A Gowers tree like space and the space of its bounded linear operators

by

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Abstract. The famous Gowers tree space is the first example of a space not containing c_0, ℓ_1 or a reflexive subspace. We present a space with a similar construction and prove that it is hereditarily indecomposable (HI) and has ℓ_2 as a quotient space. Furthermore, we show that every bounded linear operator on it is of the form $\lambda I + W$ where W is a weakly compact (hence strictly singular) operator.

1. Introduction. As is well known, B. S. Tsirelson [T] constructed the first Banach space not containing c_0 or ℓ_p for $1 \leq p < \infty$. After Tsirelson's fundamental example, the original question of whether every Banach space contains an isomorphic copy c_0 or ℓ_p was replaced by the following: Does every Banach space contain c_0, ℓ_1 or a reflexive subspace?

A classical result of R. C. James [J1] asserting that a space with an unconditional basis is either reflexive or has a subspace isomorphic to either c_0 or ℓ_1 provides an affirmative answer to the above question within the class of Banach spaces containing an unconditional basic sequence. In 1994 W. T. Gowers, based on the fundamental construction of HI spaces by Gowers and B. Maurey [GM], settled the above problem in the negative by providing a Banach space not containing c_0, ℓ_1 or a reflexive subspace. This example became known as the Gowers tree space and we hereafter denote it by GT. Gowers' famous dichotomy [G2] implies that any space sharing this property should be HI saturated, i.e. every infinite-dimensional closed subspace of it contains a hereditarily indecomposable one.

The main idea behind the GT construction is to endow each subspace of the predual with a structure that resembles the tree structure of the biorthogonal functionals of the basis of the James tree space (denoted by JT), a space not containing ℓ_1 and with nonseparable dual [J2]. In order to achieve this, Gowers combines, to some extent, the Gowers and Maurey

norm [GM] with that of JT. The key point is the way the special functionals are produced. Let us recall that the tree structure in JT occurs on the basis of the space which, through a suitable partial ordering, can be indexed as $(e_t)_{t \in 2^{<\omega}}$ with $2^{<\omega}$ denoting the dyadic tree. Then the special functionals are defined as $s^* = \sum_{t \in s} e_t^*$ for all segments $s \subset 2^{<\omega}$. In the case of GT Gowers has defined, through a coding similar to the one used in [GM], an infinitely branching tree structure of functionals which penetrates every block subspace of $\langle e_n^* : n \in \mathbb{N} \rangle$. The special functionals are defined again as sums over all segments of this tree structure. Thus, the GT construction imposes a hereditary James tree type structure in every block subspace of the predual of GT. Furthermore, there is a natural notion of disjointness that characterizes special functionals as in the case of segments in JT^*. This is used in order to include ℓ_2-sums of "disjoint" special functionals in the norming set of GT similar to the ℓ_2-convex combinations of disjoint segments in JT^*.

These so-called special combinations are on one hand essential as they do not allow ℓ_1 to embed into GT, and on the other hand they make some crucial estimations very hard. This is because the special functionals used to form an ℓ_2-special combination do not necessarily have disjoint supports. Gowers overcomes this problem by using elaborate finite combinatorics and advanced probabilistic arguments associated to the Hamming distance to provide estimates for certain averages of rapidly increasing sequences (RIS) (see Lemma 4 in [G1]). Namely, he shows that if one considers such a sequence $(x_i)_{i=1}^M$ then there would necessarily exist a choice of signs $(\epsilon_i)_{i=1}^M$ such that the norm $\|M^{-1} \sum_{i=1}^M \epsilon_i x_i\|$ is approximately $1/\sqrt{\log_2(M+1)}$. As such averages exist in every block subspace, GT cannot contain ℓ_1. In addition, it allows Gowers to show that every block sequence $(y_n)_n$ in GT has a further block subsequence $(z_n)_n$ which is not weakly null. By a classical result of W. B. Johnson and H. P. Rosenthal [JR] this implies that every infinite-dimensional subspace of GT has nonseparable dual, as GT has a boundedly complete basis, which also implies that c_0 does not embed into the space.

Gowers' deep approach, however, is in its base existential and thus cannot provide more precise estimates for the action of other types of functionals on these averages which are necessary for proving additional properties of the space (for example, that it is HI) and studying its operators. In this paper we present a slight variant of GT which we denote by $\mathfrak{X}_{gt}$ and use different techniques to investigate its properties. We fix two sequences of natural numbers $(m_j)_j$ and $(n_j)_j$ and define the norming set G_{gt} for $\mathfrak{X}_{gt}$ to be the minimal subset of $c_{00}(\mathbb{N})$ satisfying:

- G_{gt} is symmetric, closed under projections on intervals and contains the set $\{\pm e_n^* : n \in \mathbb{N}\}$.

- It is closed under the $(\mathcal{A}_{n_j}, 1/2m_j)$ operations for all j.
- It is rationally convex.
- It contains the set $\mathcal{S}$ of all (finite) special functionals.
- It contains the set $\{\sum_{i=1}^d a_i x_i^* : a_i \in \mathbb{Q}, \sum_{i=1}^d a_i^2 \leq 1, (x_i^*)_{i=1}^d \subset \mathcal{S}$, $\operatorname{ind} x_i^* \cap \operatorname{ind} x_j^* = \emptyset, i \neq j\}$.

Recall that a set $F \subset c_{00}(\mathbb{N})$ is *closed under the* $(\mathcal{A}_n, 1/m)$ *operation* if for every block sequence $f_1 < \cdots < f_d$ in F with $d \leq n$ the functional $f = \frac{1}{m} \sum_{i=1}^d f_i$ lies in F. For such a functional f we write $w(f) = m$. The set $\mathcal{S}$ of special functionals contains elements of the form $x^* = E \sum_i f_i$ where E is a finite interval of $\mathbb{N}$ and $(f_i)_i$ is a special sequence. The latter are defined through a standard coding function σ. Namely, a block sequence $(f_i)_i$ is called *special* if $w(f_1) = 2m_{j_1}$ and $w(f_{i+1}) = 2m_{\sigma(f_1,\dots,f_i)}$ for $i \geq 1$. For such a sequence we set $(\operatorname{ind} f_i)_i = \{j_i : w(f_i) = 2m_{j_i}\} = \{j_1 < \sigma(f_1) < \sigma(f_1, f_2) < \cdots\}$ and for a special functional $x^* = E \sum_i f_i$, $\operatorname{ind} x^* = \{\operatorname{ind} f_i : \operatorname{ran} f_i \cap E \neq \emptyset\}$.

The norming set of GT is defined in a similar way. The only differences are that the latter is closed under the $(\mathcal{A}_n, 1/\sqrt{\log_2(n+1)})$ operations, for all $n \in \mathbb{N}$, and that the coding function σ in GT selects weights from a lacunary subset J of the natural numbers (see [G1]).

As mentioned above, our methods differ significantly from those used by Gowers. More precisely, we start with an arbitrary *infinite* RIS $(y_n)_n$ and refine it through repeated application of classical Ramsey theory to produce a new RIS $(w_n)_n$ with strong stability properties with respect to the action of all types of functionals on its elements. In particular, starting with $j_0 \in \mathbb{N}$ and a RIS sequence $(y_n)_n$ it is shown that there exists a subsequence $(w_n)_n$ of $(y_{2n} - y_{2n-1})_n$ such that for every ℓ_2-special combination $y^* = \sum_{i=1}^d a_i x_i^*$ with $\operatorname{ind} x_i^* \geq j_0$ for all $i \leq d$, we have

$$(1) \qquad |\{k \in \mathbb{N} : |y^*(w_k)| \geq 5/m_{j_0}\}| \leq 1025 m_{j_0}^2.$$

We call a sequence satisfying (1) a j_0-separated RIS. This permits us to use a Basic Inequality to derive precise estimates for the actions of functionals on averages of a j-separated RIS. This is done by reducing evaluations to the basis of an appropriately defined auxiliary space. The main difficulty at this point is that given an ℓ_2-special combination $y^* = \sum_{i=1}^d a_i x_i^*$, the special functionals $(x_i^*)_{i=1}^d$ may have overlapping supports. After proving the Basic Inequality one can use standard arguments to establish the existence of exact pairs and dependent sequences in every block subspace of $\mathfrak{X}_{gt}$ (see, for example, [ATO]). This, in turn, implies that the space is HI and enables us to study the structure of bounded linear operators on the space as well as the properties of its predual, dual and second dual.

We believe that it is possible to apply an analogous procedure to rapidly increasing sequences in the original Gowers tree space to prove that GT shares similar properties. We also note that the proof of (1) above extends techniques which have been developed by S. A. Argyros, A. Arvanitakis and A. Tolias in [AAT], where a new class of spaces not containing c_0, ℓ_1 or a reflexive subspace is presented. Their constructions involve the method of attractors and are different from that of GT and the present one.

Our main results are the following.

THEOREM I. *The space $\mathfrak{X}_{gt}$ is HI and every bounded linear operator $T : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}$ is of the form $\lambda I + W$ where W is a strictly singular and weakly compact operator.*

THEOREM II. *The predual $(\mathfrak{X}_{gt})_*$ is HI and every bounded linear operator $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is of the form $\lambda I + W$ where W is a strictly singular and weakly compact operator.*

We also show the following.

THEOREM III. *For every infinite-dimensional closed subspace Y of $\mathfrak{X}_{gt}$ the dual space Y^* is nonseparable and contains an isomorphic copy of ℓ_2. Therefore, Y has ℓ_2 as a quotient space.*

The above result shows that *no* closed infinite subspace of $\mathfrak{X}_{gt}$ is quotient HI, where a Banach space is said to be *quotient HI* if all of its infinite-dimensional quotient spaces, over closed subspaces, are hereditarily indecomposable. A problem posed by S. A. Argyros mentioned in [F2] asks whether there exists a reflexive HI space X such that the dual of no infinite-dimensional subspace is HI. In the case of $\mathfrak{X}_{gt}$ we see, by Theorem III above, that for every such subspace Y of $\mathfrak{X}_{gt}$, Y^* is not HI. However, $\mathfrak{X}_{gt}$ is not reflexive and thus the above problem remains open. Moreover, using techniques developed in [AAT], it can be shown that every quotient of $\mathfrak{X}_{gt}$ with a w^*-closed kernel is HI. Hence, if we consider a quotient of $\mathfrak{X}_{gt}$ by a block subspace Y we find that it is HI. For more details on properties of quotient HI spaces we refer the interested reader to the work of V. Ferenczi [F1], [F2].

In addition we show the following.

THEOREM IV. *For every infinite-dimensional closed subspace Y of $\mathfrak{X}_{gt}$ its second dual space Y^{**} contains an isomorphic copy of $\ell_2(c)$, where c is the Cantor set, and thus Y^* has $\ell_2(c)$ as a quotient space.*

Theorems III and IV above illustrate the analogies between the triples $\mathfrak{X}_{gt}, \mathfrak{X}_{gt}^*, \mathfrak{X}_{gt}^{**}$ and JT, JT^*, JT^{**}. It seems peculiar, at first, that an HI space containing no reflexive subspace is in a sense "surrounded" by Hilbert spaces and that its dual and second dual share similar properties with the corresponding ones of JT, which is hereditarily ℓ_2. However striking, this phe-

nomenon is supported by deep results of S. A. Argyros, P. Dodos and V. Kanellopoulos [ADK1], [ADK2] asserting that for every separable Banach space X not containing ℓ_1 with X^* nonseparable there is a James tree type structure in X. In particular, it is shown that for such X there exist two families $\mathcal{A} = (x_t)_{t \in 2^{<\omega}} \subset B_X$ and $\mathcal{B} = (x_b^{**})_{b \in 2^\omega} \subset B_{X^{**}}$ (2^ω denotes the Cantor set) such that $\mathcal{B}$ is w^*-discrete, 1-unconditional accumulating to zero and for every $b \in 2^\omega$, $x_{b|n} \xrightarrow{w^*} x_b^{**}$.

This paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we give the definition of the norming set of the space. In Section 4 we recall the definition of (C, ϵ) rapidly increasing sequences (RIS) and then show that in every block subspace of $\mathfrak{X}_{gt}$ one can find a $(3, \epsilon)$ RIS for every $\epsilon > 0$.

In Section 5 we investigate the combinatorial properties of RIS in $\mathfrak{X}_{gt}$. We introduce the notion of j_0-separated RIS for a given $j_0 \in \mathbb{N}$, mentioned above, and show that for every $j_0 \in \mathbb{N}$, every block subspace Y of $\mathfrak{X}_{gt}$ and $\epsilon < 5/m_{j_0}$ there exists a j_0-separated $(6, \epsilon)$ RIS in Y. Once this is achieved, in the next section we verify that n_{2j_0}-averages of vectors in a j_0-separated RIS satisfy precise estimates. This process goes through the general technique called the Basic Inequality. The whole of Section 6 is devoted to the proof of the Basic Inequality. In Section 7 we use the Basic Inequality to establish the existence of dependent sequences in every block subspace of $\mathfrak{X}_{gt}$. Subsequently, we pass in Sections 8 and 9 to study the fundamental properties of the space and the space of its bounded linear operators. In these two sections we prove Theorem I stated above.

In Section 10 we study the structure of the triple $\mathfrak{X}_{gt}, \mathfrak{X}_{gt}^*, \mathfrak{X}_{gt}^{**}$. In particular we prove Theorems III and IV. These two results go through the following proposition.

PROPOSITION. *For each $i \in \mathbb{N}$ consider a 6-dependent sequence $(w_n^i, f_n^i)_n$. Assume that $\operatorname{ind} f_n^i \cap \operatorname{ind} f_n^{i'} = \emptyset$ for all $i \neq i' \in \mathbb{N}$, and set $b_i^* = \sum_n f_n^i$ for all $i \in \mathbb{N}$. Then $(b_i^*)_{i \in \mathbb{N}}$ with the $\mathfrak{X}_{gt}^*$ norm is equivalent to the standard ℓ_2 basis.*

In the proof we use a second Basic Inequality to provide the lower ℓ_2 estimates. Finally, in the last section we study the structure of the predual space $(\mathfrak{X}_{gt})_*$ and prove Theorem II.

2. Notation. Throughout, we make use of the following standard notation.

We denote by $c_{00}(\mathbb{N})$ the set $\{f : \mathbb{N} \to \mathbb{R} : f(n) \neq 0$ for finitely many $n \in \mathbb{N}\}$ and by $c_{00}^{\mathbb{Q}}(\mathbb{N})$ the set of all elements of $c_{00}(\mathbb{N})$ with rational coordinates. For every $x \in c_{00}(\mathbb{N})$ we denote by $\operatorname{supp} x$ the set $\{i \in \mathbb{N} : x(i) \neq 0\}$ and by $\operatorname{ran} x$ the minimal interval of $\mathbb{N}$ that contains $\operatorname{supp} x$.

We denote by $(e_n)_n$ the standard Hamel basis of $c_{00}(\mathbb{N})$.

Let E_1, E_2 be two nonempty finite subsets of $\mathbb{N}$. We write $E_1 < E_2$ if $\max E_1 < \min E_2$. If $x_1, x_2 \in c_{00}(\mathbb{N})$ we write $x_1 < x_2$ whenever $\operatorname{ran} x_1 < \operatorname{ran} x_2$. A sequence $(x_k)_k$ in $c_{00}(\mathbb{N})$ is called a *block sequence* if $x_i < x_j$ for all i, j with $i < j$. For a function $f : \mathbb{N} \to \mathbb{R}$ and E an interval of $\mathbb{N}$ we denote by Ef the restriction of f to E.

We say that a subset G of $c_{00}(\mathbb{N})$ is *closed under the $(\mathcal{A}_n, \theta)$-operation*, for $0 < \theta < 1$, if $\theta \sum_{i=1}^d f_i \in G$ for every block sequence $(f_i)_{i=1}^d$ in G with $d \leq n$.

We fix two sequences of natural numbers $(m_j)_j$ and $(n_j)_j$ defined recursively as follows for $j \geq 1$:

$$m_1 = 2^4, \quad m_{j+1} = m_j^5,$$

$$n_1 = 2^7, \quad n_{j+1} = (2n_j)^{s_{j+1}}, \quad s_{j+1} = \log_2(m_{j+1}^3).$$

For a set A we denote by $|A|$ the cardinality of A and by $[A]$ the set of its infinite subsets. We also denote by $2^{<\omega}$ the dyadic tree and by 2^ω the set of its infinite branches.

3. The norming set G_{gt}

DEFINITION 3.1. We say that $f \in c_{00}(\mathbb{N})$ has *weight* m_j, and we write $w(f) = m_j$, if there exists a block sequence $(f_i)_{i=1}^d$ with $\|f_i\|_\infty \leq 1$ and $d \leq n_j$ such that $f = \frac{1}{2m_j} \sum_{i=1}^d f_i$.

DEFINITION 3.2. We fix two disjoint infinite subsets Ω_1, Ω_2 of $\mathbb{N}$ and set

$$Q_s = \{(f_i)_{i=1}^d : d \in \mathbb{N}, \, f_i \in c_{00}^{\mathbb{Q}}(\mathbb{N}), f_i \neq 0, \, (f_i)_{i=1}^d \text{ is a block sequence}\}.$$

As Q_s is countable we fix an injective coding function $\sigma : Q_s \to \Omega_2$ satisfying

$$m_{\sigma(f_1,\dots,f_d)} > \max\{1/|f_i(e_l)| : i = 1, \dots, d \text{ and } l \in \operatorname{supp} f_i\} \cdot \max \operatorname{supp} f_d.$$

DEFINITION 3.3. A block sequence $(f_i)_i$ such that $(f_i)_{i=1}^n \in Q_s$ for all n is called *σ-special* or simply *special* if $w(f_1) = m_j$ for some $j \in \Omega_1$, and $w(f_{i+1}) = m_{\sigma(f_1,\dots,f_i)}$ for all $i \geq 1$.

For a given special sequence $(f_i)_i$ we will denote by $(\operatorname{ind} f_i)_i$ the sequence

$$\{j_i \in \mathbb{N} : w(f_i) = m_{j_i}\} = \{j_1 < j_2 < \cdots\}$$

where $j_1 \in \Omega_1$ and $j_{i+1} = \sigma(f_1, \dots, f_i)$ for $i \in \mathbb{N}$.

DEFINITION 3.4. An *infinite special functional* is of the form $x^* = E \sum_i f_i$ where $(f_i)_i$ is an infinite special sequence, E is an infinite interval of $\mathbb{N}$ and the sum is convergent in the pointwise topology.

A *finite special functional* is of the form $x^* = E \sum_i f_i$ where $(f_i)_i$ is an infinite special sequence and E is a finite interval of $\mathbb{N}$. The set of all finite special functionals will be denoted by $\mathcal{S}$.

For every special functional $x^* = E \sum_i f_i$ we set

$$\operatorname{ind} x^* = \{\operatorname{ind} f_i : \operatorname{ran} f_i \cap E \neq \emptyset\}.$$

We call two special functionals x_1^*, x_2^* *incomparable* if $\operatorname{ind} x_1^* \cap \operatorname{ind} x_2^* = \emptyset$.

REMARK 3.1 (Tree like property). If $(f_i)_i, (g_i)_i$ are two distinct special sequences then there exists an $i_0 \in \mathbb{N}$ such that $f_i = g_i$ for $i < i_0$, $w(f_{i_0}) = w(g_{i_0})$, $f_{i_0} \neq g_{i_0}$ and $w(f_i) \neq w(g_i)$ for $i > i_0$.

We now define the *norming set*:

DEFINITION 3.5. Let G_{gt} be the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following:

- $\pm e_n \in G_{gt}$ for all $n \in \mathbb{N}$.
- G_{gt} is closed under the $(\mathcal{A}_{n_j}, 1/2m_j)$-operation for every $j \in \mathbb{N}$.
- G_{gt} is rationally convex.
- $\mathcal{S} \subseteq G_{gt}$.
- G_{gt} contains $\{\sum_{i=1}^d a_i x_i^* : a_i \in \mathbb{Q}, \sum_{i=1}^d a_i^2 \leq 1, (x_i^*)_{i=1}^d \subseteq \mathcal{S}$ and $\operatorname{ind} x_i^* \cap \operatorname{ind} x_j^* = \emptyset$ for $i \neq j\}$.

It is clear that G_{gt} induces a norm on $c_{00}(\mathbb{N})$: we set

$$\|x\|_{gt} = \sup\{f(x) : f \in G_{gt}\} \quad \text{for all } x \in c_{00}(\mathbb{N}),$$

and we denote by $\mathfrak{X}_{gt}$ the completion of $c_{00}(\mathbb{N})$ under the norm $\| \cdot \|_{gt}$.

We also make use of the following terminology:

DEFINITION 3.6. Let $f \in G_{gt}$ and $f \neq 0$. We say that f is of

- *type 0* if $f \in \{\pm e_n : n \in \mathbb{N}\}$,
- *type I* if $f \in \{\frac{1}{2m_j}(f_1 + \cdots + f_d) : j \in \mathbb{N}, d \leq n_j, (f_i)_{i=1}^d \subset G_{gt}$ and $(f_i)_i$ is a block sequence$\}$,
- *type II* if $f \in \{\sum_{i=1}^d a_i x_i^* : a_i \in \mathbb{Q}, \sum_{i=1}^d a_i^2 \leq 1, (x_i^*)_{i=1}^d \subseteq \mathcal{S}$ and $\operatorname{ind} x_i^* \cap \operatorname{ind} x_j^* = \emptyset$ for $i \neq j\}$,
- *type III* if $f \in \{\sum_{i=1}^d r_i f_i : d \in \mathbb{N}, r_i \in \mathbb{Q}^+, f_i \in G_{gt}, \sum_{i=1}^d r_i = 1\}$.

