^\dagger$. The set of all elements of $\mathcal{A}$ that have a Moore–Penrose inverse will be denoted by $\mathcal{A}^\dagger$. An element $a \in \mathcal{A}$ such that there exists $x \in \mathcal{A}$ with $axa = a$ will be named *regular*. A basic result of the theory of Moore–Penrose inverses in C^*-algebras is that if $a \in \mathcal{A}$ then $a \in \mathcal{A}^\dagger$ if and only if a is regular (see [H-M, Kol2]).

Several characterizations of elements $a \in \mathcal{A}^\dagger$ such that $aa^\dagger = a^\dagger a$ can be found in the literature (see [Ben, Kol3]). In this paper we shall study the class of elements in a C^*-algebra that, in some sense, is complementary to the subset of $\mathcal{A}^\dagger$ composed of elements that commute with their Moore–Penrose inverses. When the C^*-algebra is the set of $n \times n$ complex matrices, it is customary to say that a matrix A is *EP* when A commutes with its Moore–Penrose inverse, which justifies the following definition.

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Definition 1.1. Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is said to be *co-EP* when $a \in \mathcal{A}^\dagger$ and $aa^\dagger - a^\dagger a$ is invertible. The subset of $\mathcal{A}$ composed of co-EP elements will be denoted by $\mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$.

Let $a \in \mathcal{A}$. Since, as is easy to see, $aa^\dagger$ and $a^\dagger a$ are projections, the study of the invertibility of $aa^\dagger - a^\dagger a$ is related to the study of the invertibility of differences of two projections in a C^*-algebra. In [Buc1, Buc2], Buckholtz characterized when $P-Q$ is invertible when P and Q are orthogonal projections of a Hilbert space. Koliha and Rakočević gave in [K-R2, Theorem 4.1] several characterizations of the invertibility of $p-q$ when p,q are nontrivial projections in a C^*-algebra. One of these characterizations uses the concept of the range projection. For the convenience of the reader we recall its definition, introduced by Koliha in [Kol4].

Definition 1.2. Let $f \in \mathcal{A}$ be an idempotent. We say that $p \in \mathcal{A}$ is a *range projection* of f if p is a projection satisfying $pf = f$ and $fp = p$.

Let us recall ([Kol4, Theorem 1.3] and [K-R3, Theorem 1.3]) that for every idempotent $f \in \mathcal{A}$ there exists a unique range projection of f, denoted by $f^\perp$, given explicitly by the Kerzman–Stein formula (see [K-S])

$$f^\perp = f(f + f^* - 1)^{-1}. \tag{1.2}$$

If f is a projection, then obviously $f^\perp = f$.

The following concept was introduced by Koliha and Rakočević in [K-R2]. It allowed them (among other things) to characterize the invertibility of $p-q$ when p and q are nontrivial projections in a C^*-algebra.

Definition 1.3. Let $e, f \in \mathcal{A}$ be idempotents. We denote by $\pi(e,f)$ an idempotent $h \in A$ (if it exists) satisfying the conditions

$$h^\perp = e^\perp, \qquad (1-h)^\perp = f^\perp.$$

If, in addition, e and f are self-adjoint, the above reduces to $h^\perp = e$ and $(1-h)^\perp = f$. In other words, if $e, f \in \mathcal{A}$ are projections such that $h = \pi(e,f)$ exists, then

$$he = e, \qquad eh = h, \qquad (1-h)f = f, \qquad f(1-h) = 1-h.$$

An element $a \in \mathcal{A}$ is *quasipolar* if 0 is an isolated singularity of the resolvent of a. Koliha proved in [Kol1, Theorem 4.2] that $a \in \mathcal{A}$ is quasipolar if and only if there exists an idempotent $p \in \mathcal{A}$ such that $ap = pa$ is quasinilpotent and $a + p \in \mathcal{A}^{-1}$. Such an idempotent is unique, and is called the *spectral idempotent* of a corresponding to 0, written a^π. In [Kol4, Theorem 3.6] it is proved that if $x \in \mathcal{A}$ then $x \in \mathcal{A}^\dagger$ implies that x^*x and xx^* are quasipolar and $x^\dagger = (x^*x + (x^*x)^\pi)^{-1}x^* = x^*(xx^* + (xx^*)^\pi)^{-1}$. Indeed, the only fact we shall use is the following:

$$x \in \mathcal{A}^\dagger \;\Rightarrow\; \text{there exist } y, z \in \mathcal{A}^{-1} \text{ such that } x^\dagger = yx^* = x^*z. \tag{1.3}$$

2. Characterizations of co-EP elements in a C^*-algebra. The main result of this section characterizes when $aa^\dagger - a^\dagger a$ is invertible if a is an element of a unital C^*-algebra that has a Moore–Penrose inverse. Before presenting this characterization, let us prove the following lemma:

LEMMA 2.1. *Let $\mathcal{A}$ be a C^*-algebra with unity* 1 *and $a \in \mathcal{A}^\dagger$. If $h \in \mathcal{A}$ is an idempotent satisfying $h^\perp = a^\dagger a$ and $(1-h)^\perp = aa^\dagger$, then $a + a^* \in \mathcal{A}^{-1}$ and*

$$ah^* = a, \quad ha = 0, \quad (a+a^*)^{-1} = h^* a^\dagger (1-h) + (1-h^*)(a^\dagger)^* h. \tag{2.1}$$

Proof. From $h^\perp = a^\dagger a$ and $(1-h)^\perp = aa^\dagger$ we have

$$a^\dagger a h = h, \quad h a^\dagger a = a^\dagger a, \quad aa^\dagger(1-h) = 1-h, \quad (1-h)aa^\dagger = aa^\dagger. \tag{2.2}$$

Recall that $aa^\dagger$ and $a^\dagger a$ are self-adjoint. Taking * in the second equality of (2.2) and premultiplying by a we have $ah^* = a$. Postmultiplying the last equality of (2.2) by a yields $ha = 0$. Now, we will prove that $a + a^*$ is invertible with inverse $h^* a^\dagger (1-h) + (1-h^*)(a^\dagger)^* h$. Indeed,

$$\begin{aligned}
(a+a^*)&[h^* a^\dagger (1-h) + (1-h^*)(a^\dagger)^* h] \\
&= ah^* a^\dagger (1-h) + a^* h^* a^\dagger (1-h) + a(1-h^*)(a^\dagger)^* h + a^*(1-h^*)(a^\dagger)^* h \\
&= aa^\dagger (1-h) + a^*(a^\dagger)^* h = 1 - h + (a^\dagger a)^* h = 1.
\end{aligned}$$

Set $u = a + a^*$ and $v = h^* a^\dagger (1-h) + (1-h^*)(a^\dagger)^* h$. Since $uv = 1$ and u and v are self-adjoint, we get $1 = 1^* = (uv)^* = v^* u^* = vu$. Therefore, $v = u^{-1}$. ∎

THEOREM 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$.
(ii) $a + a^* \in \mathcal{A}^{-1}$ *and there exists an idempotent $p \in \mathcal{A}$ such that $ap = a$ and $p^* a = 0$.*
(iii) $a - a^* \in \mathcal{A}^{-1}$ *and there exists an idempotent $p \in \mathcal{A}$ such that $ap = a$ and $p^* a = 0$.*
(iv) $aa^* + a^* a \in \mathcal{A}^{-1}$ *and $a\mathcal{A} \cap a^*\mathcal{A} = \{0\}$.*
(v) $a + a^* \in \mathcal{A}^{-1}$, $a(a+a^*)^{-1} a = a$, *and* $a^*(a+a^*)^{-1} a = 0$.
(vi) $a - a^* \in \mathcal{A}^{-1}$, $a(a-a^*)^{-1} a = a$, *and* $a^*(a-a^*)^{-1} a = 0$.
(vii) $a\mathcal{A} \oplus a^*\mathcal{A} = \mathcal{A}$.

Proof. Let 1 be the unity of $\mathcal{A}$.

(i)⇒(ii): We shall use the implication (viii)⇒(ii) of [K-R2, Theorem 4.1]. Checking the proof of that implication, we can observe that the two projections involved need not be nontrivial. Since $aa^\dagger$, $a^\dagger a$ are projections and $aa^\dagger - a^\dagger a$ is invertible, by using the aforementioned implication, $h = \pi(a^\dagger a, aa^\dagger)$ exists. By definition of $\pi(a^\dagger a, aa^\dagger)$ we get $h^\perp = a^\dagger a$ and $(1-h)^\perp = aa^\dagger$. By

Lemma 2.1 we see that $a + a^* \in \mathcal{A}^{-1}$ and by setting $p = h^*$, another appeal to Lemma 2.1 finishes the proof of (i)⇒(ii).

(ii)⇔(iii): Assume that p is an idempotent such that $ap = a$ and $p^*a = 0$. Since $(a+a^*)(2p-1) = 2ap - a + 2a^*p - a^* = a - a^*$ and $2p-1$ is invertible (because $(2p-1)^2 = 1$) we have $a + a^* \in \mathcal{A}^{-1} \Leftrightarrow a - a^* \in \mathcal{A}^{-1}$.

(iii)⇒(iv): First, note that $1 - p - p^* \in \mathcal{A}^{-1}$ because $(1-p-p^*)^2 = 1 + (p-p^*)(p-p^*)^*$. Now,

$$\begin{aligned}(a+a^*)(1-p-p^*)(a-a^*) &= [a(1-p) + a^*(1-p) - ap^* - a^*p^*](a-a^*) \\ &= [a^* - ap^* - a^*p^*](a-a^*) \\ &= a^*a - ap^*a - a^*p^*a - (a^*)^2 + ap^*a^* + a^*p^*a^* \\ &= a^*a + aa^*.\end{aligned}$$

Since the hypothesis also implies that $a + a^* \in \mathcal{A}^{-1}$ (because (ii)⇔(iii)), the previous computations show that $aa^* + a^*a \in \mathcal{A}^{-1}$.