NOTATION 3.1. For a special functional $f = E \sum_i f_i \in G_{gt}$ and $k \in \mathbb{N}$ we write

$$f_{<k} = E \sum_{w(f_i) < m_k} f_i \quad \text{and} \quad f_{\geq k} = E \sum_{w(f_i) \geq m_k} f_i.$$

Similarly, for a type II functional $y^* = \sum_{i=1}^d a_i x_i^*$ we write $\operatorname{ind} y^* = \bigcup_{i=1}^d \operatorname{ind} x_i^*$ and

$$y_{<k}^* = \sum_{i=1}^d a_i x_{i,<k}^* \quad \text{and} \quad y_{\geq k}^* = \sum_{i=1}^d a_i x_{i,\geq k}^*.$$

DEFINITION 3.7. Let $f \in G_{gt}$ with finite support and $j_0 \in \mathbb{N}$. A family $(f_a)_{a \in A}$ is called a j_0-*tree analysis* of f if:

(1) A is a finite tree, equipped with a partial ordering $\sqsubset$, with a least element denoted by 0, $f_a \in G_{gt}$ for all $a \in A$ and $f_0 = f$.
(2) For $a \in A$ maximal, $f_a \in \{\pm e_n : n \in \mathbb{N}\}$.
(3) For $a, b \in A$ with $a \sqsubset b$ we have $\operatorname{ran} f_b \subset \operatorname{ran} f_a$.
(4) For $a \in A$ not maximal we denote by S_a the set of immediate successors of a in A. If $(f_b)_{b \in S_a}$ can be written as a block sequence we assume S_a to be totally ordered as $\{b_1 < \cdots < b_{|S_a|}\}$ so that $b_i < b_j$ iff $\operatorname{ran} f_{b_i} < \operatorname{ran} f_{b_j}$ for $b_i \neq b_j \in S_a$.
(5) For $a \in A$ not maximal f_a has one of the following forms:
   - If f_a is of type I then $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$, where $|S_a| \leq n_{j_a}$ and $(f_s)_{s \in S_a}$ is a block sequence.
   - If f_a is special then $f_a = E_a \sum_i f_i$, where E_a is a finite interval of $\mathbb{N}$ and $(f_i)_i$ is a special sequence. Set $F_a = \{i \in \mathbb{N} : \operatorname{ran} f_i \cap E_a \neq \emptyset\} = \{i_1^a, \ldots, i_{d_a}^a\}$ and $S_a = \{s_1, \ldots, s_{d_a}\}$ where $f_{s_j} = E_a f_{i_j}$ and $w(f_{s_j}) = w(f_{i_j})$ for all $j \in \{1, \ldots, d_a\}$. Finally, we write $f_a = \sum_{s \in S_a} f_s$.
   - If f_a is of type II and $f_{a, < j_0} \neq 0$ and $f_{a, \geq j_0} \neq 0$ then $S_a = \{s_1, s_2\}$ and $f_a = f_{s_1} + f_{s_2}$, where $f_{s_1} = f_{a, < j_0}$ and $f_{s_2} = f_{a, \geq j_0}$. If either $f_{a, < j_0} = 0$ or $f_{a, \geq j_0} = 0$ then $f_a = \sum_{s \in S_a} a_s f_s$, where $(a_s)_{s \in S_a} \subset \mathbb{Q}$, $(f_s)_{s \in S_a}$ are special functionals with disjoint sets of indices and $\sum_{s \in S_a} a_s^2 \leq 1$.
   - If f_a is of type III then $f_a = \sum_{s \in S_a} r_s f_s$, where $r_s \in \mathbb{Q}^+$ and $\sum_{s \in S_a} r_s = 1$.

REMARK 3.2. The following can be readily established:

(1) For $f \in G_{gt}$ we have $\|f\|_\infty \leq 1$.
(2) G_{gt} is symmetric and closed under projections on intervals of $\mathbb{N}$. The Hamel basis $(e_n)_n$ is a bimonotone and normalized Schauder basis for $\mathfrak{X}_{gt}$. Moreover, $(e_n)_n$ is boundedly complete.
(3) Every $f \in G_{gt}$ is of type 0, I, II or III. However, the type is not uniquely defined.
(4) For every $j_0 \in \mathbb{N}$, every $f \in G_{gt}$ admits a j_0-tree analysis, which in general is not unique as functionals may have various types. This, however, does not play any role as the proofs work for any j_0-tree analysis that one considers.
(5) For every block sequence $(x_i)_{i=1}^d$ in $\mathfrak{X}_{gt}$ and $j \in \mathbb{N}$ such that $d \leq n_{2j}$ it follows that $\|\sum_{i=1}^d x_i\|_{gt} \geq \frac{1}{2m_{2j}} \sum_{i=1}^d \|x_i\|_{gt}$.

4. Rapidly increasing sequences in $\mathfrak{X}_{gt}$. We begin by recalling the definitions of a rapidly increasing sequence (RIS) and M-ℓ_1^k averages.

DEFINITION 4.1. Let $(x_n)_n$ be a block sequence in $\mathfrak{X}_{gt}$ and C, ϵ positive numbers. The sequence $(x_n)_n$ will be called a (C, ϵ) *RIS* if:

- $\|x_n\|_{gt} \leq C$ for all $n \in \mathbb{N}$.
- There exists a strictly increasing sequence $(j_n)_n$ of natural numbers such that $|\operatorname{supp} x_n|/m_{j_{n+1}} < \epsilon$ for all $n \in \mathbb{N}$.
- For $n \in \mathbb{N}$ and $f \in G_{gt}$ with $w(f) = m_i < m_{j_n}$ we have $|f(x_n)| \leq C/2m_i$.

DEFINITION 4.2. Let $k \in \mathbb{N}$ and $M > 0$. We call a vector $x \in \mathfrak{X}_{gt}$ an M-ℓ_1^k *average* if

- $\|x\|_{gt} > 1$.
- There exists a block sequence $(x_i)_{i=1}^k$ with $\|x_i\|_{gt} \leq M$ for all $i = 1, \ldots, k$ such that $x = \frac{1}{k} \sum_{i=1}^k x_i$.

Here we need three lemmas that establish the existence of a $(3, \epsilon)$ RIS in every block subspace of $\mathfrak{X}_{gt}$. We start with the following.

LEMMA 4.1. *Let Z be a block subspace of $\mathfrak{X}_{gt}$ and $k \in \mathbb{N}$. Then there exists an $x \in Z$ which is a 2-ℓ_1^k average.*

This lemma is an immediate consequence of Remark 3.2(5); for a detailed proof we refer the interested reader to [ATO, Lemma II.22].

The following lemma is necessary to describe the behavior of functionals with small weight acting on large ℓ_1 averages; its proof follows a standard technique which can be found in most of the articles in the relevant literature. For more details we refer to [ATO, Lemma II.23].

LEMMA 4.2. *Let $x \in \mathfrak{X}_{gt}$ be an M-ℓ_1^k average for $k \in \mathbb{N}$ and $M > 0$, and let $f \in G_{gt}$ with $w(f) = m_i$. Then*

$$|f(x)| \leq \frac{M}{2m_i}\left(1 + \frac{2n_i}{k}\right).$$

Finally, combining Lemmas 4.1, 4.2 and a simple inductive argument we obtain the following.

LEMMA 4.3. *For every $\epsilon > 0$ and any block subspace Z of $\mathfrak{X}_{gt}$ there exists a block sequence $(x_n)_n$ in Z which is a $(3, \epsilon)$ RIS and $\|x_n\|_{gt} > 1$ for all $n \in \mathbb{N}$. In addition, each x_n is a 2-$\ell_1^{n_{j_n}}$ average with $(j_n)_n$ as in Definition 4.1.*

The proof is identical to that of Proposition II.25 in [ATO] so we omit it.

5. Combinatorial properties of rapidly increasing sequences in $\mathfrak{X}_{gt}$.

In this section we establish the existence of rapidly increasing sequences that satisfy some strong combinatorial properties in every block subspace of $\mathfrak{X}_{gt}$. Before proceeding it is necessary to give a brief description of the

pointwise closure of $\mathcal{S}$, the set of all finite special functionals. Namely, we can readily establish

FACT 5.1. *Every $f \in \overline{\mathcal{S}}^{w^*}$ has one of the following forms*:

(1) *f is a finite special functional.*
(2) *f is an infinite special functional.*
(3) *f can be written as $f = \sum_{i=1}^{k} f_i$ and*

- *$(f_i)_{i=1}^{k-1}$ is a finite special sequence,*
- *f_k can be represented as $f_k = \frac{1}{2w(f_k)} \sum_{n=1}^{\infty} f_n^k$ where $(f_n^k)_n$ is a block sequence and $w(f_k) = m_{\sigma(f_1,\ldots f_{k-1})}$. In this case we set $\mathrm{ind}\, f = \mathrm{ind}((f_i)_{i=1}^{k-1}) \cup w(f_k)$.*

REMARK 5.1. It can be seen that for any two finite sequences $(a_i)_{i=1}^d$ and $(f_i)_{i=1}^d$ such that

- $\sum_{i=1}^{d} a_i^2 \le 1$,
- $f_i \in \overline{\mathcal{S}}^{w^*}$ for all $i \le d$ and $(f_i)_{i=1}^d$ have disjoint sets of indices,

the functional $\sum_{i=1}^{d} a_i f_i$ is an element of $\overline{G}_{gt}^{w^*}$.

The following definition sums up all the desired combinatorial properties of an RIS mentioned at the beginning of this section:

DEFINITION 5.1. Let $j_0 \in \mathbb{N}$ and $(x_n)_n$ be a (C, ϵ) RIS with $0 < \epsilon < 5/m_{j_0}$ and $(j_n)_n$ its associated sequence of natural numbers. We will call $(x_n)_n$ *j_0-separated* if:

- $j_1 > j_0$.
- For every functional $f \in G_{gt}$ of type I with $w(f) \ge m_{j_0}$, we have $|\{k \in \mathbb{N} : |f(x_k)| \ge 5/m_{j_0}\}| \le 1$.
- For every special functional x^* with $\mathrm{ind}\, x^* \ge j_0$, we have $|\{k \in \mathbb{N} : |x^*(x_k)| \ge 10/m_{j_0}\}| \le 2$.
- If $y^* \in G_{gt}$ is of type II with $\mathrm{ind}\, y^* \ge j_0$, then $|\{k \in \mathbb{N} : |y^*(x_k)| \ge 5/m_{j_0}\}| \le 1025 m_{j_0}^2$.

By Lemma 4.3, in every block subspace of $\mathfrak{X}_{gt}$ one can find a seminormalized $(3, \epsilon)$ RIS. The rest of this section is devoted to showing that for every block subspace Z of $\mathfrak{X}_{gt}$ and every $j_0 \in \mathbb{N}$ one can find a seminormalized $(6, \epsilon)$ RIS in Z which is additionally j_0-separated. We begin with the following general lemma.

LEMMA 5.1. *Let $(x_n)_n$ be a bounded block sequence in $\mathfrak{X}_{gt}$. Then there exists an $L \in [\mathbb{N}]$ such that the sequence $(x^*(x_n))_{n \in L}$ is convergent for every special functional x^*.*

Proof. Since $(x_n)_n$ is a block sequence we only need to consider the case of infinite special functionals. We need the following.

CLAIM. *For every $\epsilon > 0$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ and a finite collection $A = \{x_1^*, \ldots, x_l^*\}$ of infinite special functionals such that for every infinite special functional $x^* \notin A$ we have $\limsup_{n \in L} |x^*(x_n)| \leq \epsilon$.*

Proof of Claim. Suppose not. Then there exist $\epsilon > 0$ and $M \in [\mathbb{N}]$ for which we can construct a decreasing sequence $(M_i)_i$ of infinite subsets of M and a sequence $(x_i^*)_i$ of pairwise different infinite special functionals such that $|x_i^*(x_n)| > \epsilon$ for every $n \in M_j$ with $j \geq i$. Set $C = \sup\{\|x_n\|_{gt} : n \in \mathbb{N}\}$ and choose $r > C/\epsilon$. Since the functionals $x_1^*, \ldots, x_{r^2}^*$ are mutually different, by Remark 3.1 we can choose an arbitrarily large finite interval E of $\mathbb{N}$ such that the functionals $\widehat{x}_1^* = Ex_1^*, \ldots, \widehat{x}_{r^2}^* = Ex_{r^2}^*$ have mutually disjoint sets of indices $(\text{ind } \widehat{x}_i^*)_{i=1}^{r^2}$. As $(x_n)_n$ is a block sequence and E is arbitrarily large we can pick $n \in M_{r^2}$ such that $\text{supp } x_n \subset E$. Let also $a_i = (\text{sgn } x_i^*(x_n))/r$ for $i = 1, \ldots, r^2$. Set $f = \sum_{i=1}^{r^2} a_i \widehat{x}_i^*$. Then $f \in G_{gt}$ and $f(x_n) > C$. This contradiction yields the claim.

Using the claim we can inductively construct a strictly decreasing sequence $(L_n)_n$ of infinite subsets of M and a sequence $(A_n)_n$ of finite collections of infinite special functionals such that for every infinite functional $x^* \notin A_k$ we have $\limsup_{n \in L_k} |x^*(x_n)| \leq 1/k$. Thus, we can choose a diagonal set L_∞ satisfying $\limsup_{n \in L_\infty} |x^*(x_n)| = 0$ for every infinite special functional x^* with $x^* \notin \bigcup_n A_n$. Now since $\bigcup_n A_n$ is countable, using a further diagonal procedure we arrive at an infinite $L \subset L_\infty$ such that $(x^*(x_n))_{n \in L}$ is convergent for every special functional x^*. ∎

REMARK 5.2. It can be readily seen that if $(x_n)_n$ is $(3, \epsilon)$ RIS and $w_n = x_{2n-1} - x_{2n}$ for $n \in \mathbb{N}$, then we can choose $L \in [\mathbb{N}]$ such that $(w_n)_{n \in L}$ is a $(6, \epsilon)$ RIS. Using this fact in conjunction with the previous lemma we can assume that every $(x_n)_n$ which is a $(6, \epsilon)$ RIS has the additional property that

$$\lim_n x^*(x_n) = 0 \quad \text{for every special functional } x^*.$$

We will always assume this property, unless stated otherwise.

LEMMA 5.2. *Let $j_0 \in \mathbb{N}$ and $(x_n)_n$ be a $(6, \epsilon)$ RIS with $0 < \epsilon < 5/m_{j_0}$. Assume that the associated sequence $(j_n)_n$ satisfies $j_1 > j_0$. Then for every $f \in G_{gt}$ of type I with $w(f) \geq m_{j_0}$ we have $|\{k \in \mathbb{N} : |f(x_k)| \geq 5/m_{j_0}\}| \leq 1$. Moreover, if the above set is nonempty then the element it contains depends only on the weight of f.*

Proof. Let $f \in G_{gt}$ with $w(f) = m_i$ and $i \geq j_0$. Let $E_1 = \{n \in \mathbb{N} : j_n \leq i\}$ and $E_2 = \{n \in \mathbb{N} : j_n > i\}$. Set $m = \max E_1$ and $M = \min E_2$. Then $|f(x_n)| < \epsilon$ for every $n < m$ by the definition of RIS. Simultaneously, $|f(x_n)| \leq 6/2m_i \leq 3/m_{j_0}$ for every $n \geq M$. Thus, $\{k \in \mathbb{N} : |f(x_k)| \geq 5/m_{j_0}\} \subseteq \{m\}$. It is also clear that m depends only on $w(f)$. ∎

LEMMA 5.3. *Let $j_0 \in \mathbb{N}$ and $(w_n)_n$ be a $(6, \epsilon)$ RIS with $0 < \epsilon < 5/m_{j_0}$. Assume that the associated sequence $(j_n)_n$ satisfies $j_1 > j_0$. Then there exists an $L \in [\mathbb{N}]$ such that for every infinite special functional x^* with $\mathrm{ind}\, x^* \geq j_0$, we have $|\{n \in L : |x^*(w_n)| \geq 10/m_{j_0}\}| \leq 1$.*

Proof. Suppose not. Then for every $L \in [\mathbb{N}]$ there exist $(l_1, l_2) \in [L]^2$ and an infinite special functional $x^*_{(l_1, l_2)}$ such that $\mathrm{ind}\, x^*_{(l_i, l_2)} \geq j_0$ and $|x^*_{(l_1, l_2)}(w_{l_i})| \geq 10/m_{j_0}$ for $i = 1, 2$. By Ramsey's theorem there exists $L \in [\mathbb{N}]$ such that for every $(l_1, l_2) \in [L]^2$ there exists an infinite special functional $x^*_{l_1, l_2}$ with $\mathrm{ind}\, x^*_{(l_i, l_2)} \geq j_0$ such that $|x^*_{l_1, l_2}(w_{l_i})| \geq 10/m_{j_0}$ for $i = 1, 2$. Hence by passing to a subsequence we may assume that for any $n < k \in \mathbb{N}$ there exists an infinite special functional $x^*_{n,k}$ with $\mathrm{ind}\, x^*_{n,k} \geq j_0$ such that $|x^*_{n,k}(w_i)| \geq 10/m_{j_0}, i = n, k$. Let

$$x^*_{n,k} = E_{n,k} \sum_{i=1}^{\infty} f^i_{n,k}$$

where $(f^i_{n,k})_i$ is a special sequence. For every $n < k$ we set

$$o_{n,k} = \min\{i : \max \mathrm{supp}(E_{n,k} f^i_{n,k}) \geq \min \mathrm{supp}\, w_k\}.$$

Now if $(E_{n,k} f^{o_{n,k}}_{n,k})(w_n) > 5/m_{j_0}$ we set $s^*_{n,k} = \sum_{i=1}^{o_{n,k}} f^i_{n,k}$, and otherwise we set $s^*_{n,k} = \sum_{i=1}^{o_{n,k}-1} f^i_{n,k}$. We need the following

CLAIM. *There exists $d \in \mathbb{N}$ such that $|D_k| = |\{s^*_{n,k} : n < k\}| \leq d$ for every $k \geq 2$.*

Proof of Claim. Let $k \in \mathbb{N}$. Let $s^*_{n_j,k}$, $j = 1, \ldots, d$, be the distinct elements of D_k. We consider the following special functionals:

$$z^*_{n_j} = x^*_{n_j,k} - E_{n_j,k} s^*_{n_j,k}, \quad j = 1, \ldots, d.$$

We can observe that $\mathrm{ind}\, z^*_{n_j} \geq j_0$ for all $j = 1, \ldots, d$, and as $\{s^*_{n_j,k}\}_{j=1}^{d}$ are pairwise different, $\{z^*_{n_j}\}_{j=1}^{d}$ are incomparable.

Now for each $j = 1, \ldots, d$ we have $|z^*_{n_j}(w_k)| \geq 5/m_{j_0}$. Indeed, if $s^*_{n_j,k} = \sum_{i=1}^{o_{n_j,k}} f^i_{n_j,k}$ then by Lemma 5.2,

$$|z^*_{n_j}(w_k)| \geq |x^*_{n_j,k}(w_k)| - |E_{n_j,k} f^{o_{n_j,k}}_{n_j,k}(w_k)| \geq \frac{10}{m_{j_0}} - \frac{5}{m_{j_0}} = \frac{5}{m_{j_0}}.$$

If $s^*_{n_j,k} = \sum_{i=1}^{o_{n_j,k}-1} f^i_{n_j,k}$ we have

$$|z^*_{n_j}(w_k)| = |x^*_{n_j,k}(w_k)| \geq \frac{10}{m_{j_0}}.$$

Thus, if we set

$$y^* = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \text{sgn}(z_{n_j}^*(w_k)) \cdot z_{n_j}^*$$

we obtain $y^*(w_k) \geq \sqrt{d} \cdot 5/m_{j_0}$ and thus $d \leq (6/5)^2 m_{j_0}^2$, which completes the proof of the claim.

Now we can see that for every $k \geq 2$ and $n < k$ we have $|s_{n,k}^*(w_n)| \geq 5/m_{j_0}$. Thus, for every $k \geq 2$ there exists a family $\{s_{r,k}^* : r = 1, \ldots, d\}$ of special functionals such that for all $n = 1, \ldots, k-1$ there exists $r \in \{1, \ldots, d\}$ such that $|s_{r,k}^*(w_n)| \geq 5/m_{j_0}$. By passing to subsequences we may assume that $s_{r,k}^* \overset{w^*}{\to} x_r^* \in \overline{\mathcal{S}}^{w^*}$ for all $r = 1, \ldots, d$. Now for $n \in \mathbb{N}$ and $r \in \{1, \ldots, d\}$ we say that k is *r-large for* n if $k > n$ and $|s_{r,k}^*(w_n)| \geq 5/m_{j_0}$. We know that for every $n \in \mathbb{N}$ there exists $r \in \{1, \ldots, d\}$ such that the set

$$LR_n^r = \{k : k \text{ is } r\text{-large for } n\}$$

is infinite. Hence, there exist $r_0 \in \{1, \ldots, d\}$ and $M \in [\mathbb{N}]$ with $LR_m^{r_0}$ infinite for all $m \in M$. Thus, since $s_{r_0,k}^* \to x_{r_0}^*$ and $|s_{r_0,k}^*(w_m)| \geq 5/m_{j_0}$ for infinitely many k and $m \in M$, it follows that $|x_{r_0}^*(w_m)| \geq 5/m_{j_0}$ for every $m \in M$. To complete the proof we need only show that $x_{r_0}^*$ cannot be of the form $x_{r_0}^* = \sum_{i=1}^{d-1} f_i + f_\infty$ where f_∞ is an infinite functional with weight. Indeed, suppose that $x_{r_0}^*$ is of that from. If we set $s_{r_0,k}^* = \sum_{i=1}^{d_k} f_i^k$ we can assume that there exists $l \in \mathbb{N}$ such that:

- $m_l \in \text{ind } s_{r_0,k}^*$ for all k.
- If, for every k, f_k is the unique element of $\{f_i^k : i = 1, \ldots, d_k\}$ with $w(f_k) = m_l$, then $f_k \to f_\infty$.