To prove $a\mathcal{A} \cap a^*\mathcal{A} = \{0\}$, pick $x \in a\mathcal{A} \cap a^*\mathcal{A}$. There exist $u, v \in \mathcal{A}$ with $x = au = a^*v$, hence $p^*au = p^*a^*v$, and therefore $0 = a^*v$, because $p^*a = 0$ and $p^*a^* = a^*$. Thus, $x = a^*v = 0$.

(iv)⇒(v): Since $aa^* + a^*a$ is invertible, there exists $x \in \mathcal{A}$ such that

$$1 = (aa^* + a^*a)x. \tag{2.3}$$

Thus, $a = aa^*xa + a^*axa$. Since $a^*axa = a(1 - a^*xa)$ we have $a^*axa \in a\mathcal{A} \cap a^*\mathcal{A} = \{0\}$. Therefore, $a = aa^*xa$, which means that a is regular. Property (1.3) leads to $a^\dagger a\mathcal{A} \subset a^*\mathcal{A}$. Having in mind that $aa^\dagger\mathcal{A} \subset a\mathcal{A}$ and $a\mathcal{A} \cap a^*\mathcal{A} = \{0\}$ we obtain $aa^\dagger\mathcal{A} \cap a^\dagger a\mathcal{A} = \{0\}$. To prove $aa^\dagger\mathcal{A} + a^\dagger a\mathcal{A} = \mathcal{A}$, it is sufficient to show that $1 \in aa^\dagger\mathcal{A} + a^\dagger a\mathcal{A}$. By (1.3), there exist $u, v \in \mathcal{A}^{-1}$ such that $a^* = a^\dagger u$ and $a^* = va^\dagger$. Since $a^*a = (a^*a)^* = (va^\dagger a)^* = a^\dagger av^*$, from (2.3) we have

$$1 = aa^*x + a^*ax = aa^\dagger u + a^\dagger av^* \in aa^\dagger\mathcal{A} + a^\dagger a\mathcal{A}.$$

Note that the equivalence (i)⇔(ii) of [K-R2, Theorem 4.1] does not use the nontriviality of the projections involved. Thus, by that equivalence, the idempotent $h = \pi(aa^\dagger, a^\dagger a)$ exists, and so $h^\perp = aa^\dagger$ and $(1-h)^\perp = a^\dagger a$. By Lemma 2.1, we have $a + a^* \in \mathcal{A}^{-1}$. From (2.1) we get

$$a(a+a^*)^{-1}a = a[h^*a^\dagger(1-h) + (1-h^*)(a^\dagger)^*h]a = aa^\dagger a = a$$

and

$$a^*(a+a^*)^{-1}a = a^*[h^*a^\dagger(1-h) + (1-h^*)(a^\dagger)^*h]a = 0.$$

(v)⇒(vi): Set $q = (a+a^*)^{-1}a$. From the hypothesis, it is trivial to check that $q^2 = q$, $aq = a$, and $a^*q = 0$. The equalities $(a+a^*)(2q-1) = a - a^*$ and $(2q-1)^2 = 1$ lead to $a - a^* \in \mathcal{A}^{-1}$ and $(a-a^*)^{-1} = (2q-1)(a+a^*)^{-1}$. Now we have

$$a(a-a^*)^{-1}a = a(2q-1)(a+a^*)^{-1}a = a(a+a^*)^{-1}a = a$$

and

$$a^*(a-a^*)^{-1}a = a^*(2q-1)(a+a^*)^{-1}a = -a^*(a+a^*)^{-1}a = 0.$$

(vi)⇒(vii): To prove $a\mathcal{A}+a^*\mathcal{A}=\mathcal{A}$ it is sufficient to prove $1 \in a\mathcal{A}+a^*\mathcal{A}$: in fact, since $a-a^*$ is invertible, there exists $x \in \mathcal{A}$ such that $1=(a-a^*)x$, and thus $1 = ax + a^*(-x) \in a\mathcal{A}+a^*\mathcal{A}$. Now, let us prove $a\mathcal{A}\cap a^*\mathcal{A}=\{0\}$: if $y \in a\mathcal{A}\cap a^*\mathcal{A}$, there exist $u,v\in\mathcal{A}$ with $y=au=a^*v$; hence

$$y^* = v^*a = v^*a(a-a^*)^{-1}a = u^*a^*(a-a^*)^{-1}a = 0,$$

and therefore $y=0$.