Then since $(w_m)_{m \in M}$ is a RIS there exists $m_0 \in M$ such that $|f_k(w_m)| \leq 6/2m_l \leq 3/m_{j_0} < 5/m_{j_0}$ for all $m \geq m_0$. Thus $\lim_k |f_k(w_m)| = |f_\infty(w_m)| < 5/m_{j_0}$ for all $m \geq m_0$, a contradiction.

Now, by Fact 5.1, x^* is necessarily an infinite special functional such that $|x^*(w_m)| \geq 5/m_{j_0}$, which contradicts the assumption that $x^*(w_k) \to 0$ (see Remark 5.2), and the proof is complete. ∎

REMARK 5.3. Let $(w_n)_n$ be a block sequence such that for every infinite special functional x^* we have $|\{n \in \mathbb{N} : |x^*(w_n)| \geq 10/m_{j_0}\} \leq 1$. Then for every finite special functional f we have $|\{n \in \mathbb{N} : |f(w_n)| \geq 10/m_{j_0}\}| \leq 2$.

REMARK 5.4. We point out that for every $j_0 \in \mathbb{N}$ and $(x_k)_k$ a $(6, \epsilon)$ RIS with $0 < \epsilon < 5/m_{j_0}$ we can have, by passing to a subsequence, the additional property that:

- For every $f \in G_{gt}$ of type I with $w(f) > m_{j_0}$ the set $\{k \in \mathbb{N} : |f(x_k)| \geq 5/m_{j_0}^2\}$ contains at most one element.

- For every special functional x^* with $\operatorname{ind} x^* > j_0$ the set $\{k \in \mathbb{N} : |x^*(x_k)| \geq 10/m_{j_0}^2\}$ contains at most two elements.

This can be shown by applying the same techniques as in Lemmas 5.2 and 5.3.

In order to control the action of type II functionals on the elements of a $(6, \epsilon)$ RIS as above, we need the following auxiliary lemma:

LEMMA 5.4. *Let* $x \in c_{00}(\mathbb{N})$ *and* $\epsilon > 0$. *There exists* $n \in \mathbb{N}$ *such that* $|y^*(x)| < \epsilon$ *for every* $y^* = \sum_{k=1}^d \alpha_k y_k^* \in G_{gt}$ *of type II with* $\max\{|\alpha_k| : k = 1, \ldots, d\} < 1/n$.

Proof. Let $\delta = \epsilon/\|x\|_1$, where $\|x\|_1 = \sum_{n \in \operatorname{supp} x} |x(n)|$. Clearly one can choose $m_0 \in \mathbb{N}$ such that $\sum_{j=m_0+1}^\infty 1/m_j < \delta$. We pick $n \in \mathbb{N}$ such that $1/n < \epsilon/(2m_0 \|x\|_{gt})$. Let $y^* = \sum_{k=1}^d \alpha_k y_k^* \in G_{gt}$ of type II with $\max\{|\alpha_k| : k = 1, \ldots, d\} < 1/n$. For every $k = 1, \ldots, d$ we can decompose each y_k^* as $y_k^* = y_{k,1}^* + y_{k,2}^*$ where $\operatorname{ind} y_{k,1}^* \subset \{1, \ldots, m_0\}$ and $\operatorname{ind} y_{k,2}^* \subset \{m_0 + 1, \ldots\}$. Thus $y^* = \sum_{k=1}^d \alpha_k y_{k,1}^* + \sum_{k=1}^d \alpha_k y_{k,2}^*$. Observe that

$$(2) \qquad \Big| \sum_{k=1}^d \alpha_k y_{k,2}^*(x) \Big| \leq \|x\|_1 \sum_{j=m_0+1}^\infty \frac{1}{2m_j} \frac{1}{2} \leq \|x\|_1 \delta \leq \frac{\epsilon}{2}$$

and

$$(3) \qquad \Big| \sum_{k=1}^d \alpha_k y_{k,1}^*(x) \Big| \leq \|x\|_{gt} m_0 \frac{1}{n} < \frac{\epsilon}{2}.$$

Combining (2) and (3) we obtain $|y^*(x)| < \epsilon$. ∎

REMARK 5.5. Let $j_0 \in \mathbb{N}$ and $(w_n)_n$ be a $(6, \epsilon)$ RIS and suppose that $j_1 > j_0$. Let also $k \in \mathbb{N}$. If $y^* = \sum_{i=1}^d a_i y_i^*$ is a type II functional with $\operatorname{ind} y^* \geq j_0$ and we set $y_{<k}^* = y_{<j_k}^*$, then

$$|y_{<k}^*(w_k)| \leq \sum_{i=1}^d |\alpha_i| \, |y_{i,<k}^*(w_k)| \leq \sum_{i=1}^d |y_{i,<k}^*(w_k)| \leq \sum_{j_0 \leq i < j_k} \frac{6}{2m_i}$$
$$\leq \frac{3}{m_{j_0}} + \frac{1}{m_{j_0}} = \frac{4}{m_{j_0}}.$$

PROPOSITION 5.1. *Let* $j_0 \in \mathbb{N}$ *and* $(w_n)_n$ *be a block sequence of averages with increasing lengths. Then there exists an* $L \in [\mathbb{N}]$ *such that for every* $y^* \in G_{gt}$ *of type II with* $\operatorname{ind} y^* \geq j_0$ *we have*

$$|\{n \in L : |y^*(w_n)| \geq 5/m_{j_0}\}| \leq 1025 m_{j_0}^2.$$

Proof. We assume that $\lim_n x^*(w_n) = 0$ for every special functional x^*. For $\delta_1 = 1/4m_{j_0}$ there exists $j_1 \in \mathbb{N}$ with $j_1 > j_0$ such that $14/m_{j_1} < \delta_1$. For

$0 < \epsilon_1 < 5/m_{j_1}$ by Lemma 4.2 there exists $M_1 \in [\mathbb{N}]$ such that $(w_n)_{n \in M_1}$ is a $(6, \epsilon_1)$ RIS. By Lemma 5.3 and Remark 5.3 there is also an $L_1 \in [M_1]$ such that for every special x^* with $\operatorname{ind} x^* \geq j_1$ we have $|\{n \in L_1 : |x^*(w_n)| \geq 10/m_{j_1}\}| \leq 2$. Let $l_1 = \min L_1$. For δ_1 and w_{l_1}, by Lemma 5.4 we can find $r_1 \in \mathbb{N}$ such that for every $y^* = \sum_{i=1}^c \alpha_i y_i^* \in G_{gt}$ of type II with $\max\{|\alpha_i| : i = 1, \ldots, c\} < 1/r_1$ we have $|y^*(y_{l_1})| < \delta_1$. We can inductively construct a strictly decreasing sequence $(L_n)_{n \in \mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that if we set $l_n = \min L_n$ we have $L_{n+1} \subset L_n \backslash \{l_n\}$, and sequences of natural numbers $(j_n)_{n \in \mathbb{N}}$ with $j_n > j_0, n \in \mathbb{N}$ for $(r_n)_{n \in \mathbb{N}}$ and $(\delta_n)_{n \in \mathbb{N}}$ such that $(j_n)_{n \in \mathbb{N}}$, $(r_n)_{n \in \mathbb{N}}$ are strictly increasing and $\delta_{n+1} = 1/(4m_{j_0} 2^n r_n^2)$ for $n \in \mathbb{N}$ and the following hold:

- For $n \in \mathbb{N}$ and $y^* = \sum_{i=1}^c \alpha_i y_i^* \in G_{gt}$ of type II with $\max\{|\alpha_i| : i = 1, \ldots, c\} < 1/r_n$ we have $|y^*(w_{l_i})| < \delta_n$ for $i = 1, \ldots, n$.
- For $n \in \mathbb{N}$ and x^* special with $\operatorname{ind} x^* \geq j_n$ we obtain $|\{k \in L_n : |x^*(w_k)| \geq 10/m_{j_n}\}| \leq 2$.

Observe that $(w_{l_i})_{i \in \mathbb{N}}$ for $l_i \geq j_0$ is a $(6, \epsilon)$ RIS for $5/m_{j_1} < \epsilon < 5/m_{j_0}$. It can also be seen that for $n \in \mathbb{N}$ and x^* special with $\operatorname{ind} x^* \geq j_n$ we have $|\{k \in L_n : |x_{>k}^*(w_{l_k})| \geq 14/m_{j_n}\}| \leq 2$. Indeed, let $k \in \mathbb{N}$ and $|y_{>k}^*(w_{l_k})| \geq 14/m_{j_n}$. Then

$$|y^*(w_{l_k})| = |y_{>k}^*(w_{l_k}) + y_{\leq k}^*(w_{l_k})| \geq |y_{>k}^*(w_{l_k})| - |y_{\leq k}^*(w_{l_k})|$$
$$\geq \frac{14}{m_{j_n}} - \sum_{i \geq j_n} \frac{6}{m_i} \geq \frac{10}{m_{j_n}}.$$

Therefore, for all $n \in \mathbb{N}$ and $x^* \in S$ with $\operatorname{ind} y^* \geq j_n$,

$$|\{k \in L_n : |y_{>k}^*(w_{l_k})| \geq \delta_n\}| \leq 2.$$

Let $L = \{l_1 < l_2 < \cdots\}$ and $n \in \mathbb{N}$. Then for every y^* special with $\operatorname{ind} y^* \geq j_n$ we have the additional property that $|\{k \geq n : |y_{>k}^*(w_{l_k})| \geq \delta_n\}| \leq 2$. Let $d = 1025 m_{j_0}^2$. It suffices to show that if we choose $l_{p_1} < \cdots < l_{p_d}$, where $p_1 < \cdots < p_d$ are in $\mathbb{N}$, and a type II functional $y^* = \sum_{i=1}^c \alpha_i y_i^*$ with index greater than $j_0 - 1$, then there exists $k \in \{1, \ldots, d\}$ such that $|y^*(w_{l_{p_k}})| < 5/m_{j_0}$. We consider the following sets:

$$A_1 = \{i \in \{1, \ldots, c\} : |\alpha_i| \geq 1/r_{p_1}\},$$
$$A_k = \{i \in \{1, \ldots, c\} : 1/r_{p_k} \leq |\alpha_i| < 1/r_{p_{k-1}}\} \quad \text{for } 1 < k < d,$$
$$A_d = \{i \in \{1, \ldots, c\} : 1/r_{p_d} > |\alpha_i|\}.$$

We observe that $|A_k| \leq r_{p_k}^2$ for $1 \leq k < d$. For $k_0 \in \{1, \ldots, d\}$ we have

$$\Big| \sum_{i \in \bigcup_{j > k_0} A_j} \alpha_i y_i^*(w_{l_{p_{k_0}}}) \Big| \leq \delta_{p_{k_0}} \leq \delta_1 < \frac{1}{4m_{j_0}}$$

and

$$\Big| \sum_{i\in\bigcup_{j\le k_0}A_j} \alpha_i y_i^*(w_{l_{p_{k_0}}}) \Big| \le \Big| \sum_{i\in\bigcup_{j\le k_0}A_j} \alpha_i y_{i,\le p_{k_0}}^*(w_{l_{p_{k_0}}}) \Big|$$

$$+ \Big| \sum_{i\in\bigcup_{j\le k_0}A_j} \alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}}) \Big|$$

$$\le \frac{4}{m_{j_0}} + \Big| \sum_{i\in\bigcup_{j\le k_0}A_j} \alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}}) \Big|.$$

For $1 \le j < k_0 \le d$ we set $B_{j,k_0} = \{i \in A_j : |y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})| \ge \delta_{p_j+1}\}$ and $B_{k_0} = \bigcup_{j<k_0} B_{j,k_0}$. If $k_0 > j$ then $p_{k_0} \ge p_j+1$, and therefore for $i \in A_j$ there exists at most one $B_{j,k}$ that contains i. Consequently, each $i \in \{1, \ldots, c\}$ is contained in at most one B_k. Moreover,

$$\sum_{i\in\bigcup_{j<k_0}A_j\setminus B_{k_0}} |\alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})| = \sum_{j=1}^{k_0-1} \sum_{i\in A_j\setminus B_{j,k_0}} |\alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})|$$

$$\le \sum_{j=1}^{k_0-1} |A_j|\delta_{p_j+1}$$

$$\le \sum_{j=1}^{k_0-1} r_{p_j}^2 \frac{1}{4m_{j_0}2^{p_j}r_{p_j}^2} < \frac{1}{4m_{j_0}}.$$

At this point we need the following

CLAIM. *There exists $A \subset \{1, \ldots, d\}$ with $|A| \ge d/2 + 1$ such that*

$$\Big| \sum_{i\in A_k} \alpha_i y_{i,>p_k}^*(w_{l_{p_k}}) \Big| < \frac{1}{4m_{j_0}} \quad \text{for all } k \in A.$$

Proof of Claim. Suppose not. Then there exists $B \subset \{1, \ldots, d\}$ with $|B| \ge d/2$ such that $|\sum_{i\in A_k} \alpha_i y_{i,>p_k}^*(w_{l_{p_k}})| \ge 1/4m_{j_0}$ for every $k \in B$. Thus, $\sum_{i\in A_k} \alpha_i^2 \ge 1/256m_{j_0}^2$ for all $k \in B$ and

$$\sum_{i=1}^c \alpha_i^2 = \sum_{k=1}^d \sum_{i\in A_k} \alpha_i^2 \ge \frac{d}{2} \cdot \frac{1}{256m_{j_0}^2} = \frac{d}{512m_{j_0}^2} > 1,$$

a contradiction.

Furthermore, there must exist an $E \subset \{1, \ldots, d\}$ with $|E| \ge d/2+1$ such that $|\sum_{i\in B_k} \alpha_i y_{i,>p_k}^*(w_{l_{p_k}})| < 1/4m_{j_0}$ for all $k \in E$. Indeed, if this were not the case then there would exist an $F \subset \{1, \ldots, d\}$ with $|F| \ge d/2$ such that $|\sum_{i\in B_k} \alpha_i y_{i,>p_k}^*(w_{l_{p_k}})| \ge 1/4m_{j_0}$ for all $k \in F$. Therefore, $\sum_{i\in B_k} \alpha_i^2 \ge$

$1/256m_{j_0}^2$ for every $k \in F$. But this means that

$$2 \geq 2\sum_{i=1}^c \alpha_i^2 \geq \sum_{k=1}^d \sum_{i \in B_k} \alpha_i^2 \geq \frac{d}{2} \cdot \frac{1}{256m_{j_0}^2} = \frac{d}{512m_{j_0}^2},$$

a contradiction. Thus for an appropriate $k_0 \in A \cap E$ we have

$$\Big|\sum_{i=1}^c \alpha_i y_i^*(w_{l_{p_{k_0}}})\Big| \leq \Big|\sum_{i \in \bigcup_{j>k_0} A_j} \alpha_i y_i^*(w_{l_{p_{k_0}}})\Big| + \frac{4}{m_{j_0}}$$

$$+ \Big|\sum_{i \in \bigcup_{j<k_0} A_j \setminus B_{k_0}} \alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})\Big|$$

$$+ \Big|\sum_{i \in B_{k_0}} \alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})\Big|$$

$$+ \Big|\sum_{i \in A_{k_0}} \alpha_i y_{i,>p_{k_0}}^*(w_{l_{p_{k_0}}})\Big| < \frac{5}{m_{j_0}}. \quad\blacksquare$$

All the above yields the following

PROPOSITION 5.2. *Let $j_0 \in \mathbb{N}$ and $0 < \epsilon < 5/m_{j_0}$. In every block subspace Z of $\mathfrak{X}_{gt}$ there exists a $(6, \epsilon)$ RIS which is j_0-separated.*

Applying similar arguments to those in Lemma 5.3 and Proposition 5.1 the following can be readily established:

REMARK 5.6. Let $j_0 \in \mathbb{N}$ and consider a $(6, \epsilon)$ RIS $(w_n)_n$ with $\epsilon < 5/m_{j_0}$ and $j_1 > j_0$ and $\min \operatorname{supp} x_1 > m_{j_0}$ with the following property: There exists a finite set $B = \{x_1^*, \ldots, x_r^*\}$ of infinite special functionals such that $x^*(w_n) \to 0$ for every special $x^* \notin B$. Then there exists an $L \in [\mathbb{N}]$ such that

- For every special functional $x^* \notin B$ with $\operatorname{ind} x^* \geq j_0$ the set $\{k \in L : |x^*(w_k)| > 10/m_{j_0}\}$ contains at most two elements.
- For every $y^* = \sum_{i=1}^d a_i x_i^* \in G_{gt}$ of type II with $\operatorname{ind} y^* \geq j_0$ and $x_i^* \notin B$ for all $i = 1, \ldots, d$ we have $|\{k \in L : |y^*(w_n)| \geq 5/m_{j_0}\}| \leq 1025m_{j_0}^2$.

In the following section we make use of the following crucial observation.

REMARK 5.7. Let $j_0 \in \mathbb{N}$ and $(x_k)_k$ be a block sequence. We can assume that $\min \operatorname{supp} x_1 > m_{j_0}$. In this case, for every special functional x^* such that $x^* = x_{<j_0}^* + x_{\geq j_0}^*$, if we write $x_{<j_0}^* = \sum_{i=1}^d f_i$ then the set

$$\{i \in \{1, \ldots, d\} : \exists k \in \mathbb{N} \text{ such that } \operatorname{ran} f_i \cap \operatorname{ran} x_k \neq \emptyset\}$$

contains at most one element. Indeed, suppose that there exists at least one such i and set $i_0 = \min\{i : \exists k \text{ with } \operatorname{ran} f_i \cap \operatorname{ran} x_k \neq \emptyset\}$. Then $\max \operatorname{supp} f_{i_0} > m_{j_0}$ and by the definition of the coding σ we have $\sigma(f_1, \ldots, f_{i_0}) > j_0$ and hence $i_0 = d$.

6. The Basic Inequality. At this point we need to provide precise estimates for the norms of n_{j_0}-averages of the vectors in a j_0-separated $(6, \epsilon)$ RIS. As is common in the relevant literature, we do this after reducing estimates to the norms of corresponding averages of the basis of an auxiliary space $\mathrm{T}_{gt}^{j_0}$. This is done mainly in two steps. First, we make use of a Basic Inequality (Proposition 6.1), and then we enlarge the norming set of $\mathrm{T}_{gt}^{j_0}$ and provide exact estimates for the norms of n_{j_0}-averages of the basis of $\mathrm{T}_{gt}^{j_0}$. This along with the results in the previous section, which imply that for every $j_0 \in \mathbb{N}$ and every block subspace Z of $\mathfrak{X}_{gt}$ one can find $(x_k)_k$ in Z that is a j_0-separated $(6, \epsilon)$ RIS with $\epsilon < 5/m_{j_0}$ and $\min \operatorname{supp} x_1 > m_{j_0}$, leads to the existence of exact pairs and dependent sequences in every block subspace. We start with the definition of the norming set of $\mathrm{T}_{gt}^{j_0}$.

6.1. *The auxiliary space.* We note that in what follows we make use of the terminology developed above considering weights and types of functionals, as in all the following cases their meaning is quite analogous to the ones considered so far.

DEFINITION 6.1. Let $j_0 \in \mathbb{N}$. We define $W_{gt}^{j_0}$ to be the minimal subset of $c_{00}(\mathbb{N})$ with the following properties:

- $\{\pm e_n^* : n \in \mathbb{N}\} \subset W_{gt}^{j_0}$.
- $W_{gt}^{j_0}$ is closed under the $(\mathcal{A}_{2n_j}, 1/m_j)$-operation for all $j \in \mathbb{N}$.
- $W_{gt}^{j_0}$ is closed under the $(\mathcal{A}_{m_{j_0}^3}, 1/2)$ operation.
- For $d < j_0$ and $f_1, \ldots, f_d$ in $W_{gt}^{j_0}$ such that each f_i is of type I and $w(f_i) < m_{j_0}$ for all i, $w(f_i) \neq w(f_j)$ for $i \neq j$, and for $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}$ with $\sum_{i=1}^{d} \alpha_i^2 \leq 1$, we have $\sum_{i=1}^{d} \alpha_i f_i \in W_{gt}^{j_0}$. We call this last sum a *result of the (j_0, ℓ_2) operation.*
- $W_{gt}^{j_0}$ is rationally convex.

The set $W_{gt}^{j_0}$ induces the following norm on $c_{00}(\mathbb{N})$:

$$\|x\|_{W_{gt}^{j_0}} = \sup\{|f(x)| : f \in W_{gt}^{j_0}\} \quad \text{for } x \in c_{00}(\mathbb{N}).$$

The completion of $(c_{00}(\mathbb{N}), \| \cdot \|_{W_{gt}^{j_0}})$ is denoted by $\mathrm{T}_{gt}^{j_0}$. The next step is to estimate the norms of n_{j_0}-averages of the basis of $\mathrm{T}_{gt}^{j_0}$. However, in this case the presence of ℓ_2 convex combinations in the tree analysis of a functional $f \in W_{gt}^{j_0}$ impedes the direct use of standard techniques developed in the past (see for example [AT, Remark 3.18]). In order to achieve the desired estimates we need to enlarge $W_{gt}^{j_0}$ to a set G^{j_0} defined below that contains only type I functionals and their convex combinations. This enlargement, however, results in slightly worse estimates compared to the ones obtained in [AT, Remark 3.18].

We start with the definition of the larger norming set:

DEFINITION 6.2. We define G^{j_0} to be the minimal subset of $c_{00}(\mathbb{N})$ with the following properties:

(1) $\{\pm e_k : k \in \mathbb{N}\} \subset G^{j_0}$.
(2) G^{j_0} is closed under the $(\mathcal{A}_{2n_j}, 1/\sqrt{m_j})$ operations for $j < j_0$ and under the $(\mathcal{A}_{2n_j}, 1/m_j)$ operations for $j \geq j_0$.
(3) G^{j_0} is closed under the $(\mathcal{A}_{m_{j_0}^3}, 1/2)$ operation.
(4) G^{j_0} is rationally convex.

There is an alternative way to define the sets $W_{gt}^{j_0}$ and G^{j_0} using a recursive construction. Namely, we set $A_0 = B_0 = \{\pm e_k : k \in \mathbb{N}\}$ and $W_0 = G_0 = \text{conv}_{\mathbb{Q}}(A_0)$. Let $n \in \mathbb{N}$ and suppose that A_n, W_n, B_n, G_n have been defined for $k \leq n$. We then define A_{n+1} to be the union of A_n and the set of $f \in c_{00}(\mathbb{N})$ of one of the following forms:

$$f = \frac{1}{m_j} \sum_{i=1}^{d} f_i, \quad d \leq 2n_j, \ (f_i)_i \text{ is a block sequence, } f_i \in W_n,$$

$$f = \frac{1}{2} \sum_{i=1}^{d} f_i, \quad d \leq m_{j_0}^3, \ (\text{supp } f_i)_i \text{ are mutually disjoint,}$$

$$f = \sum_{i=1}^{d} \alpha_i f_i, \quad \alpha_i \in \mathbb{Q}, \ \sum_{i=1}^{d} \alpha_i^2 \leq 1, \ d \leq j_0, \ (f_i)_i \text{ are type I with}$$
$$w(f_i) \leq m_{j_0} \text{ and } f_i \in W_n,$$

and we set $W_{n+1} = \text{conv}_{\mathbb{Q}}(A_{n+1})$. Analogously we define B_{n+1} to be the union of B_n and the set of $f \in c_{00}(\mathbb{N})$ of one of the following forms:

$$f = \frac{1}{\sqrt{m_j}} \sum_{i=1}^{d} f_i, \quad d \leq 2n_j, \ (f_i)_i \text{ is a block sequence } f_i \in W_n, \ j < j_0.$$

$$f = \frac{1}{m_j} \sum_{i=1}^{d} f_i, \quad d \leq 2n_j, \ (f_i)_i \text{ is a block sequence } f_i \in W_n, \ j \geq j_0.$$

$$f = \frac{1}{2} \sum_{i=1}^{d} f_i, \quad d \leq m_{j_0}^3, \ (\text{supp } f_i)_i \text{ are mutually disjoint,}$$

and we set $G_{n+1} = \text{conv}_{\mathbb{Q}}(B_{n+1})$. We can see that $W_{gt}^{j_0} = \bigcup_n W_n$ and $G^{j_0} = \bigcup_n G_n$. The following lemma establishes the connection between the sets $W_{gt}^{j_0}$ and G^{j_0}.

LEMMA 6.1. *The set $W_{gt}^{j_0}$ is a subset of G^{j_0}.*

Proof. We use induction to prove that $W_n \subseteq G_n$ for every $n \in \mathbb{N}$. For $n = 0$ this is obvious. Let $n \in \mathbb{N}$ and suppose that $W_n \subseteq G_n$. To prove that $W_{n+1} \subseteq G_{n+1}$ it is enough to show that $A_{n+1} \subseteq G_{n+1}$. Let $f \in A_{n+1}$. If $f \in A_n$ then clearly $f \in G_{n+1}$ by the inductive hypothesis. If not, we distinguish the following cases:

- $f = \frac{1}{m_j} \sum_{i=1}^d f_i$, $(f_i)_{i=1}^d$ is a block sequence, $f_i \in W_n$ and $d \leq 2n_j$. If $m_j \geq m_{j_0}$ then since $f_i \in W_n$ we have $f \in B_{n+1} \subseteq G_{n+1}$. If $m_j < m_{j_0}$ then f can be written as $f = \frac{1}{\sqrt{m_j}} \big(\frac{1}{\sqrt{m_j}} \sum_{i=1}^d f_i \big)$. Now since $\big(\frac{1}{\sqrt{m_j}} \sum_{i=1}^d f_i \big) \in B_{n+1}$ we have $f \in G_{n+1}$.

- $f = \frac{1}{2} \sum_{i=1}^d f_i$, $f_i \in W_n$, $(\operatorname{supp} f_i)_i$ are mutually disjoint and $d \leq m_{j_0}^3$. Then since $f_i \in W_n \subseteq G_n$ for all $i = 1, \dots, d$, we have $f \in B_{n+1} \subseteq G_{n+1}$.

- $f = \sum_{i=1}^d a_i f_i$, $d \leq j_0$, $\sum_{i=1}^d a_i^2 \leq 1$, $w(f_i) < m_{j_0}$, $w(f_i) \neq w(f_j)$ for $i \neq j$ and $f_i \in W_n$ for all i. Then we can rewrite f as $f = \sum_{i=1}^d \frac{|a_i|}{\sqrt{m_i}} f_i'$ where $f_i' = \frac{1}{\sqrt{m_i}} \sum_{j=1}^{2n_i} \operatorname{sgn}(a_i) f_i^j$ for all i and $f_i^j \in W_n$ for all i, j. Thus $f_i' \in B_{n+1}$ for all i as all the sets we consider are symmetric. Now since the functionals $(f_i)_i$ have pairwise different weights we obtain $\sum_{i=1}^d |a_i|/\sqrt{m_i} \leq 1$ and thus $f \in G_n$.

The induction is complete. ∎

We define the tree analysis for a functional $f \in G^{j_0}$ as follows:

DEFINITION 6.3. Let $f \in G^{j_0}$, $f \neq 0$. A family $(f_a)_{a \in A}$ with $f_a \in G^{j_0}$ for all $a \in A$ is called a *tree analysis* of f if:

- A is a finite tree with a least element denoted by 0 and $f_0 = f$.
- For $a, b \in A$ with $a \sqsubset b$ we have $\operatorname{ran} f_b \subset \operatorname{ran} f_a$.
- For $a \in A$ maximal we have $f_a \in \{\pm e_n : n \in \mathbb{N}\}$.
- For $a \in A$ not maximal, if we denote by S_a the immediate successors of a in A then f_a has one of the following forms:

$$f_a = \frac{1}{\sqrt{m_{j_a}}} \sum_{s \in S_a} f_s, \quad |S_a| \leq 2n_{j_a}, \ (f_s)_{s \in S_a} \text{ block}, \ m_{j_a} < m_{j_0},$$

$$f_a = \frac{1}{m_{j_a}} \sum_{s \in S_a} f_s, \quad |S_a| \leq 2n_{j_a}, \ (f_s)_{s \in S_a} \text{ block}, \ m_{j_a} \geq m_{j_0},$$

$$f_a = \frac{1}{2} \sum_{s \in S_a} f_s \quad |S_a| \leq m_{j_0}^3, \ (\operatorname{supp} f_s)_{s \in S_a} \text{ mutually disjoint},$$

$$f_a = \sum_{s \in S_a} q_s f_s, \quad \sum_{s \in S_a} q_s = 1, \ q_s \in \mathbb{Q}^+.$$

Before providing estimates for averages of the basis of $\mathrm{T}_{gt}^{j_0}$ we need the following fact that will allow us to consider only functionals in G^{j_0} such that convex combinations do not appear in their tree analysis.

FACT 6.1. *Let $G_1^{j_0}$ be the minimal subset of $c_{00}(\mathbb{N})$ that satisfies* (1)–(3) *of Definition* 6.2. *Then every $f \in G^{j_0}$ with weight $w(f)$ can be written as $\sum_{i=1}^{d} \lambda_i f_i$ with $(f_i)_{i=1}^{d} \subset G_1^{j_0}$, $w(f_i) = w(f)$ for all i and $\sum_{i=1}^{d} \lambda_i = 1$ with $\lambda_i \in \mathbb{Q}^{+}$ for all i.*

The proof of this fact uses standard arguments similar to the ones in Lemma 3.15 in [AT] and so we omit it.

LEMMA 6.2. *Let $j_0 \in \mathbb{N}$ with $j_0 \geq 2$. Let also $g \in G_1^{j_0}$ and $k_1 < \cdots < k_{n_{j_0}}$ be a sequence of natural numbers. Then*

$$\left| g\left(\frac{1}{n_{j_0}} \sum_{r=1}^{n_{j_0}} e_{k_r} \right) \right| \leq \begin{cases} 2/\sqrt{m_i}\, m_{j_0} & \text{if } w(g) = m_i,\ i < j_0, \\ 1/m_i & \text{if } w(g) = m_i,\ i \geq j_0. \end{cases}$$

Here we make the convention $w(g) = 1/2$ if g is of the form $f = \frac{1}{2} \sum_{i=1}^{d} g_i$ where $(g_i)_i$ have disjoint supports and $d \leq m_{j_0}^3$.

For the proof we refer to Lemma 3.16 and Proposition 3.19 in [AT]. We can readily see that by Fact 6.1 we obtain exactly the same estimates for functionals in G^{j_0}.

We use the following piece of notation:

DEFINITION 6.4. Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_{gt}$, $j_0 \in \mathbb{N}$ and $f \in G_{gt}$ with a j_0-tree analysis $(f_a)_{a \in A}$. For each $k \in \mathbb{N}$ we denote by A_k the set of all $a \in A$ such that:

- $\operatorname{ran} f_a \cap \operatorname{ran} x_k \neq \emptyset$.
- For every $b \sqsubseteq a$ with $b \in S_u$ such that $f_u \in \mathcal{S}$ or f_u is of type I we have $\operatorname{ran} f_u \cap \operatorname{ran} x_k = \operatorname{ran} f_b \cap \operatorname{ran} x_k$.
- There exists no $b \sqsubset a$ such that $b \in S_u$, f_u is of type II and $f_b = f_{u, \geq j_0}$.
- Either f_a is of type 0, type I or special and $\operatorname{ran} f_b \cap \operatorname{ran} x_k \neq \operatorname{ran} f_a \cap \operatorname{ran} x_k$ for every $b \in S_a$, or $f_a = f_{u, \geq j_0}$ and $a \in S_u$ and f_u is of type II.

DEFINITION 6.5. Let $(x_k)_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{gt}$, $j_0 \in \mathbb{N}$, $f \in G_{gt}$ and $(f_a)_{a \in A}$ a j_0-tree analysis of f. Let $a \in A$. We set $D_a = \bigcup_{b \sqsupseteq a} \{k : b \in A_k\}$ and $E_a = \{k : a \in A_k\}$.

REMARK 6.1. Let $f \in G_{gt}$ and $(f_a)_{a \in A}$ be a j_0-tree analysis of f. Let also $(x_k)_{k \in \mathbb{N}}$ be a block sequence and $k \in \mathbb{N}$. Then we can establish the following properties of A_k:

- If $a_1, a_2 \in A_k$ then the nodes a_1, a_2 are incomparable.
- For $a \in A$ not maximal, if $(f_s)_{s \in S_a}$ is a block sequence then so is $(D_s)_{s \in S_a}$.

- Let $a \in A$ be such that f_a is of type II and $f_a = f_{s_1} + f_{s_2}$, where $f_{s_1} = f_{a,<j_0}$ and $f_{s_2} = f_{a,\geq j_0}$. Then for $i,j = 1,2$, if there is $k \in D_{s_i}$ such that $\operatorname{ran} x_k \cap \operatorname{ran} f_{s_j} \neq \emptyset$ then $k \in D_{s_j}$.
- For every k such that $\operatorname{ran} x_k \cap \operatorname{supp} f \neq \emptyset$ there exists $a \in A$ such that $k \in D_a$. In particular, $D_0 = \{k \in \mathbb{N} : \operatorname{supp} f \cap \operatorname{ran} x_k \neq \emptyset\}$.

For the proof we refer to Lemma 4.6 in [AT].

PROPOSITION 6.1 (Basic Inequality). *Let* $j_0 \in \mathbb{N}$, $j_0 \geq 3$ *and* $(x_k)_k$ *a* j_0-*separated* (C, ϵ) *RIS with* $\min \operatorname{supp} x_1 > m_{j_0}$, $0 < \epsilon < 5/m_{j_0}$ *and* $C \geq 1$. *Let also* $(\lambda_k)_k$ *be an arbitrary finite sequence of scalars. Then for every* $f \in G_{gt}$ *of type I such that* $w(f) < m_{j_0}$ *there exist* $g_1, g_2, g_3 \in c_{00}(\mathbb{N})$ *with nonnegative coordinates satisfying*

- $g_j \in G^{j_0}$ *and* $w(g_j) = w(f)$ *for* $j = 1,2$,
- $\|g_3\|_\infty \leq \frac{1}{w(f)} \cdot \frac{10}{m_{j_0}}$,

such that

$$\left| f\left(\sum_k \lambda_k x_k \right) \right| \leq 4C \left(\frac{1}{2} g_1 + \frac{1}{2} g_2 + g_3 \right) \left(\sum_k |\lambda_k| e_k \right).$$

Proof. Let $f \in G_{gt}$ be of type I with $w(f) < m_{j_0}$ and $(f_a)_{a \in A}$ a j_0-tree analysis of f. We will recursively construct for each $a \in A$ functionals $g_1^a, g_2^a, g_3^a \in c_{00}(\mathbb{N})$ such that

- $\operatorname{supp} g_i^a \subseteq D_a$ for $i = 1,2,3$ and $g_i^a \in G^{j_0}$.
- $\|g_3^a\|_\infty \leq 10/m_{j_0}$ and if f_a is of type I with $w(f_a) = m_{j_a} < m_{j_0}$ then $\|g_3^a\|_\infty \leq \frac{1}{m_{j_a}} \cdot \frac{5}{m_{j_0}}$ and $w(f_a) = w(g_1^a) = w(g_2^a)$.
- $\left| f_a\left(\sum_{k \in D_a} \lambda_k x_k \right) \right| \leq 4C(g_1^a + g_2^a + g_3^a)\left(\sum_{k \in D_a} |\lambda_k| e_k \right).$

The proof is by induction. Let $a \in A$ be maximal. Then if $D_a = \emptyset$ we set $g_1^a = g_2^a = g_3^a = 0$. If $D_a \neq \emptyset$ we can see that D_a is a singleton, say $D_a = \{k_a\}$. We set $g_1^a = e_{k_a}^*$, $g_2^a = 0$, $g_3^a = 0$ and the inequality is easily verified.

Let $a \in A$ be not maximal and suppose g_1^b, g_2^b, g_3^b have been defined for every $b \in A$ with $b \sqsupset a$ according to our inductive hypotheses. We distinguish the following cases:

CASE 1: f_a is of type I with $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$ and $j_a \geq j_0$. By Lemma 5.2 there exists at most one $k_a \in D_a$ such that $|f_a(x_{k_a})| > 5/m_{j_0}$. Suppose without loss of generality that such a k_a exists. We set $g_1^a = \frac{1}{2} e_{k_a}^*$, $g_2^a = 0$ and $g_3^a = \frac{5}{m_{j_0}} \sum_{k \in D_a \setminus \{k_a\}} e_k^*$. Then

$$\left| f_a\Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq |f_a(\lambda_{k_a} x_{k_a})| + \left| f_a\Big(\sum_{k \in D_a \setminus \{k_a\}} \lambda_k x_k \Big) \right|$$

$$\leq 4C(g_1^a + g_2^a + g_3^a)\Big(\sum_{k \in D_a} |\lambda_k| e_k \Big).$$

CASE 2: f_a is of type I with $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$ and $j_a < j_0$. We enumerate S_a as $\{s_1^a < \cdots < s_r^a\}$; we know that $|S_a| \leq n_{j_a}$. We can see that $D_a = E_a \cup \bigcup_{s \in S_a} D_s$ and $|E_a| \leq n_{j_a}$. By Remark 6.1 we find that $(D_{s_i^a})_{i=1}^r$ are successive subsets of $\mathbb{N}$, and thus $(g_1^{s_i})_{i=1}^r$, $(g_2^{s_i})_{i=1}^r$ and $(g_3^{s_i})_{i=1}^r$ are block sequences. Now since $(x_k)_k$ is a j_0-separated (C, ϵ) RIS we have $|f_a(x_k)| \leq C/2m_{j_a}$ for all k. Hence,

$$\left| f_a\Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq \left| f_a\Big(\sum_{k \in E_a} \lambda_k x_k \Big) \right| + \left| \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s\Big(\sum_{k \in \bigcup_{s \in S_a} D_s} \lambda_k x_k \Big) \right|.$$

But

$$\left| f_a\Big(\sum_{k \in E_a} \lambda_k x_k \Big) \right| \leq \frac{C}{2m_{j_a}} \sum_{k \in E_a} |\lambda_k|$$

and

$$\left| \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s\Big(\sum_{k \in \bigcup_{s \in S_a} D_s} \lambda_k x_k \Big) \right| = \left| \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s\Big(\sum_{k \in D_s} \lambda_k x_k \Big) \right|.$$

Thus

$$\left| f_a\Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right|$$

$$\leq 4C\Big(\frac{1}{2\sqrt{m_{j_a}}} \sum_{s \in S_a} g_1^s + \frac{1}{2\sqrt{m_{j_a}}} \sum_{s \in S_a} g_2^s + \frac{1}{2m_{j_a}} \sum_{s \in S_a} g_3^s \Big)\Big(\sum_{k \in \bigcup_{s \in S_a} D_s} |\lambda_k| e_k \Big)$$

$$+ \frac{C}{2\sqrt{m_{j_a}}} \sum_{k \in E_a} |\lambda_k|.$$

We set

$$g_1^a = \frac{1}{\sqrt{m_{j_a}}}\Big(\sum_{k \in E_a} e_k^* + \sum_{s \in S_a} g_1^s \Big), \quad g_2^a = \frac{1}{\sqrt{m_{j_a}}} \sum_{s \in S_a} g_2^s, \quad g_3^a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} g_3^s.$$

In what follows we actually use the following stronger inequality:

$$(4) \qquad \left| f_a\Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq 4C\Big(\frac{1}{2} g_1^a + \frac{1}{2} g_2^a + g_3^a \Big)\Big(\sum_{k \in D_a} |\lambda_k| e_k \Big).$$

In addition we observe that $w(g_1^a) = w(g_2^a) = w(f_a)$ and $g_1^a, g_2^a \in G^{j_0}$. The latter holds as $|E_a| + |S_a| \leq 2n_{j_a}$ and by Remark 6.1 the family $\{e_k^* : k \in E_a\} \cup \{g_1^s : s \in S_a\}$ consists of successive functionals. Finally, as $\|g_3^s\|_\infty \leq 10/m_{j_0}$

for all $s \in S_a$ and $(\text{supp } g_2^s)_{s \in S_a}$ are successive we have the crucial property

$$\|g_3^a\|_\infty \leq \frac{1}{2} \frac{1}{m_{j_a}} \frac{10}{m_{j_0}}.$$

CASE 3: f_a is a type III functional, i.e. $f_a = \sum_{s \in S_a} r_s f_s$ with $r_s \in \mathbb{Q}^+$ and $\sum_{s \in S_a} r_s = 1$. In this case we set $g_i^a = \sum_{s \in S_a} r_s g_i^s$ for $i = 1, 2, 3$ and all the desired properties can be readily verified.

CASE 4: f_a is a special functional. Then $f_a = \sum_{s \in S_a} f_s$ where each f_s is as in Definition 3.7. We set $S_a^1 = \{s \in S_a : \text{ind } f_s < j_0\}$ and $S_a^2 = S_a \setminus S_a^1$. Observe that $|S_a^1| \leq j_0$. Let $k_a \in \mathbb{N}$ be such that there exist $s_1 \in S_a^1$ and $s_2 \in S_a^2$ satisfying $\text{ran } x_{k_a} \cap \text{ran } f_{s_i} \neq \emptyset$ for $i = 1, 2$. We can assume that such a k_a exists. We define

$$D_{\leq j_0} = \{k \in D_a : x_k < x_{k_a}\}, \quad D_{>j_0} = \{k \in D_a : x_k > x_{k_a}\}$$

Also by Remark 5.7 the set $\{s \in S_a^1 : \exists k \in D_{\leq j_0}, \text{ran } f_s \cap \text{ran } x_k \neq \emptyset\}$ contains at most one element, say s_0. We note that

$$D_a = D_{\leq j_0} \cup D_{>j_0} \cup \{k_a\}$$

and we have the following estimates:

$$\left| f_a \Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| = \left| f_a \Big(\sum_{k \in D_{\leq j_0}} \lambda_k x_k + \sum_{k \in D_{>j_0}} \lambda_k x_k + \lambda_{k_a} x_{k_a} \Big) \right|$$

$$\leq 2C \frac{1}{2} |\lambda_{k_a}| + \left| f_{s_0} \Big(\sum_{k \in D_{s_0}} \lambda_k x_k \Big) \right| + \left| f_a \Big(\sum_{k \in D_{>j_0}} \lambda_k x_k \Big) \right|$$

and by our inductive hypothesis and inequality (4),

$$\left| f_{s_0} \Big(\sum_{k \in D_{s_0}} \lambda_k x_k \Big) \right| \leq 4C \Big(\frac{1}{2} g_1^{s_0} + \frac{1}{2} g_2^{s_0} + g_3^{s_0} \Big) \Big(\sum_{k \in D_{s_0}} |\lambda_k| e_k \Big).$$

As $(x_k)_k$ is j_0-separated, the set $L_a = \{k \in D_{>j_0} : |(\sum_{s \in S_a^2} f_s)(x_k)| \geq 10/m_{j_0}\}$ contains at most two elements, hence

$$\left| f_a \Big(\sum_{k \in D_{>j_0}} \lambda_k x_k \Big) \right| \leq C \sum_{k \in L_a} |\lambda_k| + \frac{10}{m_{j_0}} \sum_{k \in D_{>j_0} \setminus L_a} |\lambda_k|$$

$$\leq 4C \Big(\frac{1}{2} \sum_{k \in L_a} e_k^* + \frac{1}{2} \frac{10}{m_{j_0}} \sum_{k \in D_{>j_0} \setminus L_a} e_k^* \Big) \Big(\sum_{k \in D_{>j_0}} |\lambda_k| e_k \Big).$$

Finally, we have

$$\left| f_a\Big(\sum_{k\in D_a} \lambda_k x_k \Big) \right| \le 4C\left(\frac{1}{2}\, e^*_{k_a} \right)(|\lambda_{k_a}|e_{k_a})$$

$$+ 4C\left(\frac{1}{2}\, g_1^{s_0} + \frac{1}{2}\, g_2^{s_0} + g_3^{s_0} \right)\Big(\sum_{k\in D_{s_0}} |\lambda_k|e_k \Big)$$

$$+ 4C\,\frac{1}{2}\Big(\sum_{k\in L_a} e^*_k + \frac{10}{m_{j_0}} \sum_{k\in D_{>j_0}\setminus L_a} e^*_k \Big)\Big(\sum_{k\in D_{>j_0}} |\lambda_k|e_k \Big)$$

$$= 4C\Big(g_1^a + \frac{1}{2}\, g_2^a + g_3^a \Big)\Big(\sum_{k\in D_a} |\lambda_k|e_k \Big),$$

where we have set

$$g_1^a = \frac{1}{2}\Big(g_1^{s_0} + e^*_{k_a} + \sum_{k\in L_a} e^*_k \Big), \quad g_2^a = g_2^{s_0},$$

$$g_3^a = \frac{5}{m_{j_0}} \sum_{k\in D_{>j_0}\setminus L_a} e^*_k + g_3^{s_0}.$$

We can see that $g_1^a \in W_{gt}^{j_0}$ since the set $\{g_1^{s_0}\}\cup\{e^*_{k_a}\}\cup\{e^*_k : k\in L_a\}$ consists of successive functionals and has cardinality almost 4, which is less than $m_{j_0}^3$. At the same time $\|g_3^a\|_\infty \le 10/m_{j_0}$ and of course $g_2^a = \frac{1}{2}g_2^{s_0} \in G^{j_0}$. In addition $\mathrm{supp}(g_i^a) \subset D_a$ for $i=1,2,3$ and $\mathrm{supp}\, g_i^a \cap \mathrm{supp}\, g_3^a = \emptyset$ for $i=1,2$. We point out that if $\mathrm{ind}\, f_a < j_0$ then the functionals g_i^a for $i=1,2,3$ have the following important properties:

P1. $g_i^a = \frac{1}{2}g_i^{s_0}$ for $i=1,2$.