(vii)⇒(i): Since $\mathcal{A}=a\mathcal{A}+a^*\mathcal{A}$, we have $1=ax+a^*y$ for some $x,y\in\mathcal{A}$. Thus, $a = axa + a^*ya$. Hence $a^*ya = a(1-xa) \in a\mathcal{A}\cap a^*\mathcal{A}=\{0\}$. Therefore, $a=axa$, which means that a is regular. By the equivalence (i)⇔(ii) of [K-R2, Theorem 4.1], the idempotent $h=\pi(a^\dagger a, aa^\dagger)$ exists, and thus $h^\perp = a^\dagger a$ and $(1-h)^\perp = aa^\dagger$. By the Kerzman–Stein formula (1.2) we get

$$\begin{aligned} aa^\dagger - a^\dagger a &= (1-h)^\perp - h^\perp \\ &= (1-h)[(1-h)+(1-h)^*-1]^{-1} - h[h+h^*-1]^{-1} = (1-h-h^*)^{-1}. \end{aligned}$$

This implies that $aa^\dagger - a^\dagger a$ is invertible. ∎

Corollary 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$. The idempotent p in conditions* (ii) *and* (iii) *of the preceding theorem is unique and satisfies*

$$p = [\pi(a^\dagger a, aa^\dagger)]^*. \tag{2.4}$$

Proof. From $(a+a^*)p = ap + a^*p = a+0 = a$ and the invertibility of $a+a^*$ we get $p=(a+a^*)^{-1}a$, proving the uniqueness. Now, (2.4) follows from the proof of the preceding theorem. ∎

The following corollary collects some useful formulæ.

Corollary 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra with unity* 1. *If $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$ and $p = [\pi(a^\dagger a, aa^\dagger)]^*$, then:*

(i) $[p^*]^\perp = a^\dagger a$ *and* $[1-p^*]^\perp = aa^\dagger$.
(ii) $p = (a+a^*)^{-1}a = (a-a^*)^{-1}a$.
(iii) $(aa^\dagger - a^\dagger a)^{-1} = 1-p-p^*$.
(iv) $(aa^\dagger - a^\dagger a)^{-1} = (a+a^*)^{-1}(aa^*+a^*a)(a-a^*)^{-1}$.
(v) $(aa^\dagger - a^\dagger a)^{-1} = (a+a^*)^{-1}(aa^*-a^*a)(a+a^*)^{-1}$
$= (a-a^*)^{-1}(aa^*-a^*a)(a-a^*)^{-1}$.

Proof. Items (i), (iii), and (iv) are distilled from the proof of Theorem 2.2. Item (ii) follows from the proof of Corollary 2.3. Item (v) follows by mimicking the proof of (iii)⇒(iv) of Theorem 2.2. ∎

In the setting of Hilbert spaces, Buckholtz proved that if R and K are closed subspaces of a Hilbert space $\mathcal{H}$, and P_R and P_K denote the orthogonal projections onto these subspaces, then $P_R - P_K$ is invertible if and only if there exists an idempotent M with range R and kernel K (see [Buc1, Buc2]). Moreover, $(P_R - P_K)^{-1} = M + M^* - I$. Observe that the formula in Corollary 2.4(iii) is a version of this in the C^*-algebra setting when the projections are $aa^\dagger$ and $a^\dagger a$.

Example 2.5. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of bounded operators in $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ be invertible and $\alpha, \beta \in \mathbb{R}$ be such that $\alpha^2 + \beta^2 = 1$ and $\beta \neq 0$. We consider the Hilbert space $\mathcal{H} \times \mathcal{H}$ endowed with the inner product $\langle(\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\rangle = \langle\mathbf{x},\mathbf{u}\rangle + \langle\mathbf{y},\mathbf{v}\rangle$. Define the operator R in $\mathcal{H} \times \mathcal{H}$ by $R(\mathbf{x},\mathbf{y}) = (\alpha T\mathbf{x} + \beta T\mathbf{y}, \mathbf{0})$. By checking (1.1), it is a textbook exercise to prove that R is Moore–Penrose invertible and $R^\dagger(\mathbf{x},\mathbf{y}) = (\alpha T^{-1}\mathbf{x}, \beta T^{-1}\mathbf{x})$. Thus we can compute $R^\dagger R - RR^\dagger$ obtaining

$$(R^\dagger R - RR^\dagger)(\mathbf{x},\mathbf{y}) = \beta(-\beta\mathbf{x} + \alpha\mathbf{y}, \alpha\mathbf{x} + \beta\mathbf{y}).$$

We can easily check that the operator $S \in \mathcal{B}(\mathcal{H} \times \mathcal{H})$ given by $S(\mathbf{x},\mathbf{y}) = (-\mathbf{x} + \alpha\beta^{-1}\mathbf{y}, \alpha\beta^{-1}\mathbf{x} + \mathbf{y})$ is the inverse of $R^\dagger R - RR^\dagger$. Hence R is co-EP. Furthermore, if we define $P \in \mathcal{B}(\mathcal{H} \times \mathcal{H})$ by $P(\mathbf{x},\mathbf{y}) = (\mathbf{0}, \alpha\beta^{-1}\mathbf{x} + \mathbf{y})$, we get $P^2 = P$ and $RP = R$. The computation

$$\begin{aligned}\langle P(\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\rangle &= \langle(\mathbf{0}, \alpha\beta^{-1}\mathbf{x} + \mathbf{y}),(\mathbf{u},\mathbf{v})\rangle \\ &= \langle\mathbf{x}, \alpha\beta^{-1}\mathbf{v}\rangle + \langle\mathbf{y},\mathbf{v}\rangle = \langle(\mathbf{x},\mathbf{y}),(\alpha\beta^{-1}\mathbf{v},\mathbf{v})\rangle\end{aligned}$$

shows that $P^*(\mathbf{x},\mathbf{y}) = (\alpha\beta^{-1}\mathbf{y}, \mathbf{y})$, which implies $P^*R = 0$. Therefore, P is the idempotent given by Theorem 2.2.