P2. $\|g_3^{s_0}\|_\infty \le \dfrac{1}{m_{j_{s_0}}}\dfrac{10}{m_{j_0}}$.

CASE 5: f_a is a type II functional. We distinguish the following subcases:

SUBCASE A: $\mathrm{ind}\, f_a < j_0$ and $f_a = \sum_{s\in S_a} a_s f_s$ where $\sum_{s\in S_a} a_s^2 \le 1$ and $(f_s)_s$ are special functionals with disjoint sets of indices. Then

$$\left| \Big(\sum_{s\in S_a} a_s f_s \Big)\Big(\sum_{k\in D_a} \lambda_k x_k \Big) \right| \le \sum_{s\in S_a} |a_s|\left| f_s\Big(\sum_{k\in D_s} \lambda_k x_k \Big) \right|$$

$$\le 4C\Big(\sum_{s\in S_a} |a_s|\Big(g_1^s + \frac{1}{2}\, g_2^s + g_3^s \Big) \Big)\Big(\sum_{k\in D_s} |\lambda_k|e_k \Big).$$

We set $g_1^a = \sum_{s\in S_a} |a_s|g_1^s$, $g_2^a = \sum_{s\in S_a} |a_s|g_2^s$ and $g_3^a = \sum_{s\in S_a} |a_s|g_3^s$. According to Properties P1 and P2 established in the previous case and as $\mathrm{ind}\, f_s < j_0$ for $s\in S_a$ we have the following:

- $g_i^a \in G^{j_0}$ for $i = 1, 2$. This is based on the observation that $w(g_1^s) = w(g_2^s)$ for all $s \in S_a$ and as $(\operatorname{ind} f_s)_s$ are all smaller than j_0 and mutually disjoint it follows that g_1^a, g_2^a are both the result of the (j_0, ℓ_2) operation.

- $\|g_3^a\|_\infty \leq \sum_{s \in S_a} \|g_3^s\|_\infty \leq \sum_{s \in S_a} \frac{1}{2} \frac{1}{m_{j_s}} \frac{10}{m_{j_0}} \leq \frac{5}{m_{j_0}}.$

Finally, we have the following stronger inequality:

$$\left| f_a \Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq 4C \Big(g_1^a + \frac{1}{2} g_2^a + g_3^a \Big) \Big(\sum_{k \in D_a} |\lambda_k| e_k \Big).$$

We note that the $\frac{1}{2}$ in front of g_2^a is crucial for the last subcase of the type II functionals.

SUBCASE B: $\operatorname{ind} f_a \geq j_0$ and $f_a = \sum_{s \in S_a} a_s f_s$ where $\sum_{s \in S_a} a_s^2 \leq 1$ and $(f_s)_s$ are special functionals with disjoint sets of indices. Then by Remark 6.1, either $D_a = \emptyset$ in which case we set $g_1^a = g_2^a = g_3^a = 0$, or $D_a = E_a$. If $D_a \neq \emptyset$ we set $L_a = \{k \in D_a : |f_a(x_k)| \geq 5/m_{j_0}\}$ and as $(x_k)_k$ is j_0-separated it follows that $|L_a| \leq 1025 m_{j_0}^2$. We set $g_1^a = 0$, $g_2^a = \frac{1}{2} \sum_{k \in L_a} e_k^*$ and $g_3^a = \frac{5}{m_{j_0}} \sum_{k \in D_a \setminus L_a} e_k^*$. We can see that $g_2^a \in G^{j_0}$, $\|g_3^a\|_\infty \leq 5/m_{j_0}$ and $\operatorname{supp} g_i^a \subseteq D_a$.

The following inequality is straightforward:

$$\left| f_a \Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq 4C \Big(\frac{1}{2} g_2^a + g_3^a \Big) \Big(\sum_{k \in D_a} |\lambda_k| e_k \Big).$$

We note again that g_2^a is multiplied by $\frac{1}{2}$ for later use.

SUBCASE C: f_a is of the form $f_a = f_{s_1} + f_{s_2}$ where $f_{s_1} = f_{a,<j_0}$ and $f_{s_2} = f_{a,\geq j_0}$. By Remark 6.1, $D_a = D_{s_1} \cup D_{s_2}$ and for every $k \in D_a$, $\operatorname{ran} x_k \cap \operatorname{ran} f_{s_i} \neq \emptyset$ if and only if $k \in D_{s_i}$, for $i = 1, 2$. Thus we set $g_1^a = g_1^{s_1}$, $g_2^a = \frac{1}{2}(g_2^{s_2} + g_2^{s_1})$. The functionals $g_2^{s_2}, g_2^{s_1}$ are elements of G^{j_0} and since g_2^a is their convex combination we conclude that $g_2^a \in G^{j_0}$. Finally, set $g_3^a = g_3^{s_1} + g_3^{s_2}$. The estimates take the following form:

$$\left| f_a \Big(\sum_{k \in D_a} \lambda_k x_k \Big) \right| \leq \left| f_{s_1} \Big(\sum_{k \in D_{s_1}} \lambda_k x_k \Big) \right| + \left| f_{s_2} \Big(\sum_{k \in D_{s_2}} \lambda_k x_k \Big) \right|$$

$$\leq 4C \Big(\frac{1}{2} g_1^{s_1} + \frac{1}{2} g_2^{s_1} + g_3^{s_1} \Big) \Big(\sum_{k \in D_{s_1}} |\lambda_k| e_k \Big)$$

$$+ 4C \Big(\frac{1}{2} g_2^{s_2} + g_3^{s_2} \Big) \Big(\sum_{k \in D_{s_2}} |\lambda_k| e_k \Big)$$

$$\leq 4C \left(\frac{1}{2} g_1^{s_1} + \frac{1}{2} g_2^{s_1} + g_3^{s_1} \right) \left(\sum_{k \in D_a} |\lambda_k| e_k \right)$$

$$+ 4C \left(\frac{1}{2} g_2^{s_2} + g_3^{s_2} \right) \left(\sum_{k \in D_a} |\lambda_k| e_k \right)$$

$$\leq 4C (g_1^a + g_2^a + g_3^a) \left(\sum_{k \in D_a} |\lambda_k| e_k \right).$$

Moreover, $g_i^a \in G^{j_0}$ for $i = 1, 2$, $\|g_3^a\|_\infty \leq 10/m_{j_0}$ and $\operatorname{supp} g_i^a \subseteq D_a$.

The induction is complete. ∎

7. Consequences of the Basic Inequality and exact pairs. In this section we analyze the consequences of the Basic Inequality. In particular, we recall the definitions of exact pairs and dependent sequences and then prove that every block subspace of $\mathfrak{X}_{gt}$ contains a dependent sequence. We start with

LEMMA 7.1. *Let $j_0 \in \mathbb{N}$ with $j_0 \geq 3$. Let also $(x_k)_k$ be a (C, ϵ) RIS which is j_0-separated. Then for every choice of natural numbers $k_1 < \cdots < k_{n_{j_0}}$ we have*

$$\left\| \frac{1}{n_{j_0}} \sum_{i=1}^{n_{j_0}} x_{k_i} \right\|_{gt} \leq \frac{15C}{m_{j_0}}.$$

Proof. We set $x = \frac{1}{n_{j_0}} \sum_{i=1}^{n_{j_0}} x_{k_i}$. Let $f \in G_{gt}$. We distinguish the following cases:

- $f \in \{\pm e_k^* : k \in \mathbb{N}\}$. Then $|f(x)| \leq C/n_{j_0} \leq 15C/m_{j_0}$.
- $w(f) \geq m_{j_0}$. Then

$$|f(x)| \leq \frac{C + (n_{j_0} - 1)\frac{5}{m_{j_0}}}{n_{j_0}} \leq \frac{6C}{m_{j_0}}.$$

- $w(f) = m_i < m_{j_0}$. Then by the Basic Inequality there exist g_1, g_2, g_3 with $g_j \in G^{j_0}$, $w(g_j) = m_i$ for $j = 1, 2$ and $\|g_3\|_\infty \leq \frac{1}{m_i} \cdot \frac{10}{m_{j_0}}$ such that

$$|f(x)| \leq 4C \left(\frac{1}{2} g_1 + \frac{1}{2} g_2 + g_3 \right) \left(\frac{1}{n_{j_0}} \sum_{i=1}^{n_{j_0}} e_{k_i} \right).$$

Thus, by Lemma 6.2 we have

$$|f(x)| \leq 4C \left(\frac{4}{\sqrt{m_i} \, m_{j_0}} + \frac{1}{m_i} \frac{10}{m_{j_0}} \right) \leq \frac{8C}{m_{j_0}}.$$

- f is of type II, $f = \sum_{i=1}^d a_i x_i^*$ where $(x_i^*)_{i=1}^d$ are special functionals with disjoint sets of indices. We write $f = f_{<j_0} + f_{\geq j_0}$. For $f_{\geq j_0}$ we

have

$$|\{k \in \mathbb{N} : |f_{\geq j_0}(x_k)| \geq 5/m_{j_0}\}| \leq 1025 m_{j_0}^2$$

and thus

$$|f_{\geq j_0}(x)| \leq \frac{C \cdot 1025 m_{j_0}^2 + \frac{5}{m_{j_0}}(n_{j_0} - 1025 m_{j_0}^2)}{n_{j_0}} \leq \frac{6C}{m_{j_0}}.$$

For $f_{<j_0}$ we have $|f_{<j_0}(x)| \leq \sum_{i=1}^d |x_{i,<j_0}^*(x)| \leq \sum_{i \in A} |f_i(x)|$ where $f_i \in G_{gt}$ are of type I and $w(f_i) = m_i < m_{j_0}$ and $|A| < m_{j_0}$. By the Basic Inequality we obtain

$$f_{<j_0}(x) \leq 4C \sum_{i \in A} \left(\frac{2}{\sqrt{m_i} m_{j_0}} + \frac{10}{m_i m_{j_0}} \right) \leq \frac{8C}{m_{j_0}}.$$

Finally, $|f(x)| \leq 15C/m_{j_0}$.

- f is of type III with $f = \sum_{i=1}^d q_i f_i$ such that $q_i \in \mathbb{Q}^+$, $\sum_i q_i = 1$ and f_i is not of type III for every $i = 1, \ldots, d$. Using the previous cases we have

$$|f(x)| \leq \sum_{i=1}^d q_i \frac{15C}{m_{j_0}} \leq \frac{15C}{m_{j_0}}. \qquad \blacksquare$$

An immediate consequence of the above lemma and Proposition 5.2 is

COROLLARY 7.1. *The space $\mathfrak{X}_{gt}$ does not contain an isomorphic copy of ℓ_1.*

In the rest of this section we define exact pairs and prove that one can find a (C, j) exact pair for each $j \in \mathbb{N}$.

DEFINITION 7.1. Let $x \in \mathfrak{X}_{gt}$ and $\phi \in G_{gt}$. The pair (x, ϕ) is called a (C, j) *exact pair* for some $C \geq 1$ and $j \in \mathbb{N}$ if:

- $\|x\|_{gt} \leq 30C$.
- ϕ is of type I and $w(\phi) = m_j$.
- $\operatorname{ran} \phi = \operatorname{ran} x$ and $\phi(x) = 1$.
- If $f \in G_{gt}$ is of type I with $w(f) = m_i < m_j$ then $|f(x)| \leq 100C/\sqrt{m_i}$.

PROPOSITION 7.1. *Let Z be a block subspace of $\mathfrak{X}_{gt}$ and $j \in \mathbb{N}$. Then there exists a $(6, j)$ exact pair (w, f) with $w \in Z$.*

Proof. By Proposition 5.2 there exists a j-separated $(6, \epsilon)$ RIS $(x_k)_k$ with $0 < \epsilon < 5/m_j$ and $x_k \in Z$ for all k. Additionally, each x_k is a $2\text{-}\ell_1^{n_{j_k}}$ average and thus $\|x_k\|_{gt} > 1$. We can also assume that $\lim_k x^*(x_k) = 0$ for every special x^*. We choose $(f_k)_k \subset G_{gt}$ such that $\operatorname{ran} f_k = \operatorname{ran} x_k$ and $f_k(x_k) = 1$ for all k. Set

$$w = \frac{2m_j}{n_j} \sum_{k=1}^{n_j} x_k, \qquad f = \frac{1}{2m_j} \sum_{k=1}^{n_j} f_k.$$

Then (w, f) is a $(6, j)$ exact pair. Indeed, by the choice of $(f_k)_k$ we have $\operatorname{ran} f = \operatorname{ran} w$ and $f(w) = 1$. Moreover, Lemma 7.1 yields $|g(w)| \leq 100C/\sqrt{m_i}$ for every $g \in G_{gt}$ with $w(g) = m_i < m_j$. ∎

DEFINITION 7.2. A sequence of pairs $(w_k, f_k)_{k \in \mathbb{N}}$ with $w_k \in \mathfrak{X}_{gt}$ is said to be *C-dependent* if:

- (w_k, f_k) is a (C, j_k) exact pair for every $k \in \mathbb{N}$.
- $(f_k)_k$ is a special sequence with $w(f_k) = m_{j_k}$.

The last proposition of this section establishes the existence of a 6-dependent sequence in every subspace of $\mathfrak{X}_{gt}$ with the additional property that the sequence is weakly Cauchy. The existence of a 6-dependent sequence in every subspace of $\mathfrak{X}_{gt}$ is an immediate consequence of Proposition 7.1. To prove that it is weakly Cauchy we need the following lemma which describes the structure of $\mathfrak{X}_{gt}^*$ and for which we give a short proof. For a detailed exposition, see [ATO, Proposition II.26].

LEMMA 7.2. *The dual of the space* $\mathfrak{X}_{gt}$ *can be described as follows*:

$$\mathfrak{X}_{gt}^* = \overline{\langle \{e_n^* : n \in \mathbb{N}\} \cup \{f : f \text{ is an infinite special functional}\}\rangle}^{\|\cdot\|}.$$

Proof. Suppose otherwise and set

$$Z = \overline{\langle \{e_n^* : n \in \mathbb{N}\} \cup \{f : f \text{ is an infinite special functional}\}\rangle}^{\|\cdot\|}.$$

Then there exists a functional $x^* \in \mathfrak{X}_{gt}^* \setminus Z$ of norm 1. Thus, there exists $x^{**} \in \mathfrak{X}_{gt}^{**}$ such that $x^{**}(x^*) = \theta > 0$ and $x^{**}(f) = 0$ for every $f \in Z$. We may assume that $\|x^{**}\| \leq 1$. By Corollary 7.1 and the Odell–Rosenthal theorem [OR] there exists a sequence $(x_n)_n$ in $\mathfrak{X}_{gt}$ with $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ such that $x_n \xrightarrow{w^*} x^{**}$. By passing to a subsequence we may assume that $x^*(x_n) > \theta/2$ for all $n \in \mathbb{N}$. In addition, as $e_k^* \in Z$ for all $k \in \mathbb{N}$, using a sliding hump argument, we may suppose that $(x_n)_n$ is in fact a block sequence. Using the fact that $x^*(x_n) > \theta/2$ we can construct a block sequence $(y_n)_n$ of successive ℓ_1 averages of $(x_n)_n$ with increasing lengths and $x^*(y_n) > \theta/2$ for all n. Thus, by passing to subsequences, we assume that $(y_n)_n$ is a $(6, \epsilon)$ RIS. As $f(y_n) \to 0$ for every infinite special functional f, we can pass to a further subsequence and suppose that $(y_n)_n$ is j-separated for j satisfying $60/m_j < \theta/2$. Thus, setting $y = \frac{1}{n_j} \sum_{i=1}^{n_j} y_i$, we obtain the following contradictory facts: $\|y\| \geq x^*(y) > \theta/2$, and by Lemma 7.1, $\|y\| \leq 60/m_j < \theta/2$. This completes the proof. ∎

PROPOSITION 7.2. *Let Y and Z be block subspaces of $\mathfrak{X}_{gt}$. Then there exists a 6-dependent sequence $(w_k, f_k)_k$ with $w_{2k-1} \in Y$ and $w_{2k} \in Z$ for all $k \in \mathbb{N}$. In addition, $(w_{2k-1} - w_{2k})_k$ is weakly null.*

Proof. By repeated application of Proposition 7.1 we can inductively construct a 6-dependent sequence $(w_k, f_k)_k$ with $w_{2k-1} \in Y$ and $w_{2k} \in Z$ for all $k \in \mathbb{N}$. Each w_k is of the form

$$w_k = \frac{2m_{j_k}}{n_{j_k}} \sum_{i \in F_k} y_i$$

where $|F_k| = n_{j_k}$. Furthermore, we assume that for each k the sequence $(y_i)_{i \in F_k}$ is a j_k-separated $(6, \epsilon)$ RIS with $0 < \epsilon < 5/m_{j_k}^2$ which also satisfies the conclusion of Remark 5.4. Now consider a special functional $f = \sum_i g_i$ where $(g_i)_i$ is a special sequence different from $(f_i)_i$. By Remark 3.1 there exists $r \in \mathbb{N}$ such that

$$f_i = g_i \quad \text{for } i = 1, \ldots, r, \qquad f_{r+1} \neq g_{r+1}, \qquad w(f_{r+1}) = w(g_{r+1}),$$

while $w(f_i) \neq w(g_i)$ for $i > r + 1$. Let $\varepsilon > 0$. We pick $l \in \mathbb{N}$ with $l > r + 1$ and $1/m_{j_l} < \varepsilon$. Set $i_0 = \min\{i \in \mathbb{N} : \text{ind}\, g_i > j_l\}$. We choose $k_0 \in \mathbb{N}$ such that

$$\frac{m_{j_{k_0}}}{n_{j_{k_0}}} < \varepsilon, \qquad \text{ran}\, g_{i_0} < \text{ran}\, w_{2k_0-1}, \qquad \text{ind}\, g_{i_0} < j_{2k_0-1}.$$

For $k \geq k_0$ we have

$$|g(w_k)| \leq \sum_{m_{j_l} < w(g_i) = m_i < m_{j_k}} |g_i(w_k)| + |x^*(w_k)|$$

where $\text{ind}\, x^* > j_k$. However,

$$\sum_{m_{j_l} < w(g_i) = m_i < m_{j_k}} |g_i(w_k)| \leq \sum_{j_l < i < j_k} \frac{600}{m_i} < 600\varepsilon$$

and by Remark 5.4,

$$|x^*(w_k)| \leq \frac{m_{j_k}}{n_{j_k}} \left(8 + \frac{10}{m_{j_k}^2}(n_{j_k} - 2) \right) \leq 18\varepsilon.$$

Therefore, $\lim_k (\sum_i g_i)(w_k) = 0$ for every special sequence $(g_i)_i$ distinct from $(f_i)_i$. Consequently, Lemma 7.2 shows that $(w_{2k-1} - w_{2k})_k$ is weakly null. ∎

REMARK 7.1. In the proof of the above proposition we have seen that $\lim_k x^*(w_k) = 0$ for every dependent sequence $(f_k, w_k)_k$ and every special functional x^* distinct from $\sum_i f_i$. Using this along with Remark 5.6 we can obtain, for every $j_0 > 3$, an $L \in [\mathbb{N}]$ such that:

- For every special x^* such that $x^* \neq \sum_i f_i$ and $\text{ind}\, x^* \geq j_0$ the set $\{k \in L : |x^*(w_k)| > 10/m_{j_0}\}$ contains at most two elements.
- For every $y^* = \sum_{j=1}^d a_j x_j^* \in G_{gt}$ of type II with $\text{ind}\, y^* \geq j_0$ and $x_j^* \neq \sum_i f_i$ for all j we have $|\{k \in L : |y^*(w_k)| \geq 5/m_{j_0}\}| \leq 1025m_{j_0}^2$.