3. A distinguished subset of co-EP elements in a C^*-algebra. If $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$, then the idempotent $\pi(a^\dagger a, aa^\dagger)$ exists. In this section we characterize the elements $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$ such that $\pi(a^\dagger a, aa^\dagger)$ is a projection.

Definition 3.1. Let $\mathcal{A}$ be a unital C^*-algebra. We denote by $\mathcal{A}_{\mathsf{co}}^{\mathsf{ep}\perp}$ the subset of $\mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$ consisting of the elements a such that $\pi(a^\dagger a, aa^\dagger)$ is self-adjoint.

We shall use the following notation: If $X, Y \subset \mathcal{A}$, then

$$X \perp Y \;\Leftrightarrow\; x^*y = 0 \;\forall (x,y) \in X \times Y.$$

It is evident (by the C^*-identity) that $X \perp Y$ implies that $X \cap Y \subset \{0\}$.

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra with unity* 1 *and $a \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}\perp}$.
(ii) $a \in \mathcal{A}^\dagger$ *and* $aa^\dagger + a^\dagger a = 1$.
(iii) $a \in \mathcal{A}^\dagger$ *and* $a\mathcal{A} = \{x \in \mathcal{A} : ax = 0\}$.

(iv) $a\mathcal{A} \perp a^*\mathcal{A}$ *and* $a\mathcal{A} + a^*\mathcal{A} = \mathcal{A}$.
(v) $a \in \mathcal{A}^\dagger$ *and* $(aa^\dagger - a^\dagger a)^2 = 1$.

Proof. (i)⇒(ii): Let p be the idempotent given in Theorem 2.2. We shall prove $aa^\dagger = 1 - p$ and $a^\dagger a = p$. By Corollary 2.3, we have $\pi(a^\dagger a, aa^\dagger) = p^*$. Applying the definition of $\pi(\cdot,\cdot)$ we have $(p^*)^\perp = a^\dagger a$ and $(1-p^*)^\perp = aa^\dagger$. Since $p = p^*$ we obtain $p = a^\dagger a$ and $1 - p = aa^\dagger$.

(ii)⇒(iii): Postmultiplying $aa^\dagger + a^\dagger a = 1$ by a leads to $a^\dagger a^2 = 0$, which by premultiplying by a yields $a^2 = 0$, and this implies that $a\mathcal{A} \subset \{x \in \mathcal{A} : ax = 0\}$. To prove the opposite inclusion, pick $x \in \mathcal{A}$ with $ax = 0$; then from $1 = aa^\dagger + a^\dagger a$ we get $x = (aa^\dagger + a^\dagger a)x = aa^\dagger x \in a\mathcal{A}$.

(iii)⇒(iv): Since $a \in a\mathcal{A} = \{x \in \mathcal{A} : ax = 0\}$, we obtain $a^2 = 0$. Since for any $x, y \in \mathcal{A}$ we have $(ax)^*(a^*y) = x^*(a^2)^*y = 0$, we get $a\mathcal{A} \perp a^*\mathcal{A}$. To prove $a\mathcal{A} + a^*\mathcal{A} = \mathcal{A}$, it is sufficient to prove $1 \in a\mathcal{A} + a^*\mathcal{A}$: in fact, from $a^\dagger a - 1 \in \{x \in \mathcal{A} : ax = 0\} = a\mathcal{A}$, there exists $u \in \mathcal{A}$ such that $a^\dagger a - 1 = au$. Thus, $1 = a^\dagger a - au = (a^\dagger a)^* - au = a(-u) + a^*(a^\dagger)^* \in a\mathcal{A} + a^*\mathcal{A}$.

(iv)⇒(v): Since $a\mathcal{A} \perp a^*\mathcal{A}$ we have $a^2 = 0$. By using (1.3) we get the existence of $y, z \in \mathcal{A}^{-1}$ such that $a^\dagger = ya^*$ and $a^\dagger = a^*z$, and therefore $(a^\dagger)^2 = ya^*a^*z = 0$. Since $a\mathcal{A} \oplus a^*\mathcal{A} = \mathcal{A}$, by Theorem 2.2 and Corollary 2.3, there exists a unique idempotent p such that $ap = a$ and $p^*a = 0$. Observe that $a^\dagger a$ is an idempotent and

$$a(a^\dagger a) = a, \qquad (a^\dagger a)^* a = a^\dagger a^2 = 0,$$

thus, the uniqueness of p yields $p = a^\dagger a$. Furthermore, $1 - aa^\dagger$ is another idempotent and

$$a(1 - aa^\dagger) = a - a^2 a^\dagger = a, \qquad (1 - aa^\dagger)^* a = (1 - aa^\dagger)a = 0.$$

Again, the uniqueness of p leads to $p = 1 - aa^\dagger$. Thus,

$$(aa^\dagger - a^\dagger a)^2 = aa^\dagger aa^\dagger - a(a^\dagger)^2 a - a^\dagger a^2 a^\dagger + a^\dagger aa^\dagger a = aa^\dagger + a^\dagger a = 1 - p + p = 1.$$

(v)⇒(i): The hypothesis $(aa^\dagger - a^\dagger a)^2 = 1$ entails that $aa^\dagger - a^\dagger a$ is invertible, and by Corollary 2.4 we get $(aa^\dagger - a^\dagger a)^{-1} = 1 - p - p^*$, where p is the idempotent obtained in Theorem 2.2. Therefore, $(1 - p - p^*)^2 = (aa^\dagger - a^\dagger a)^{-2} = 1$. Now we have

$$1 = (1 - p - p^*)^2 = 1 - p - p^* + pp^* + p^*p.$$

Thus, $p + p^* = pp^* + p^*p$, which easily leads to $(p - p^*)(p - p^*)^* = 0$. The C^*-identity yields $p = p^*$. ■

COROLLARY 3.3. *If* $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}_\perp}$, *then the projector* p *given in Theorem* 2.2 *is* $a^\dagger a$.

In case $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}_\perp}$, the following corollary gives some formulæ that relate $a^\dagger$ to $(a + a^*)^{-1}$ and $(a - a^*)^{-1}$.

Corollary 3.4. *If* $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}\perp}$, *then:*

(i) $a^\dagger = (a + a^*)^{-1} a (a + a^*)^{-1}$.
(ii) $a^\dagger = (a + a^*)^{-1} a (a - a^*)^{-1}$.
(iii) $a^\dagger = (a - a^*)^{-1} a (a + a^*)^{-1}$.
(iv) $a^\dagger = (a - a^*)^{-1} a (a - a^*)^{-1}$.

Proof. Observe that by (1.3) there exists $y \in \mathcal{A}^{-1}$ such that $a^\dagger = ya^*$. From the proof of Theorem 3.2, one finds that $a^2 = 0$. Hence $a^\dagger a^* = ya^*a^* = y(a^2)^* = 0$. Furthermore, $a^*a^\dagger a = a^*(a^\dagger a)^* = (a^\dagger aa)^* = 0$. Therefore,

$$(a + a^*) a^\dagger (a + a^*) = a + aa^\dagger a^* + a^*a^\dagger a + a^*a^\dagger a^* = a.$$

The remaining assertions are proved in a similar way. ∎

Example 3.5. This is a continuation of Example 2.5. If we set $\alpha = 0$ and $\beta = 1$ we get $R(\mathbf{x}, \mathbf{y}) = (T\mathbf{y}, \mathbf{0})$ and $R^\dagger(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, T^{-1}\mathbf{x})$. Obviously,

$$\begin{aligned}(RR^\dagger + R^\dagger R)(\mathbf{x}, \mathbf{y}) &= R(\mathbf{0}, T^{-1}\mathbf{x}) + R^\dagger(T\mathbf{y}, \mathbf{0}) \\ &= (\mathbf{x}, \mathbf{0}) + (\mathbf{0}, \mathbf{y}) = (\mathbf{x}, \mathbf{y}).\end{aligned}$$

Thus, the operator R satisfies item (ii) of Theorem 3.2. Another way of seeing this is by setting $\alpha = 0$ and $\beta = 1$ in the expression for the idempotent P obtained in Example 2.5: we get $P(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{y})$, which is obviously self-adjoint.

4. Limits of sequences of co-EP elements in a C^*-algebra. In this section we shall research the following problem. Let $(a_m)_{m=1}^\infty$ be a convergent sequence in a C^*-algebra and $a = \lim_{n\to\infty} a_n$. We ask:

(a) if $a_n \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$ for all $n \in \mathbb{N}$, when $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$?
(b) if $a_n \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}\perp}$ for all $n \in \mathbb{N}$, when $a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}\perp}$?

We shall introduce some notation before answering these questions. To motivate the following definition, let us recall that the minimal angle between two nonzero subspaces $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$ is the number $\theta \in [0, \pi/2]$ for which $\cos\theta = \|P_{\mathcal{M}} P_{\mathcal{N}}\|$, where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are the orthogonal projectors onto $\mathcal{M}$ and $\mathcal{N}$, respectively (see [Mey, Chapter 5]).

Definition 4.1. Let p and q be two projections in a C^*-algebra. The *angle between* p *and* q is defined to be the number $\theta_{p,q} \in [0, \pi/2]$ such that $\cos\theta_{p,q} = \|pq\|$.

By observing that $\|pq\| = \|(pq)^*\|$, one sees that $\theta_{p,q} = \theta_{q,p}$.