8. The basic properties of $\mathfrak{X}_{gt}$. In this section we establish the basic properties of $\mathfrak{X}_{gt}$. In particular, we show that every infinite-dimensional closed subspace of $\mathfrak{X}_{gt}$ has nonseparable dual and that $\mathfrak{X}_{gt}$ is hereditarily indecomposable (HI).

DEFINITION 8.1. Let $j_0 \in \mathbb{N}$ and Y be a block subspace of $\mathfrak{X}_{gt}$. A sequence $(w_t, f_t, j_t)_{t \in 2^{<\omega}}$ of triples will be called a *special tree* in Y if:

- For every $t \in 2^{<\omega}$, (w_t, f_t) is a (C, j_t) exact pair for some $C \geq 1$ and $w_t \in Y$.
- For every $t \in 2^{<\omega}$ we have $j_t = \sigma((f'_t)_{t' \sqsubset t})$.
- For every $t \in 2^{<\omega}$ we have $w_{t \frown 0} < w_{t \frown 1}$ and $w_{t'} < w_t$ for all $t' \in 2^{<\omega}$ with $|t'| < |t|$.

If moreover $j_\emptyset > j_0$ and $\min \operatorname{supp} w_\emptyset > m_{j_0}$ then the sequence will be called a j_0-*special tree*.

THEOREM 8.1. *Let Y be a closed infinite-dimensional subspace of $\mathfrak{X}_{gt}$. Then Y^* is nonseparable.*

Proof. We can reduce the problem to the case of an arbitrary block subspace Y. We shall construct an uncountable set $A \subset \mathfrak{X}_{gt}^*$ such that $\|x^*|_Y - y^*|_Y\| \geq \delta$ for all $x^*, y^* \in A$ and an appropriate δ. By recursive applications of Proposition 7.1 we can construct a special tree $T = (w_t, f_t, j_t)_{t \in 2^{<\omega}}$ such that $w_t \in Y$ for all $t \in 2^{<\omega}$. Let $A = \{\sum_{t \sqsubset b} f_t : b$ is a branch of $2^{<\omega}\}$. Let b_1, b_2 be two different branches of $2^{<\omega}$ and g_{b_1}, g_{b_2} the corresponding elements of A. Since $b_1 \neq b_2$ there exists $t \in b_1 \setminus b_2$. Hence,

$$\|g_{b_1}|_Y - g_{b_2}|_Y\| \geq \frac{(g_{b_1} - g_{b_2})(w_t)}{\|w_t\|_{gt}} \geq \frac{1}{30 \cdot 6}.$$

Thus if we set $\delta = 1/180$ the proof is complete. ∎

THEOREM 8.2. *The space $\mathfrak{X}_{gt}$ is HI.*

Proof. Let Z and Y be two infinite-dimensional block subspaces of $\mathfrak{X}_{gt}$ and let $\epsilon > 0$. According to Proposition 7.2 there exists a 6-dependent sequence $(w_k, f_k)_k$ such that $w_{2k-1} \in Y$ and $w_{2k} \in Z$ for all k and in addition $(w_{2k-1} - w_{2k})_k$ is weakly null. By Mazur's theorem there exists an $n_0 \in \mathbb{N}$ and a sequence $(\lambda_i)_{i=1}^{n_0}$ of scalars with $\lambda_i \in \mathbb{R}^+$ and $\sum_{i=1}^{n_0} \lambda_i = 1$ such that $\|\sum_{i=1}^{n_0} \lambda_i(w_{2i-1} - w_{2i})\|_{gt} < \epsilon$. We set $y = \sum_{i=1}^{n_0} \lambda_i w_{2i-1} \in Y$ and $z = \sum_{i=1}^{n_0} \lambda_i w_{2i} \in Z$. We observe that $\|y - z\|_{gt} < \epsilon$ while if we set $f = \sum_{i=1}^{2n_0} f_i$ then $f \in G_{gt}$ since $(f_i)_i$ is a special sequence and $\|y + z\|_{gt} \geq f(y + z) = 2$. Thus, $\|y - z\|_{gt} < \epsilon < \epsilon \|y + z\|_{gt}$ and we have shown that $\mathfrak{X}_{gt}$ is HI. ∎

9. The space of bounded linear operators on $\mathfrak{X}_{gt}$. In this section we study the structure of operators on $\mathfrak{X}_{gt}$. In particular, we show that every bounded linear operator $T : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}$ is of the form $\lambda I + W$, where I is

the identity operator and W a weakly compact operator. We start with the following central lemma:

LEMMA 9.1. *Let Y be a block subspace of $\mathfrak{X}_{gt}$ and $T : Y \to \mathfrak{X}_{gt}$ be a bounded linear operator. Suppose that $(y_n)_n$ is a block sequence of 2-ℓ_1 averages with increasing lengths in Y such that $(T(y_n))_n$ is also a block sequence. Then $\lim_n \operatorname{dist}(Ty_n, \mathbb{R}\, y_n) = 0$.*

Proof. Suppose not. Then there exist $L \in [\mathbb{N}]$ and $\delta > 0$ such that $\operatorname{dist}(Ty_n, \mathbb{R}\, y_n) > \delta$ for all $n \in L$. By the Hahn–Banach theorem there exists $\phi_n \in B_{\mathfrak{X}_{gt}^*}$ such that $\phi_n(y_n) = 0$ and $\phi_n(Ty_n) > \delta$ for all $n \in L$. As $B_{\mathfrak{X}_{gt}^*} = \overline{G}_{gt}^{w^*}$, we may assume that $\phi_n \in G_{gt}$ for all $n \in L$ and $\operatorname{ran}\phi_n \subset \operatorname{ran}Ty_n$. Now since $(y_n)_{n \in L}$ is a sequence of 2-ℓ_1 averages with increasing lengths one can inductively construct in Y a 6-dependent sequence $(w_k, f_k)_k$ such that each w_k is of the form

$$w_k = \frac{2m_{j_k}}{n_{j_k}} \sum_{i \in F_k} y_i$$

and $|F_k| = n_{j_k}$. As, by Proposition 7.2, $(w_k)_k$ is weakly Cauchy, there exists $n_0 \in \mathbb{N}$ and a convex combination u_{n_0} of the form

$$u_{n_0} = \lambda_1(w_{2k_1-1} - w_{2k_1}) + \cdots + \lambda_{n_0}(w_{2k_{n_0}-1} - w_{2k_{n_0}}), \quad k_1 < \cdots < k_{n_0}.$$

such that $\|u_{n_0}\| < \frac{1}{\|T\|} \cdot \frac{\delta}{2}$. We observe that $\|T(u_{n_0})\| < \delta/2$. Set

$$w_k^* = \frac{1}{2m_{j_k}} \sum_{i \in F_k} \phi_i$$

and observe that $w_k^* \in G_{gt}$, $\operatorname{ran}w_k^* \subset \operatorname{ran}Ty_k$ and $w_k^*(Ty_k) \geq \delta/2$. Finally, setting $f = w_{2k_1-1}^* + \cdots + w_{2k_{n_0}-1}^*$ we obtain $f(T(u_{n_0})) \geq \delta/2$. This is a contradiction which completes the proof. ∎

We need a slight modification of the above lemma; we omit the proof as it is quite similar to the one above.

LEMMA 9.2. *Let Y be a block subspace of $\mathfrak{X}_{gt}$ and $T : Y \to \mathfrak{X}_{gt}$ a bounded linear operator. Suppose that $(y_n)_n$ is a block sequence of 2-ℓ_1 averages with increasing lengths in Y such that*:

- $\|y_n\| > \delta > 0$.
- $(T(w_n))_n$ *is also a block sequence, where $w_n = y_{2n-1} - y_{2n}$ for all n.*

Then $\lim_n \operatorname{dist}(Tw_n, \mathbb{R}\, w_n) = 0$.

PROPOSITION 9.1. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{gt}$. Then every bounded linear operator $T : Y \to \mathfrak{X}_{gt}$ takes the form $\lambda I + S$ where S is a strictly singular operator.*

For the proof we refer to [AT, Theorem IV.12]. The next proposition concludes the investigation of the structure of the bounded linear operators on $\mathfrak{X}_{gt}$. Its proof is similar to the proof of Theorem 9.4 in [ATO], but we include it for completeness.

PROPOSITION 9.2. *Every bounded linear operator* $T : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}$ *is of the form* $\lambda I + W$ *where* W *is a weakly compact operator.*

Proof. Let $T : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}$ be a bounded linear operator and suppose that it is not weakly compact. We shall show that T is not strictly singular. Since T is not weakly compact there exists a normalized sequence $(x_n)_n$ such that $(Tx_n)_n$ has no weakly convergent subsequence. However, since $\mathfrak{X}_{gt}$ does not contain ℓ_1 we may assume that $(Tx_n)_n$ is nontrivial weakly Cauchy. Denote by $y^{**} \in \mathfrak{X}_{gt}^{**} \setminus \mathfrak{X}_{gt}$ the w^*-limit of $(Tx_n)_n$ and assume also that $x^{**} \in \mathfrak{X}_{gt}^{**} \setminus \mathfrak{X}_{gt}$ is the w^*-limit of $(x_n)_n$. Obviously $y^{**} = T^{**}x^{**}$.

As the basis $(e_n)_n$ is boundedly complete we may assume that $x_n = x + u_n$ for all $n \in \mathbb{N}$ where $x = \sum_i x^{**}(e_i^*)e_i$ and $(u_n)_n$ is a block sequence. We observe that $u_n \xrightarrow{w^*} x^{**} - x$ and $Tu_n \xrightarrow{w^*} y^{**} - Tx$. Thus, we may assume that $x = 0$ and $(x_n)_n$ is a block sequence. Similarly, we may assume that there exists a block sequence $(z_n)_n$ and $z \in \mathfrak{X}_{gt}$ such that $Tx_n = z + z_n$ for each n. We set $\theta = \operatorname{dist}(y^{**}, \mathfrak{X}_{gt})$. If we write $z = \sum_{n=1}^{\infty} a_n e_n$ we know that there exists $n_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{n=n_0+1}^{\infty} a_n e_n \Big\| < \theta/4.$$

We claim that there exists $y^* \in B_{\mathfrak{X}_{gt}^*}$ such that $y^{**}(y^*) > 3\theta/4$ and $y^*|_{\operatorname{span}\{e_1,\dots,e_{n_0}\}} = 0$. To see this we set

$$w_1 = \sum_{n=1}^{n_0} a_n e_n, \qquad w_2 = \sum_{n=n_0+1}^{\infty} a_n e_n.$$

As $\|y^{**} - w_1\| \geq \operatorname{dist}(y^{**}, \mathfrak{X}_{gt})$ we may choose $x^* \in B_{\mathfrak{X}_{gt}^*}$ such that $|y^{**}(x^*) - x^*(w_1)| > 3\theta/4$. We set $y^* = P_{[n_0+1,\infty)}^*(x^*)$, where $P_{[n_0+1,\infty)}^*$ denotes the canonical projection onto the interval $[n_0 + 1, \infty)$ associated to the basis $(e_n^*)_n$ of the predual. Observe now that $y^*|_{\operatorname{span}\{e_1,\dots,e_{n_0}\}} = 0$ and

$$y^{**}(y^*) = \lim_n y^*(Tx_n) = \lim_n y^*(z + z_n) = y^*(w_1 + w_2) + \lim_n y^*(z_n)$$
$$= x^*(w_2) + \lim_n y^*(z_n).$$

Since $(z_n)_n$ is a block sequence it can be readily seen that $\lim_n y^*(z_n) = \lim_n x^*(z_n)$. Therefore, $y^{**}(y^*) = x^*(w_2) + \lim_n x^*(z_n)$, and as $\lim_n x^*(z_n) = y^{**}(x^*) - x^*(z)$, we obtain

$$|y^{**}(y^*)| = |y^{**}(x^*) - x^*(w_1)| > 3\theta/4.$$

As $y^*(Tx_n) \to y^{**}(y^*)$ we may also assume that $y^*(Tx_n) = y^*(z + z_n) > 3\theta/4$ for all n. Since $|y^*(z)| < \theta/4$ we obtain

$$y^*(z_n) > \theta/2 \quad \text{for all } n.$$

Pick $x^* \in B_{\mathfrak{X}^*_{gt}}$ such that $x^{**}(x^*) > \delta > 0$ and suppose also that $x^*(x_n) > \delta$ for all $n \in \mathbb{N}$. We inductively construct a block sequence $(y_n)_n$ of $(x_n)_n$ such that $(y_n)_n$ are 2-ℓ_1 averages with increasing lengths. Now as $(y_n)_n$ are convex combinations of $(x_n)_n$ we can see that for all n we have

$$\|y_n\| \geq x^*(y_n) > \delta, \quad y^*(Ty_n) > 3\theta/4.$$

In addition we observe that there exists a block sequence $(v_n)_n$ of convex combinations of $(z_n)_n$ such that for all n we have

$$Ty_n = z + v_n,$$

which gives

$$\|v_n\| \geq y^*(Ty_n) - y^*(z) > \theta/2$$

for all n. We set $w_n = y_{2n-1} - y_{2n}$ and observe that $(Tw_n)_n$ is a block sequence. Hence, by Lemma 9.2, $\mathrm{dist}(Tw_n, \mathbb{R}\,w_n) \to 0$, and thus there exists a sequence $(\lambda_n)_n$ of reals such that

$$\|Tw_n - \lambda_n w_n\| \to 0.$$

We can see that the sequence $(\lambda_n)_n$ is bounded and assume that $\lambda_n \to \lambda \neq 0$. We set $s_k = w_k/\|w_k\|$ and pass to a subsequence to obtain

$$\sum_{k=1}^{\infty} \|Ts_k - \lambda s_k\| < |\lambda|/2.$$

We claim that if we set $F = \mathrm{span}\{s_k : k \in \mathbb{N}\}$ then T restricted to F is an isomorphic embedding and hence T is not strictly singular. Indeed, let $x \in F$ with $\|x\| = 1$ be of the form $x = \sum_{k=1}^{\infty} b_k s_k$. Since $|b_k| \leq 1$ for each k, we have

$$\|Tx - \lambda x\| \leq \sum_{k=1}^{\infty} |b_k|\,\|Ts_k - \lambda s_k\| \leq |\lambda|/2.$$

Therefore, $\|Tx\| \geq |\lambda|/2$. $\blacksquare$

10. The James tree structure of $\mathfrak{X}_{gt}$. In this section we show that ℓ_2 is contained in both the dual and second dual of $\mathfrak{X}_{gt}$. The basic tool for proving this is Proposition 10.1 asserting that a sequence of incomparable special functionals constructed through dependent sequences is equivalent to the standard ℓ_2 basis. The proof of the proposition is based on Proposition 10.2 which is another Basic Inequality. It provides estimates for ℓ_2 sums of certain averages of vectors of the corresponding dependent sequences. After establishing the above propositions we apply them to indicate the

similarities in the structure of the triples $\mathfrak{X}_{gt}, \mathfrak{X}_{gt}^*, \mathfrak{X}_{gt}^{**}$ and JT, JT^*, JT^{**}, where JT denotes the James tree space. Among these applications we find that every subspace of $\mathfrak{X}_{gt}$ has ℓ_2 as a quotient space, and its dual contains ℓ_2. Moreover, it is shown that $\ell_2(c)$ (c denotes the Cantor set) is isomorphic to a subspace of $\mathfrak{X}_{gt}^{**}$, and $\mathfrak{X}_{gt}^*$ has $\ell_2(c)$ as a quotient space. We start with

PROPOSITION 10.1. *For all* $i \in \mathbb{N}$ *consider a* 6-*dependent sequence* $(w_n^i, f_n^i)_n$. *Assume that* $\operatorname{ind} f_n^i \cap \operatorname{ind} f_n^{i'} = \emptyset$ *for all* $i \neq i' \in \mathbb{N}$ *and set* $b_i^* = \sum_n f_n^i$ *for all* $i \in \mathbb{N}$. *Then* $(b_i^*)_{i \in \mathbb{N}}$ *is equivalent to the standard* ℓ_2 *basis.*

Proof. Let $0 < \epsilon < 1$, $d \in \mathbb{N}$ and consider scalars $a_1, \ldots, a_d$ such that $\sum_{i=1}^d a_i^2 = 1$. Clearly, by the definition of the norming set, $\|\sum_{i=1}^d a_i b_i^*\| \leq 1$. To complete the proof we shall show that

$$\frac{1}{14400} \leq \Big\| \sum_{i=1}^d a_i b_i^* \Big\|.$$

For that, first choose $j_0 \in \mathbb{N}$ which satisfies the following conditions:

$$d \cdot 5 \cdot 720^2 < m_{j_0}, \quad 2d < m_{j_0} - 2, \quad d \sum_{i=1}^d |a_i| < m_{j_0}, \quad 1/m_{j_0} < \epsilon,$$

and then a finite sequence $\{l_t^i : 1 \leq i \leq d, \ 1 \leq t \leq n_{j_0}\}$ such that if we set $x_{(t,i)} = w_{l_t^i}^i$ then $(x_{(t,i)})_{(t,i)}$ has the following properties:

P1. $w(f_{l_t^i}^i) > m_{j_0}$ for every (t, i).

P2. $\min \operatorname{supp} x_{(t,i)} > m_{j_0}$ and $\|x_{(t,i)}\| \leq 180$ for all t, i. In addition, there exists a sequence $(j_{(t,i)})_{t,i}$ of natural numbers such that $j_0 < j_{(t,i)} < j_{(t,i+1)}$ for all $i = 1, \ldots, d-1$ and $j_{(t,d)} < j_{(t+1,1)}$ for all $t \in \mathbb{N}$, and for all $f \in G_{gt}$ with $w(f) = m_l$, $l < j_{(t,i)}$ we have $|f(x_{(t,i)})| < 600/\sqrt{m_l}$. Furthermore,

$$\frac{|\operatorname{supp} x_{(t,i)}|}{m_{j_{(t,i+1)}}} < \frac{1}{\sqrt{m_{j_0}}}, \quad t \in \mathbb{N}, \ i = 1, \ldots, d-1,$$

$$\frac{|\operatorname{supp} x_{(t,d)}|}{m_{j_{(t+1,1)}}} < \frac{1}{\sqrt{m_{j_0}}}.$$

P3. For every special functional $x^* \notin \{b_1^*, \ldots, b_d^*\}$ with $\operatorname{ind} x^* \geq j_0$, and every i, at most two t satisfy $|x^*(x_{(t,i)})| > 1202/\sqrt{m_{j_0}}$.

P4. For every $y^* = \sum_{k=1}^d a_i x_k^* \in G_{gt}$ of type II with $\operatorname{ind} y^* \geq j_0$ and $x_k^* \notin \{b_1^*, \ldots, b_d^*\}$ for all $k = 1, \ldots, d$, and for every i, the cardinality of $\{t : |y^*(x_{(t,i)})| \geq 602/\sqrt{m_{j_0}}\}$ is at most $5 \cdot 720^2 m_{j_0}$.

P5. For every $i \neq i'$ we have $|b_{i'}^*(x_{(t,i)})| \leq 1/m_{j_0}^2$ for all $t = 1, \ldots, n_{j_0}$.

P6. $(x_{(t,i)})_{(t,i)}$ ordered lexicographically is a block sequence.

The choice of such a sequence $(x_{(t,i)})_{(t,i)}$ is possible through the use of Remark 7.1 and Proposition 7.2. Set

$$z_i = \frac{1}{n_{j_0}} \sum_{t=1}^{n_{j_0}} x_{(t,i)} \quad \text{for } i = 1, \ldots, d.$$

Property P5 yields

$$\sum_{i=1}^{d} a_i b_i^* \Big(\sum_{i=1}^{d} a_i z_i \Big) \geq 1 - \epsilon.$$

It remains to show that $\|\sum_{i=1}^{d} a_i z_i\|_{gt}$ is bounded by a constant. This is done in the following two lemmas where a second Basic Inequality is stated and proved. Namely, we shall show that

(5)
$$\Big\| \sum_{i=1}^{d} a_i z_i \Big\|_{gt} \leq 14400. \quad \blacksquare$$

The auxiliary space is defined through the following norming set:

DEFINITION 10.1. Let $k_{(t,i)} = \min \operatorname{supp} x_{(t,i)}$ and $s_i = \{k_{(t,i)} : t = 1, \ldots, n_{j_0}\}$ for $i = 1, \ldots, d$. We denote by D the minimal subset of $c_{00}(\mathbb{N})$ satisfying:

- D contains the set $\{\sum_{i=1}^{d} \sum_j \lambda_{i,j} s_{i,j}^* : \sum_{i,j} \lambda_{i,j}^2 \leq 1, \lambda_{(i,j)} \in \mathbb{Q}$ and $(s_{i,j}^*)_j$ are disjoint subsets of $s_i\} \cup \{e_n^* : n \in \mathbb{N}\}$.
- D is closed under the $(\mathcal{A}_{2n_j}, 1/m_j)$ operations for all $j \in \mathbb{N}$.
- D is closed under the $(\mathcal{A}_{m_{j_0}^2}, 1/2)$ operation.
- For every sequence $(f_k)_{k=1}^r$ with $r < j_0$, f_k of type I, $w(f_k) < m_{j_0}$ and $w(f_k) \neq w(f_{k'})$ for all $k \neq k' < d$, we have $\sum_{k=1}^{d} a_k f_k \in D$ whenever $\sum_{k=1}^{r} a_k^2 \leq 1$.
- D is rationally convex.

We use an enlarged norming set D' that contains D, as in Section 5, defined as follows.

DEFINITION 10.2. Let $k_{(t,i)} = \min \operatorname{supp} x_{(t,i)}$ and $s_i = \{k_{(t,i)} : t = 1, \ldots, d\}$. We consider the minimal subset D' of $c_{00}(\mathbb{N})$ that satisfies:

- D' contains the set $\{\sum_{i=1}^{d} \sum_j \lambda_{i,j} s_{i,j}^* : \sum_{i,j} \lambda_{i,j}^2 \leq 1, \lambda_{(i,j)} \in \mathbb{Q}$ and $(s_{i,j}^*)_j$ are disjoint subsets of $s_i\} \cup \{e_n^* : n \in \mathbb{N}\}$.
- D' is closed under the $(\mathcal{A}_{2n_j}, 1/\sqrt{m_j})$ operations for all $j < j_0$ and under the $(\mathcal{A}_{2n_j}, 1/m_j)$ operations for all $j \geq j_0$.
- D' is closed under the $(\mathcal{A}_{m_{j_0}^2}, 1/2)$ operation.
- D' is rationally convex.