Theorem 4.2. *Let* $\mathcal{A}$ *be a unital* C^**-algebra and* $(a_n)_{n=1}^\infty$ *a sequence of elements in* $\mathcal{A}_{\mathsf{co}}^{\mathsf{ep}}$ *that converges to* a. *Then the following conditions are equivalent:*

(i) $a \in \mathcal{A}^\dagger$ *and* $\lim_{n\to\infty} a_n^\dagger = a^\dagger$ *and there exist* $\theta_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that* $\theta_{a_n a_n^\dagger, a_n^\dagger a_n} > \theta_0$ *for all* $n \geq n_0$.

(ii) $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$.

Proof. (i)⇒(ii): Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n^\dagger = a^\dagger$, we have

$$\lim_{n\to\infty} (a_n a_n^\dagger - a_n^\dagger a_n) = aa^\dagger - a^\dagger a. \tag{4.1}$$

Since $a_n \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$, there exist idempotents p_n obtained in Theorem 2.2. By Corollary 2.4, we have $(a_n a_n^\dagger - a_n^\dagger a_n)^{-1} = 1 - p_n - p_n^*$. Observe that p_n is not a trivial idempotent (if $p_n = 0$, then $a_n = a_n p_n = 0$; if $p_n = 1$, then $a_n = p_n^* a_n = 0$; in both cases, $a_n = 0 \notin \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$, contrary to the hypothesis). By [K-R2, Lemma 2.3] we have $\|1-p_n-p_n^*\| = \|p_n\| = \|p_n^*\|$, and by [K-R2, Theorem 3.1] we have $\|p_n^*\| = (1-\|(p_n^*)^\perp(1-p_n^*)^\perp\|^2)^{-1/2}$. Moreover, we use item (i) of Corollary 2.4 to obtain $\|(p_n^*)^\perp(1-p_n^*)^\perp\| = \|(a_n^\dagger a_n)(a_n a_n^\dagger)\| = \cos\theta_{a_n^\dagger a_n, a_n a_n^\dagger}$. Thus by Corollary 2.4(iii) we get

$$\|(a_n a_n^\dagger - a_n^\dagger a_n)^{-1}\| = \frac{1}{\sin\theta_{a_n a_n^\dagger, a_n^\dagger a_n}}.$$

By assumption, the sequence $(\|(a_n a_n^\dagger - a_n^\dagger a_n)^{-1}\|)_{n=1}^\infty$ is bounded. Hence (4.1) implies that $aa^\dagger - a^\dagger a \in \mathcal{A}^{-1}$, i.e., $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$.

(ii)⇒(i): To prove this, we only have to prove that

$$\lim_{n\to\infty} a_n^\dagger = a^\dagger. \tag{4.2}$$

In fact: if (4.2) is true, then $\lim_{n\to\infty}(a_n a_n^\dagger - a_n^\dagger a_n) = aa^\dagger - a^\dagger a$. Thus (recall that $a \in \mathcal{A}^{\mathsf{ep}}_{\mathsf{co}}$ by hypothesis), $\lim_{n\to\infty}(a_n a_n^\dagger - a_n^\dagger a_n)^{-1} = (aa^\dagger - a^\dagger a)^{-1}$. Therefore, the sequence $(\|(a_n a_n^\dagger - a_n^\dagger a_n)^{-1}\|)_{n=1}^\infty$ is bounded, and as in the proof of (i)⇒(ii), there exist $\theta_0 > 0$ and $n_0 \in \mathbb{N}$ such that $\theta_{a_n a_n^\dagger, a_n^\dagger a_n} > \theta_0$ for all $n \geq n_0$.

Let us prove (4.2): Let p_n and p be the idempotents obtained in Theorem 2.2, i.e.,

$$p_n = (a_n + a_n^*)^{-1} a_n \quad \forall n \in \mathbb{N}, \qquad p = (a + a^*)^{-1} a. \tag{4.3}$$

Hence

$$\lim_{n\to\infty} p_n = p. \tag{4.4}$$

Let us remark that the invertibility of $a_n a_n^\dagger - a_n^\dagger a_n$ implies the invertibility of $a_n a_n^\dagger + a_n^\dagger a_n$ and that (see [K-R1, Theorem 3.5])

$$\begin{gathered}(a_n a_n^\dagger + a_n^\dagger a_n)^{-1} = 1 - p_n - p_n^* + 2p_n p_n^*,\\ (aa^\dagger + a^\dagger a)^{-1} = 1 - p - p^* + 2pp^*.\end{gathered} \tag{4.5}$$

Also, from Corollary 2.4(iii) we have

$$(4.6)\qquad (a_n a_n^\dagger - a_n^\dagger a_n)^{-1} = 1 - p_n - p_n^*, \qquad (aa^\dagger - a^\dagger a)^{-1} = 1 - p - p^*.$$

Clearly, (4.4)–(4.6) imply

$$\lim_{n\to\infty} (a_n a_n^\dagger - a_n^\dagger a_n)^{-1} = (aa^\dagger - a^\dagger a)^{-1},$$
$$\lim_{n\to\infty} (a_n a_n^\dagger + a_n^\dagger a_n)^{-1} = (aa^\dagger + a^\dagger a)^{-1}.$$

Hence,

$$(4.7)\quad \lim_{n\to\infty} (a_n a_n^\dagger - a_n^\dagger a_n) = aa^\dagger - a^\dagger a, \qquad \lim_{n\to\infty} (a_n a_n^\dagger + a_n^\dagger a_n) = aa^\dagger + a^\dagger a.$$

Therefore, $\lim_{n\to\infty} a_n a_n^\dagger = aa^\dagger$, i.e., $\lim_{n\to\infty} a_n^\dagger = a^\dagger$ (see, e.g., [Kol4, Theorem 3.7] or [Rak, Theorem 2.2]). ∎

Example 4.3. This is a continuation of Example 2.5. Define $R_n(\mathbf{x},\mathbf{y}) = (\alpha_n T\mathbf{x} + \beta_n T\mathbf{y}, 0)$, where $\alpha_n = \cos(1/n)$ and $\beta_n = \sin(1/n)$ for $n \in \mathbb{N}$ and $L(\mathbf{x},\mathbf{y}) = (T\mathbf{x}, \mathbf{0})$. The following elementary facts can be easily checked:

(a) $\lim_{n\to\infty} R_n = L$.
(b) L is Moore–Penrose invertible and $L^\dagger(\mathbf{x},\mathbf{y}) = (T^{-1}\mathbf{x}, \mathbf{0})$.
(c) $LL^\dagger - L^\dagger L = 0$, which implies that the subset of co-EP elements in a C^*-algebra is not always closed.
(d) $\lim_{n\to\infty} R_n^\dagger = L^\dagger$.
(e) Let $U_n = R_n(R_n^\dagger)^2 R_n$. If $\theta_n \in [0,\pi/2]$ is the angle between $R_n R_n^\dagger$ and $R_n^\dagger R_n$, then by using $\cos^2\theta_n = \|(R_n R_n^\dagger)(R_n^\dagger R_n)\|^2 = \|U_n\|^2 = \|UU^*\|$ we get $\theta_n = 1/n$.

This example shows that the condition "there exist $\theta_0 > 0$ and $n_0 \in \mathbb{N}$ such that $\theta_{a_n a_n^\dagger, a_n^\dagger a_n} > \theta_0$ for all $n \ge n_0$" in Theorem 4.2 cannot be removed.

Example 4.4. This is a continuation of Example 3.5. Let $R_n(\mathbf{x},\mathbf{y}) = (\mathbf{y}/n, \mathbf{0})$ for $n \in \mathbb{N}$. It is evident that $\lim_{n\to\infty} R_n = 0$ and $R_n^\dagger(\mathbf{x},\mathbf{y}) = (\mathbf{0}, n\mathbf{x})$, which shows that $(R_n^\dagger)_{n=0}^\infty$ does not converge. This proves that the condition $\lim_{n\to\infty} a_n^\dagger = a^\dagger$ in Theorem 4.2 cannot be removed.

Theorem 4.5. *Let $\mathcal{A}$ be a unital C^*-algebra and $(a_n)_{n=1}^\infty$ a sequence of elements in $\mathcal{A}_{\mathsf{co}}^{\mathsf{ep}_\perp}$ that converges to $a \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) *$a \in \mathcal{A}^\dagger$ and $\lim_{n\to\infty} a_n^\dagger = a^\dagger$.*
(ii) *$a \in \mathcal{A}_{\mathsf{co}}^{\mathsf{ep}_\perp}$.*

Proof. (i)⇒(ii): Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n^\dagger = a^\dagger$, and moreover $\lim_{n\to\infty}(a_n a_n^\dagger + a_n^\dagger a_n) = 1$ (this last relation is guaranteed by Theorem 3.2), we obtain $aa^\dagger + a^\dagger a = 1$. The conclusion follows again by Theorem 3.2.

(ii)⇒(i): By Corollary 3.4 we have, for every $n \in \mathbb{N}$,

$$a_m^\dagger = (a_m + a_m^*)^{-1} a_m (a_m + a_m^*)^{-1}, \qquad a^\dagger = (a + a^*)^{-1} a (a + a^*)^{-1}. \tag{4.8}$$

Since $\lim_{n\to\infty} a_n = a$, we have $\lim_{n\to\infty} a_n^* = a^*$. Recall that the function $\phi : \mathcal{A}^{-1} \to \mathcal{A}^{-1}$ given by $\phi(x) = x^{-1}$ is continuous. By Theorem 2.2, for each $n \in \mathbb{N}$ we have $a_n + a_n^* \in \mathcal{A}^{-1}$ and $a + a^* \in \mathcal{A}^{-1}$. Hence $\lim_{n\to\infty}(a_n + a_n^*)^{-1} = (a + a^*)^{-1}$. From (4.8) we have $\lim_{n\to\infty} a_n^\dagger = a^\dagger$, which finishes the proof. ∎

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