For each $i \in \{1, \ldots, d\}$ we set

$$\widetilde{z}_i = \frac{1}{n_{j_0}} \sum_{t=1}^{n_{j_0}} e_{k_{(t,i)}}.$$

Before proceeding we need a slight modification of Definition 3.7.

NOTATION 10.1. For every functional $y^* = \sum_{k=1}^r \beta_k x_k^* \in G_{gt}$ of type II we set

$$I_{y^*} = \{k \in \{1, \ldots r\} : x_k^* = Eb_i^* \text{ for some } i \in \{1, \ldots d\}$$
$$\text{and some interval } E \text{ of } \mathbb{N}\}.$$

DEFINITION 10.3. Let $f \in G_{gt}$ and $j_0 \in \mathbb{N}$ and $(b_i^*)_i$ be a finite collection of infinite special functionals. A family $(f_a)_{a \in A}$ is called a j_0-*tree analysis* of f *with respect to* $(b_i^*)_i$ if:

(1) A is a finite tree with a least element denoted by 0 and $f_a \in G_{gt}$ for all $a \in A$ with $f_0 = f$.

(2) For $a, b \in A$ with $a < b$ we have $\operatorname{ran} f_b \subset \operatorname{ran} f_a$.

(3) For $a \in A$ maximal we have $f_a \in \{\pm e_n : n \in \mathbb{N}\}$.

(4) For $a \in A$ not maximal, if we denote by S_a the immediate successors of a in A then f_a has one of the following forms:

- If f_a is of type I then $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$, $|S_a| \le n_{j_a}$ and $(f_s)_{s \in S_a}$ is a block sequence.
- If f_a is special then $f_a = E_a \sum_i f_i$, where E_a is a finite interval of $\mathbb{N}$ and $(f_i)_i$ is a special sequence. We set $F_a = \{i \in \mathbb{N} : \operatorname{ran} f_i \cap E_a \ne \emptyset\} = \{i_1^a, \ldots, i_{d_a}^a\}$ and $S_a = \{s_1, \ldots, s_{d_a}\}$ where $f_{s_j} = E_a f_{i_j}$ and $w(f_{s_j}) = w(f_{i_j})$ for all $j \in \{1, \ldots, d_a\}$. Finally, we write $f_a = \sum_{s \in S_a} f_s$.
- If f_a is of type II with $f_a = \sum_{k=1}^r \beta_k x_k^*$ then $S_a = \{s_1, s_2, s_3\}$ and $f_a = f_{s_1} + f_{s_2} + f_{s_3}$, where $f_{s_1} = \sum_{k \in I_a^c} \beta_k x_{k,<j_0}^*$, $f_{s_2} = \sum_{k \in I_a} \beta_k x_k^*$ and $f_{s_3} = \sum_{k \in I_a^c} \beta_k x_{k,\ge j_0}^*$. In addition, as in Definition 3.7, if two of the functionals f_{s_i}, $i = 1, 2, 3$, are zero then $f_a = \sum_{s \in S_a} a_s f_s$, where $(a_s)_{s \in S_a} \subset \mathbb{Q}$, $(f_s)_{s \in S_a}$ are special functionals with disjoint sets of indices and $\sum_{s \in S_a} a_s^2 \le 1$.
- If f_a is of type III then $f_a = \sum_{s \in S_a} r_s f_s$, $r_s \in \mathbb{Q}^+$ and $\sum_{s \in S_a} r_s = 1$.

Now let $f \in G_{gt}$ and $(f_a)_{a \in A}$ be a tree analysis of f as above. Define $A_{(t,i)}$ to be the set of all $a \in A$ such that:

- $\operatorname{ran} f_a \cap \operatorname{ran} x_{(t,i)} \ne \emptyset$.
- For every $b \sqsubseteq a$ with $b \in S_u$ such that $f_u \in \mathcal{S}$ or f_u is of type I we have $\operatorname{ran} f_u \cap \operatorname{ran} x_{(t,i)} = \operatorname{ran} f_b \cap \operatorname{ran} x_{(t,i)}$.
- There exists no $b \sqsubset a$ such that $f_b = Eb_i^*$ for an interval E and $i \in \{1, \ldots, d\}$ or $b \in S_u$, f_u is of type II and $f_b = \sum_{k \in I_a^c} \beta_k x_{k, \ge j_0}^*$.

- Either f_a is of type 0 or I or special and $\operatorname{ran} f_b \cap \operatorname{ran} x_{(t,i)} \neq \operatorname{ran} f_a \cap \operatorname{ran} x_{(t,i)}$ for every $b \in S_a$, or $f_a = \sum_{k \in I_a^c} \beta_k x_{k,\geq j_0}^*$, or $f_a = \sum_{k \in I_a} \beta_k x_k^*$ and $a \in S_u$ and f_u is of type II.

For each $a \in A$ we set

$$D_a^i = \{t \in \{1, \ldots, n_{j_0}\} : \exists b \sqsupset a \text{ with } b \in A_{(t,i)}\}, \quad i = 1, \ldots, d,$$

$$D_a = \bigcup_{i=1}^d D_a^i,$$

$$E_a^i = \{t \in \{1, \ldots, n_{j_0}\} : a \in A_{(t,i)}\}, \quad E_a = \bigcup_{i=1}^d E_a^i,$$

$$F_a = \{i \in \{1, \ldots, d\} : D_a^i \neq \emptyset\}, \quad H_a = \{i \in \{1, \ldots, d\} : E_a^i \neq \emptyset\}.$$

PROPOSITION 10.2. *Let $f \in G_{gt}$. Then there exist $g_1, g_2, g_3, g_4 \in c_{00}(\mathbb{N})$ with nonnegative coordinates satisfying*

$$g_1, g_2, g_3 \in D', \quad \|g_4\|_\infty \leq \frac{1204}{\sqrt{m_{j_0}}},$$

such that

$$\left| f\left(\sum_{i=1}^d a_i z_i \right) \right| \leq 4C(g_1 + g_2 + g_3 + g_4)\left(\sum_{i=1}^d |a_i| \widetilde{z_i} \right) \quad (C = 1200).$$

Proof. We observe that

$$\sum_{i=1}^d a_i z_i = \frac{1}{n_{j_0}} \left(\sum_{t=1}^{n_{j_0}} \sum_{i=1}^d b_{(t,i)} x_{(t,i)} \right)$$

where $b_{(t,i)} = a_i$ for $t = 1, \ldots, n_{j_0}$. Let $f \in G_{gt}$ and $(f_a)_{a \in A}$ be a tree analysis of f. We will recursively construct for each $a \in A$ functionals $g_1^a, g_2^a, g_3^a, g_4^a \in c_{00}(\mathbb{N})$ such that:

- $\operatorname{supp} g_i^a \subseteq D_a$ for $i = 1, 2, 3, 4$ and $g_i^a \in D'$.
- $\|g_4^a\|_\infty \leq 1204/\sqrt{m_{j_0}}$.

- $$\left| f_a\left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right|$$

$$\leq 4C(g_1^a + g_2^a + g_3^a + g_4^a)\left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right).$$

In case f_a is of type I with $w(f_a) = m_{j_a} < m_{j_0}$ we have the stronger conditions

$$\|g_4^a\|_\infty \leq \frac{1}{m_{j_a}} \frac{1204}{\sqrt{m_{j_0}}}, \quad w(f_a) = w(g_1^a) = w(g_3^a) = w(g_3^a).$$

The proof is by induction. We present the proof without considering restrictions to intervals, as for those one can apply the same techniques used in Proposition 6.1.

Let $a \in A$ be a maximal node. Then if $D_a = \emptyset$ we set $g_i^a = 0$ for $i = 1, 2, 3, 4$. If $D_a \neq \emptyset$ we can see that D_a is a singleton, say $D_a = \{(t_a, i_a)\}$. We set $g_1^a = e_{k_{(t_a, i_a)}}^*, g_2^a = 0, g_3^a = g_4^a = 0$ and the inequality is easily verified.

Let $a \in A$ be nonmaximal and suppose that $(g_i^b)_{i=1}^4$ have been defined for every $b \in A$ with $b \sqsupset a$ according to our inductive hypotheses. We distinguish the following cases:

CASE 1: f_a is of type I with $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$ and $j_a \geq j_0$. By the choice of $(x_{(t,i)})$ there exists at most one $(t_a, i_a) \in D_a$ such that $|f_a(x_{(t_a, i_a)})| > 601/\sqrt{m_{j_0}}$. Suppose without loss of generality that such a (t_a, i_a) exists. We set $g_1^a = \frac{1}{2} e_{k_{(t_a, i_a)}}^*$, $g_2^a = 0, g_3^a = 0$, and $g_4^a = \frac{601}{\sqrt{m_{j_0}}} \sum_{(t,i) \in D_a \setminus \{(t_a, i_a)\}} e_{k_{((t,i)}}^*$. The inequalities are easily verified.

Moreover, we can see that $g_1^a, g_2^a, g_3^a \in D'$, $\|g_4^a\|_\infty \leq 601/\sqrt{m_{j_0}}$ and $\operatorname{supp} g_i^a \subseteq D_a$, $i = 1, 2, 3, 4$.

CASE 2: f_a is of type I with $f_a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} f_s$ and $j_a < j_0$. We enumerate S_a as $\{s_1^a < \cdots < s_r^a\}$; we know that $|S_a| \leq n_{j_a}$. We can see that $D_a = E_a \cup \bigcup_{s \in S_a} D_s$ and $|E_a| \leq n_{j_a}$. By Remark 6.1, $(D_{s_i^a})_{i=1}^r$ are successive subsets of $\mathbb{N}$ and thus $(g_1^{s_i})_{i=1}^r$, $(g_2^{s_i})_{i=1}^r$, $(g_3^{s_i})_{i=1}^r$ and $(g_4^{s_i})_{i=1}^r$ are block sequences. By the choice of $(x_{(t,i)})$ we have $|f_a(x_{(t,i)})| \leq 600/\sqrt{m_{j_a}}$ for all (t, i). Set

$$g_1^a = \frac{1}{\sqrt{m_{j_a}}} \Big(\sum_{(t,i) \in E_a} e_{k_{(t,i)}}^* + \sum_{s \in S_a} g_1^s \Big),$$

$$g_2^a = \frac{1}{\sqrt{m_{j_a}}} \sum_{s \in S_a} g_2^s, \quad g_3^a = \frac{1}{\sqrt{m_{j_a}}} \sum_{s \in S_a} g_3^s, \quad g_4^a = \frac{1}{2m_{j_a}} \sum_{s \in S_a} g_4^s.$$

We obtain the following stronger inequality:

$$(6) \quad \left| f_a \Big(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \Big) \right|$$

$$\leq 4C \Big(\frac{1}{2} g_1^a + \frac{1}{2} g_2^a + \frac{1}{2} g_3^a + g_4^a \Big) \Big(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \Big).$$

We can also verify that $w(g_1^a) = w(g_2^a) = w(g_3^a) = w(f_a)$. At the same time

$$\|g_4^a\|_\infty \leq \frac{1}{2} \frac{1204}{m_{j_a} \sqrt{m_{j_0}}}.$$

CASE 3: f_a is a type III functional, i.e. $f_a = \sum_{s \in S_a} r_s f_s$ with $r_s \in \mathbb{Q}^+$ and $\sum_{s \in S_a} r_s = 1$. In this case we set $g_i^a = \sum_{s \in S_a} r_s g_i^s$ for $i = 1, \ldots, 4$. All the desired properties can be readily verified.

CASE 4: f_a is a special functional. Then $f_a = \sum_{s \in S_a} f_s$. We distinguish the following subcases:

SUBCASE (i): ind $f_a \cap \bigcup_i$ ind $b_i^* = \emptyset$. Set $S_a^1 = \{s \in S_a : \text{ind } f_s < j_0\}$ and $S_a^2 = S_a \setminus S_a^1$. We can observe that $|S_a^1| \leq j_0$. Let (t_a, i_a) be such that there exist $s_1 \in S_a^1$ and $s_2 \in S_a^2$ satisfying $\operatorname{ran} x_{(t_a, i_a)} \cap \operatorname{ran} f_{s_i} \neq \emptyset$ for $i = 1, 2$. We can assume that such a (t_a, i_a) exists. We define

$$D_{<j_0} = \{(t, i) \in D_a : x_{(t,i)} < x_{(t_a, i_a)}\}, D_{>j_0} = \{(t, i) \in D_a : x_{(t,i)} > x_{(t_a, i_a)}\}.$$

By Remark 5.7 the set $\{s \in S_a^1 : \exists (t, i) \in D_{<j_0}, \operatorname{ran} f_s \cap \operatorname{ran} x_{(t,i)} \neq \emptyset\}$ contains at most one element. We assume that all the aforementioned sets are nonempty and we set $\{s \in S_a^1 : \exists (t, i) \in D_{<j_0}, \operatorname{ran} f_s \cap \operatorname{ran} x_{(t,i)} \neq \emptyset\} = \{s_0\}$. We have

$$D_a = D_{<j_0} \cup D_{>j_0} \cup \{(t_a, i_a)\}.$$

Set $L_a = \{(t, i) \in D_{>j_0} : |f_a(x_{(t,i)})| > 1202/\sqrt{m_{j_0}}\}$. Then L_a contains at most two elements. We set

$$g_1^a = \frac{1}{2}\left(g_1^{s_0} + e_{k_{(t_a, i_a)}}^* + \sum_{(t,i) \in L_a} e_{k_{(t,i)}}^*\right),$$

$$g_i^a = g_i^{s_0} \quad \text{for } i = 2, 3,$$

$$g_4^a = g_4^{s_0} + \frac{601}{\sqrt{m_{j_0}}} \sum_{(t,i) \in D_{>j_0} \setminus L_a} e_{k_{(t,i)}}^*.$$

The desired properties of the functionals g_i^a, $i = 1, 2, \ldots$, can be readily verified.

In addition we record the following stronger inequality:

$$(7) \quad \left|f_a\left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)}\right)\right|$$

$$\leq 4C\left(g_1^a + \frac{1}{2}g_2^a + \frac{1}{2}g_3^a + g_4^a\right)\left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}}\right).$$

SUBCASE (ii): There exists $i_0 \in \{1, \ldots, d\}$ and an interval E such that $x^* = E b_{i_0}^*$. By the bimonotonicity of the basis of $\mathfrak{X}_{gt}$ for each (t, i) we have

$$|f_a(x_{(t,i)})| \leq |b_{i_0}^*(x_{(t,i)})|.$$

Moreover, by the definition of each $A_{(t,i)}$ we know that $g_i^s = 0$ for all $s \in S_a$. Suppose that $D_a \neq \emptyset$. By the choice of $(x_{(t,i)})_{(t,i)}$, if we set

$$g_2^a = s_{i_0}^* = \sum_{t=1}^{n_{j_0}} e_{k_{(t,i_0)}}^*, \qquad g_i^a = 0 \quad \text{for } i = 1, 3,$$

$$g_4^a = \frac{1}{m_{j_0}^2} \sum_{(t,i) \in D_a^{i_0}} e_{k_{(t,i_0)}}^*,$$

we obtain all the desired properties and in addition

$$\left| f_a \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right|$$

$$\leq 4C \left(g_1^a + \frac{1}{2} g_2^a + g_3^a + g_4^a \right) \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right).$$

Note again that the $\frac{1}{2}$ in front of g_2^a is important for this case. Finally, $\|g_4^a\|_\infty \leq 1/m_{j_0}^2$.

CASE 5: f_a is a type II functional. We have the following subcases:

SUBCASE A: ind $f_a < j_0$ and $f_a = \sum_{s \in S_a} a_s f_s$ where $\sum_{s \in S_a} a_s^2 \leq 1$ and $(f_s)_s$ are special functionals with disjoint sets of indices. By the previous case each g_i^s for $i = 1, 2, 3$ is a functional in D' with weight and all these weights are different. Moreover,

$$\|g_4^s\|_\infty \leq \frac{1}{2w(g_1^s)} \frac{1204}{\sqrt{m_{j_0}}}$$

for all $s \in S_a$. Hence if we set

$$g_i^a = \sum_{s \in S_a} a_s g_i^s \quad \text{for } i = 1, 2, 3, \qquad g_4^a = \sum_{s \in S_a} a_s g_4^s,$$

we find that $g_i^a \in D'$ for $i = 1, 2, 3$ and

$$\|g_4^s\|_\infty \leq \frac{1}{2} \sum_{s \in S_a} \frac{1}{m_{j_s}} \frac{1204}{\sqrt{m_{j_0}}} \leq \frac{602}{\sqrt{m_{j_0}}}.$$

Finally, by (7),

$$(8) \quad \left| f_a \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right|$$

$$\leq 4C \left(\frac{1}{2} g_1^a + \frac{1}{2} g_2^a + \frac{1}{2} g_3^a + g_4^a \right) \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right).$$

We note that the coefficient $\frac{1}{2}$ in front of g_i^a, $i = 1, 2, 3$, is crucial for this subcase.

SUBCASE B: ind $f_a \geq j_0$ and $f_a = \sum_{s \in S_a} a_s f_s$ where $\sum_{s \in S_a} a_s^2 \leq 1$, $(x_s^*)_s$ are special functionals with disjoint sets of indices and incomparable to the

functionals $(b_i^*)_{i=1}^d$. Then either $D_a = \emptyset$ in which case we set $g_i^a = 0$ for $i = 1, \ldots, 4$, or $D_a = E_a$. If $D_a \neq \emptyset$ we set $L_a^i = \{t \in D_a^i : |f_a(x_{(t,i)})| \geq 602/\sqrt{m_{j_0}}\}$. We know that $|L_a^i| \leq 5 \cdot 720^2 m_{j_0}$ for all $i = 1, \ldots, d$. We set $L_a = \bigcup_i L_a^i$,

$$g_1^a = g_2^a = 0, \quad g_3^a = \frac{1}{2} \sum_{(t,i) \in L_a} e_{k_{(t,i)}}^*, \quad g_4^a = \frac{602}{\sqrt{m_{j_0}}} \sum_{k \in D_a \setminus L_a} e_{k_{(t,i)}}^*.$$

We can see that $g_3^a \in D'$ and $\|g_4^a\|_\infty \leq 602/\sqrt{m_{j_0}}$, and supp $g_i^a \subseteq D_a$ for all $i = 1, \ldots, 4$. Finally,

$$\left| f_a \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right|$$
$$\leq 4C \left(g_1^a + g_2^a + \frac{1}{2} g_3^a + g_4^a \right) \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right).$$

SUBCASE C: f_a is of the form $f_a = \sum_{s \in S_a} a_s f_s$ and for every $s \in S_a$ there exists $i_s \in \{1, \ldots, d\}$ such that $f_s = E_s b_{i_s}^*$. We set $g_1^a = g_3^a = 0$, $g_2^a = \sum_{s \in S_a} a_s g_2^s$ and $g_4^a = \sum_{s \in S_a} g_4^s$. Then $g_2^a \in D'$ as the g_2^s are of the form $s_{i_s}^*$ and have disjoint supports. The following inequality holds:

$$\left| f_a \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right|$$
$$\leq 4C \left(g_1^a + \frac{1}{2} g_2^a + g_3^a + g_4^a \right) \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right)$$

We can also observe that $\|g_4^a\|_\infty \leq 1/m_{j_0}^2$.

SUBCASE D: $f_a = f_{s_1} + f_{s_2} + f_{s_3}$ where $(f_{s_i})_{i=1}^3$ are as in Definition 10.3. We have $D_a = D_{s_1} \cup D_{s_2} \cup D_{s_3}$ and for every $(t,i) \in D_a$, ran $x_{(t,i)} \cap$ ran $f_{s_i} \neq \emptyset$ if and only if $(t,i) \in D_{s_i}$ for $i = 1, 2, 3$. Thus if we set $g_1^a = \frac{1}{2} g_1^{s_1}$, $g_2^a = \frac{1}{2}(g_2^{s_2} + g_2^{s_1})$, $g_3^a = \frac{1}{2}(g_3^{s_1} + g_3^{s_3})$ and $g_4^a = g_4^{s_1} + g_4^{s_2} + g_4^{s_3}$ we obtain $g_i^a \in D'$ for $i = 1, 2, 3$ and $\|g_4^a\|_\infty \leq 1204/\sqrt{m_{j_0}}$, and supp $g_i^a \subseteq D_a$ for $i = 1, 2, 3, 4$. In addition, it can be readily verified that

$$\left| f_a \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} b_{(t,i)} x_{(t,i)} \right) \right| \leq 4C(g_1^a + g_2^a + g_3^a + g_4^a) \left(\frac{1}{n_{j_0}} \sum_{(t,i) \in D_a} |b_{(t,i)}| e_{k_{(t,i)}} \right).$$

The induction is complete. ■

An immediate consequence of Proposition 10.1 is

PROPOSITION 10.3. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{gt}$. Then Y^* contains an isomorphic copy of ℓ_2.*

Proof. By repeated application of Proposition 7.1 one can construct for each $i \in \mathbb{N}$ a 6-dependent sequence $(y_n^i, f_n^i)_n$ such that $y_n^i \in Y$ for all $n, i \in \mathbb{N}$ and $f_n^i \in Y^*$ with the additional property that the functionals $f^i = \sum_n f_n^i$ are incomparable. By Proposition 10.1 the sequence $(f^i)_i \subset Y^*$ is equivalent to the standard ℓ_2 basis. ∎

The above proposition implies that no subspace of $\mathfrak{X}_{gt}$ is quotient HI. More precisely, we have

THEOREM 10.1. *Every closed infinite-dimensional subspace of $\mathfrak{X}_{gt}$ has ℓ_2 as a quotient space.*

At this point we would like to illustrate the differences in behavior between quotients of $\mathfrak{X}_{gt}$ by an arbitrary subspace Y and those by block subspaces. In particular, one can show that for every w^*-closed subspace Z of $\mathfrak{X}_{gt}$ with infinite codimension the quotient $\mathfrak{X}_{gt}/Z$ is HI. The proof requires the next two lemmas.

LEMMA 10.1. *Let Z be a w^*-closed subspace of $\mathfrak{X}_{gt}$ and Y a closed subspace of $\mathfrak{X}_{gt}$ such that $Z \subset Y$ and Y/Z is infinite-dimensional. Then for all $\epsilon > 0$ and $m, k \in \mathbb{N}$ there exists a 2-ℓ_1^k average x in $\langle e_i : i \geq m \rangle$ with $\mathrm{dist}(x, Y) < \epsilon$ and $f \in B_{(\mathfrak{X}_{gt})*}$ such that $\mathrm{dist}(f, Z^\perp) < \epsilon$ and $f(x) > 1$, where we have set $Z^\perp = \{f \in (\mathfrak{X}_{gt})_* : f(z) = 0 \text{ for all } z \in Z\}$.*

LEMMA 10.2. *Let Y and Z be as in the previous lemma. Then for every $j \in \mathbb{N}$ and every $\epsilon > 0$ there exists a $(6, j)$ exact pair (y, f) with $\mathrm{dist}(y, Y) < \epsilon$ and $\mathrm{dist}(f, Z^\perp) < \epsilon$.*

We omit the proofs as they are identical to those of Lemmas 2.19 and 2.20 in [AAT]. The above yields

PROPOSITION 10.4. *Let Z be an infinite-dimensional w^*-closed subspace of $\mathfrak{X}_{gt}$ of infinite codimension. Then $\mathfrak{X}_{gt}/Z$ is HI.*

Proof. Let $Q : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}/Z$ be the quotient map. Let Y_1, Y_2 be two subspaces of $\mathfrak{X}_{gt}$ such that Z is a subspace of $Y_1 \cap Y_2$ and is of infinite codimension in each Y_i. Let $\epsilon > 0$ and choose a sequence $(\epsilon_k)_k$ of positive numbers such that $\sum_k \epsilon_k < \epsilon/2$. Then, by Lemmas 10.2 and 7.2, there exists a 6-dependent sequence $(x_k, f_k)_k$ such that

$$\mathrm{dist}(x_{2k-1}, Y_1) < \frac{\epsilon_k}{2\|Q\|}, \quad \mathrm{dist}(x_{2k}, Y_2) < \frac{\epsilon_k}{2\|Q\|}, \quad \mathrm{dist}(f_k, Z^\perp) < \epsilon_k,$$

for all $k \in \mathbb{N}$, and in addition $(x_{2k-1} - x_{2k})_k$ is weakly null. Choose a convex combination $u = \lambda_1 x_{2k_1-1} - w_{2k_1} + \cdots + \lambda_n x_{2k_n-1} - w_{2k_n}$ with $k_1 < \cdots < k_n$ with $\|u\|_{\mathfrak{X}_{gt}} < \epsilon/2\|Q\|$. Set $w_1 = \sum_{i=1}^n \lambda_i x_{2k_i-1}$, $w_2 = \sum_{i=1}^n \lambda_k x_{2k_i}$ and $\widehat{w_i} = Q(w_i)$ for $i = 1, 2$. By the choice of w_i, we obtain $\|\widehat{w_1} - \widehat{w_2}\| < \epsilon/2$. Set $f = \sum_k x_k^* \in G_{gt}$ and observe that $\mathrm{dist}(f, Z^\perp) < \epsilon/2$ so we may choose $x^* \in Z^\perp$ such that $\|f - x^*\| < \epsilon/2$. Moreover, $\|\widehat{w_1} + \widehat{w_2}\| \geq x^*(\widehat{w_1} + \widehat{w_2}) \geq$

$f(\widehat{w}_1 + \widehat{w}_2) - \epsilon/2 = 1 - \epsilon/2$. This shows that there exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $\|Q(y_1) - Q(y_2)\| < \epsilon\|Q(y_1) + Q(y_2)\|$, which implies that $\mathfrak{X}_{gt}/Z$ is HI. ∎

We now show that the second dual of every infinite-dimensional closed subspace of $\mathfrak{X}_{gt}$ contains an isomorphic copy of $\ell_2(2^\omega)$.

PROPOSITION 10.5. *Let Y be a block subspace of $\mathfrak{X}_{gt}$. Then Y^{**} contains an isomorphic copy of $\ell_2(2^\omega)$.*

Proof. By recursive application of Lemma 7.1 we construct a special tree $T = (w_t, f_t, j_t)_{t \in 2^{<\omega}}$ in Y with the additional property that for every $n \in \mathbb{N}$ if we order the set $\{t \in 2^{<\omega} : |t| = n\}$ lexicographically as $(t_i)_{i=1}^{2^n}$ then $w_{t_i} < w_{t_i'}$ whenever $t_i <^{\text{lex}} t_i'$ and $|t_i| = |t_i'| = n$. We know by Proposition 7.2 that for each $b \in 2^\omega$ the sequence $(w_{b|n})_n$ is nontrivial weakly Cauchy. We set

$$w_b^{**} = \lim_n w_{b|n}, \qquad w_b^* = \sum_n f_{b|n}, \qquad b \in 2^\omega,$$

where the limits are taken with respect to the w^* topology in Y^{**} and Y^* respectively. We claim that the family $(w_b^{**})_{b \in 2^\omega}$ generates $\ell_2(c)$. Let $F = \{b_1, \ldots, b_d\}$ be a finite subset of 2^ω and $a_1, \ldots, a_d$ scalars with $\sum_{i=1}^d a_i^2 = 1$. Notice that for $b_1 \neq b_2 \in 2^\omega$ we have $w_{b_1}^{**}(w_{b_2}^*) = 0$ as $\lim_n w_{b_1}^*(w_{b_2|n}) = 0$ while $w_{b_1}^{**}(w_{b_1}^*) = 1$, by the choice of the special tree. Therefore, by choosing $n_0 \in \mathbb{N}$ such that the functionals $w_{b_i|>n_0}^* = \sum_{n=n_0+1}^\infty f_{b_i|n}$ are mutually incomparable we see that $\|\sum_{i=1}^d a_i w_{b_i|n>n_0}^*\| \leq 1$ and

$$\Big\| \sum_{i=1}^d a_i w_{b_i}^{**} \Big\| \geq \Big(\sum_{i=1}^d a_i w_{b_i}^{**} \Big) \Big(\sum_{i=1}^d a_i w_{b_i|n>n_0}^* \Big) = \sum_{i=1}^d a_i^2.$$

Now as in the proof of Proposition 10.1 we can construct for each $i \in \{1, \ldots, d\}$ a sequence $(z_n^i)_n$ of successive averages of $(w_{b_i|n})_n$ so that by Proposition 10.2, for every $n \in \mathbb{N}$,

$$\Big\| \sum_{i=1}^d a_i z_n^i \Big\|_{gt} \leq 14400.$$

As for each $i = 1, \ldots, d$ the sequence $(z_n^i)_n$ w^*-converges again to $w_{b_i}^{**}$ we deduce that

$$\Big\| \sum_{i=1}^d a_i w_{b_i}^{**} \Big\| \leq 14400. \quad ∎$$

11. Properties of the predual $(\mathfrak{X}_{gt})_*$. In this section we study the structure of $(\mathfrak{X}_{gt})_*$. We show that this space is HI and that every bounded

linear operator $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is of the form $\lambda I + S$ where S is strictly singular. We start with

DEFINITION 11.1. Let $k \in \mathbb{N}$ and x^* be a finitely supported vector in $(\mathfrak{X}_{gt})_*$. We say that x^* is an M-c_0^k *vector* if:

- There exist $x_1^* < \cdots < x_k^* \in \langle e_n^* : n \in \mathbb{N} \rangle$ with $x^* = x_1^* + \cdots + x_k^*$.
- $\|x_i^*\| > 1/M$ for each $i = 1, \ldots, k$.
- $\|x^*\| \leq 1$.

The following lemma is an application of Ramsey's theorem. For a detailed exposition we refer to [ATO].

LEMMA 11.1. *Let Z be a block subspace of $(\mathfrak{X}_{gt})_*$ and $k \in \mathbb{N}$. There exists a block sequence $(z_n^*)_n \subset Z$ such that for every $i_1 < \cdots < i_k$ the sum $z_{i_1}^* + \cdots + z_{i_k}^*$ is a 2-c_0^k vector.*

We also make use of

LEMMA 11.2. *Let $f \in \langle e_n^* : n \in \mathbb{N} \rangle$ with $\|f\| \leq 1$ and $\epsilon > 0$. Then there exists $g \in G_{gt}$ with $\|f - g\| < \epsilon$ and $\operatorname{ran} g \subset \operatorname{ran} f$.*

THEOREM 11.1. *The space $(\mathfrak{X}_{gt})_*$ is HI.*

Proof. Let Z and U be block subspaces of $(\mathfrak{X}_{gt})_*$ and let $\epsilon > 0$. We shall show that there exist $g_Z \in Z$ and $h_U \in U$ such that

$$\|g + h\| < \epsilon \|g - h\|.$$

To do so we will construct a 6-dependent sequence $(w_k, f_k)_k$ such that $\operatorname{dist}(Z, f_{2k-1}) < \epsilon_{2k-1}$ and $\operatorname{dist}(U, f_{2k}) < \epsilon_{2k}$ where $\epsilon_k > 0$ and $\sum_k \epsilon_k < 1/2$. Let $j_1 \in \Omega_1$. There exist $z_{1,1}^* < \cdots < z_{1,n_1}^*$ in Z such that $z_1^* = z_{1,1}^* + \cdots + z_{1,n_1}^*$ is a 2-$c_0^{n_1}$ vector. Since $\|z_{1,i}^*\| > 1/2$ we can choose $z_{1,i} \in B_{\mathfrak{X}_{gt}}$ for $i = 1, \ldots, n_1$ such that $z_{1,i}^*(z_{1,i}) > 1/2$ and $\operatorname{ran} z_{1,i} = \operatorname{ran} z_{1,i}^*$. We set

$$z_1 = \frac{1}{n_1} (z_{1,1} + \cdots + z_{1,n_1})$$

and observe that $z_1^*(2z_1) > 1$ and $\operatorname{ran} z_1 = \operatorname{ran} z_1^*$. By Lemma 11.2 there exists $g_1 \in G_{gt}$ such that $\operatorname{ran} g_1 \subset \operatorname{ran} z_1^*$ and

$$\|z_1^* - g_1\| < \min \left\{ \frac{2m_{j_1}}{n_{j_1}} \epsilon_1, z_1^*(z_1) - \frac{1}{2} \right\}.$$

Observe that $g_1(2z_1) > 1$ and $\operatorname{dist}(g_1, Z) < \frac{2m_{j_1}}{n_{j_1}} \cdot \epsilon_1$.

Proceeding similarly we construct a double sequence $(2z_i, g_i)_i$ such that each $2z_i$ is a 2-$\ell_1^{n_i}$ average and $(n_i)_i$ is strictly increasing. By passing to a subsequence we may assume that $(2z_i)_i$ is j_1-separated and thus if we set

$$w_1 = \frac{2m_{j_1}}{n_{j_1}} \sum_{i=1}^{n_{j_1}} 2z_i, \qquad f_1 = \frac{1}{2m_{j_1}} \sum_{i=1}^{n_{j_1}} g_i,$$

then (w_1, f_1) is a $(6, j_1)$ exact pair and $\text{dist}(f_1, Z) \leq \epsilon_1$. In an analogous manner, we inductively construct a 6-dependent sequence $(w_k, f_k)_k$ such that $\text{dist}(Z, f_{2k-1}) < \epsilon_{2k-1}$ and $\text{dist}(U, f_{2k}) < \epsilon_{2k}$. The sequence $(w_{2k-1} - w_{2k})_k$ is weakly null and thus we can choose a finite convex combination

$$u_{n_0} = \lambda_1(w_{2k_1-1} - w_{2k_1}) + \cdots + \lambda_{n_0}(w_{2k_{n_0}-1} - w_{2k_{n_0}})$$

with $\|u_{n_0}\| < (2+\epsilon)/2\epsilon$. Set

$$f = f_{2k_1-1} + \cdots + f_{2k_{n_0}}, \quad g = \sum_{i=1}^{n_0} f_{2k_i-1}, \quad h = \sum_{i=1}^{n_0} f_{2k_i}.$$

We observe that $\text{dist}(Z, g) < 1/2$ and $\text{dist}(U, h) < 1/2$. Hence, there exist g_Z, h_U in Z, U respectively such that $\|g - g_Z\| < 1/2$ and $\|h - h_U\| < 1/2$. Observe that $\|g_Z + h_U\| < 2$ and $\|g_Z - h_U\| \geq \|g - h\| - 1$. Moreover,

$$\|g - h\| \geq \frac{(g-h)(u_{n_0})}{\|u_{n_0}\|} = \frac{2}{\|u_{n_0}\|} > \frac{2}{\epsilon} + 1.$$

Thus $\|g_Z - h_U\| > 2/\epsilon$ and the proof is complete. ∎

11.1. *The space of operators on* $(\mathfrak{X}_{gt})_*$. We now show that every bounded linear operator $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is of the form $\lambda I + W$, with W a weakly compact operator. We begin by showing that each $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is of the form $\lambda I + S$ with S strictly singular and then we prove that every strictly singular operator on this space is weakly compact. The techniques involved are quite similar to the ones used in [ATO]. For the results stated without proof we refer the interested reader to Paragraph IV.2 in [ATO]. We start with the following general lemma:

LEMMA 11.3. *Let X be an HI space with a basis $(e_n)_n$, and $T : X \to X$ a bounded linear operator on X. Suppose that T is not of the form $\lambda I + S$ with S strictly singular. Then there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\text{dist}(T(z), \langle z \rangle) \geq \delta \|z\|$ for every $z \in \langle e_n : n \geq n_0 \rangle$.*

LEMMA 11.4. *Let $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ be a bounded linear operator with $\|T\| = 1$. Suppose that $(Te_n^*)_n$ is a block sequence. If T is not of the form $\lambda I + S$ then for every $k \in \mathbb{N}$ and any block subspace Z of $(\mathfrak{X}_{gt})_*$ there exist $z^* \in Z$ with $\|z^*\| \leq 1$ and $z \in \mathfrak{X}_{gt}$ which is a $(2/\delta)$-ℓ_1^k average such that*

$$z^*(z) = 0, \quad (Tz^*)(z) > 1, \quad \text{ran } z \subset \text{ran } z^* \cup \text{ran } Tz^*.$$

Proof. Since T is not of the form $\lambda I + S$ with S strictly singular, Lemma 11.3 shows that there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\text{dist}(T(f), \langle f \rangle) \geq \delta \|f\|$ for every $f \in \langle e_n^* : n \geq n_0 \rangle$. Let Z be a block subspace of $(\mathfrak{X}_{gt})_*$ and $k \in \mathbb{N}$. By Lemma 11.1 there exists a block sequence $(z_i^*)_i$ in Z such that for any $i_1 < \cdots < i_k$ the element $z_{i_1}^* + \cdots + z_{i_k}^*$ is a 2-c_0^k vector. In addition

we suppose that $\operatorname{ran} z_1^* > n_0$. Furthermore, our assumptions yield

$$\operatorname{dist}(Tz_i^*, \langle z_i^* \rangle) \geq \delta \|z_i^*\| > \delta/2$$

for all i, and since the basis of $\mathfrak{X}_{gt}$ is boundedly complete and bimonotone, for every i there exists $z_i \in \mathfrak{X}_{gt}$ such that

$$\|z_i\| = 1, \quad z_i^*(z_i) = 0, \quad (Tz_i^*)(z_i) > 1, \quad \operatorname{ran} z_i \subset \operatorname{ran} z_i^* \cup \operatorname{ran} Tz_i^*.$$

As $(Te_n^*)_n$ has been assumed to be a block sequence, we can choose $i_i < \cdots < i_k$ such that $(\operatorname{ran} z_{i_j}^* \cup \operatorname{ran} Tz_{i_j}^*)_{j=1}^k$ is a block sequence. It is clear that the vectors

$$z = \frac{1}{k}\left(\frac{2}{\delta} z_{i_1} + \cdots + \frac{2}{\delta} z_{i_k}\right) \quad \text{and} \quad z^* = z_{i_1}^* + \cdots + z_{i_k}^*$$

satisfy the conclusion of the lemma. ∎

PROPOSITION 11.1. *Let $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ be a bounded linear operator. Then T is of the form $\lambda I + S$ with S a strictly singular operator.*

Proof. It is enough to consider an operator T such that $\|T\| = 1$. Suppose that T is not of the desired form and choose $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\operatorname{dist}(T(f), \langle f \rangle) \geq \delta \|f\| \quad \text{for every } f \in \langle e_n^* : n \geq n_0 \rangle.$$

Using the previous lemma and the fact that $e_n^* \xrightarrow{w} 0$ we can construct a double sequence $(z_k, z_k^*)_k$ such that:

- $z_k^*(z_k) = 0$ and $(Tz_k^*)(z_k) > 1$ for all $k \in \mathbb{N}$.
- $\operatorname{ran} z_k \subset \operatorname{ran} z_k^* \cup \operatorname{ran} Tz_k^*$.
- $(\operatorname{ran} z_k^* \cup \operatorname{ran} Tz_k^*)_k$ is a block sequence.
- Each z_k is a $(2/\delta)$-ℓ_1^k average.

Furthermore, by Lemma 11.2 we can assume, up to a sufficiently small perturbation, that $z_k^* \in G_{gt}$ for all $k \in \mathbb{N}$. Let now $j_1 \in \Omega_1$. We can assume by passing to subsequences if necessary that $(z_k)_k$ is j_1-separated. Thus if we set

$$w_1 = \frac{2m_{j_1}}{n_{j_1}} \sum_{k=1}^{n_{j_1}} z_k, \quad f_1 = \frac{1}{2m_{j_1}} \sum_{k=1}^{n_{j_1}} z_k^*,$$

we deduce that the pair (w_1, f_1) has the following properties:

P1. $\|w_1\|_{gt} \leq 300 \cdot 2/\delta$.
P2. f_1 is of type I and $w(f_1) = m_{j_1}$.
P3. $\operatorname{ran} f_1 \subset \operatorname{ran} w_1 \cup \operatorname{ran} Tw_1$ and $f_1(w_1) = 0$.
P4. If $\phi \in G_{gt}$ is of type I with $w(\phi) = m_i < m_{j_1}$ then $|\phi(w_1)| \leq \frac{72}{\sqrt{m_i}} \cdot \frac{2}{\delta}$.

Proceeding similarly we construct a double sequence $(w_i, f_i)_i$ such that the corresponding properties P1–P4 are satisfied for all $i \in \mathbb{N}$ and moreover

$(f_i)_i$ is a special sequence. A slight adaptation of the proof of Proposition 7.2 would also yield $(w_i)_i$ weakly null. Thus, pick a convex combination

$$u_{k_0} = \lambda_1 w_{i_1} + \cdots + \lambda_{k_0} w_{i_{k_0}}$$

such that $\|u_{k_0}\| < 1/2$ and compute

$$\Big\| T\Big(\sum_{l=1}^{i_{k_0}} f_l\Big)\Big\| \geq \frac{T(\sum_{l=1}^{i_{k_0}} f_l)(u_{k_0})}{\|u_{k_0}\|} = \frac{T(f_{i_1} + \cdots + f_{i_{k_0}})(u_{k_0})}{\|u_{k_0}\|} > \frac{1}{\|u_{k_0}\|} > 2$$

and at the same time since $\sum_{l=1}^{i_{k_0}} f_l \in G_{gt}$ and $\|T\| = 1$ we have

$$\Big\| T\Big(\sum_{l=1}^{i_{k_0}} f_l\Big)\Big\| \leq 1,$$

which is clearly a contradiction. ∎

We now show that every strictly singular operator $S : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is in addition weakly compact. We start with

PROPOSITION 11.2. *Let $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ be a strictly singular operator. Then $T^* : \mathfrak{X}_{gt} \to \mathfrak{X}_{gt}$ is also strictly singular.*

Proof. By Proposition 9.2, $T^* = \lambda I + W$ where W is strictly singular and weakly compact. To show that T^* is strictly singular we only need to prove that $\lambda = 0$. Consider $W^* : \mathfrak{X}_{gt}^* \to \mathfrak{X}_{gt}^*$. Then $W^* = T^{**} - \lambda I_{\mathfrak{X}_{gt}^*}$. The operator W^* restricted to $(\mathfrak{X}_{gt})_*$ is weakly compact. It is easily seen that every nonstrictly singular weakly compact operator must be an isomorphism on a reflexive subspace, and as $(\mathfrak{X}_{gt})_*$ does not contain a reflexive subspace, we conclude that W^* restricted to $(\mathfrak{X}_{gt})_*$ must be strictly singular. However, since T^{**} restricted to $(\mathfrak{X}_{gt})_*$ is equal to T we see that $W^* - T^{**} : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ is strictly singular. Therefore, $\lambda = 0$ and thus T^* is strictly singular. ∎

The above yields

THEOREM 11.2. *Let $T : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ be a bounded linear operator. Then $T = \lambda I + W$ where W is a strictly singular and weakly compact operator.*

Proof. Proposition 11.1 yields $\lambda \in \mathbb{R}$ and $S : (\mathfrak{X}_{gt})_* \to (\mathfrak{X}_{gt})_*$ strictly singular such that $T = \lambda I + S$. By Proposition 11.2, S^* is strictly singular and thus weakly compact. Thus, S is also weakly compact. ∎

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