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The theorem of the complement for a quasi subanalytic set

by

ABDELHAFED ELKHADIRI (Kénitra)

Dedicated to Professor Jean-Claude Tougeron

Abstract. Let $X \subset (\mathbb{R}^n, 0)$ be a germ of a set at the origin. We suppose X is described by a subalgebra, $C_n(M)$, of the algebra of germs of C^∞ functions at the origin (see 2.1). This algebra is quasianalytic. We show that the germ X has almost all the properties of germs of semianalytic sets. Moreover, we study the projections of such germs and prove a version of Gabrielov's theorem.

Introduction. The aim of this paper is to study germs, at the origin in $\mathbb{R}^n$, of some sets defined as finite unions of sets of the form

$$\{x \mid \varphi_0(x) = 0,\, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0\},$$

where $\varphi_0, \ldots, \varphi_q$ are elements of a subalgebra, say $C_n(M)$, of the algebra of C^∞ germs at the origin. We will call those germs quasi semianalytic germs and their projections quasi subanalytic. We suppose that our algebra contains the germs of real-analytic functions at the origin and is quasianalytic, that is, if $f \in C_n(M)$ is such that its Taylor series at the origin, say $T_0 f$, is zero, then the germ f is null. It is well known [3] that the Weierstrass division theorem does not hold in $C_n(M)$, and we do not know if this algebra is noetherian or not; so we cannot completely follow the methods used in the classical case, i.e. when $C_n(M)$ is the algebra of analytic germs, to study quasi semianalytic germs and their projections.

By using elementary blowings-up of $\mathbb{R}^n$ with smooth center, we can prove that by a finite number of blowings-up we can transform any $f \in C_n(M)$, modulo a product by an invertible element in $C_n(M)$, to a monomial (Proposition 7). This implies that $C_n(M)$ is topologically noetherian, that is, every decreasing sequence of germs is stationary. This property is enough for us

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to extend some well known properties of semianalytic germs (stratification, locally finite number of connected components, ...) to the quasi semianalytic germs. We also prove that the closure and each connected component of a quasi semianalytic germ are quasi semianalytic. The Tarski–Seidenberg theorem is not true in this class of germs, so in Section 8 we study the quasi subanalytic germs. The main results are Theorem 7, which gives a uniform bound of the number of connected components of the fibers of a projection restricted to a bounded quasi subanalytic set, and Lemma 7, which shows that the dimension of quasi semianalytic germs is well behaved.

Finally, we prove the complement theorem for quasi subanalytic germs. This theorem is also proved in [10] by J.-P. Rolin, P. Speissegger and A. J. Wilkie. Our approach is different. The normalization algorithm used in Section 2 of [10] is more complicated than the proof of our Proposition 7, and our way of introducing the class of functions is more convenient. We also have a theory of quasi semianalytic germs (Theorems 5, 6). Moreover, we prove the Łojasiewicz inequalities for functions in this class in the same way that was used in [11] for the Gevrey class.

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1. Background. Let n be a positive integer, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $x = (x_1, \ldots, x_n)$ the canonical coordinates on $\mathbb{R}^n$.

We use the standard notations: $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, and a preorder on $\mathbb{N}^n$ is defined by $\alpha = (\alpha_1, \ldots, \alpha_n) \le \beta = (\beta_1, \ldots, \beta_n) \Leftrightarrow \alpha_i \le \beta_i$, $\forall i = 1, \ldots, n$.

We say that a real function, m, of one real variable is C^∞ *for* $t \gg 0$ if there is $b > 0$ such that m is C^∞ in the interval $[b, \infty[$.

In all the following, m will be a C^∞ function for $t \gg 0$ with $m, m', m'' > 0$, $\lim_{t\to\infty} m'(t) = \infty$ and there is $\delta > 0$ such that $m''(t) \le \delta$ for $t \gg 0$. We put

$$M(t) = e^{m(t)}.$$

If $U \subset \mathbb{R}^n$ is an open subset, $\mathcal{E}(U)$ denotes the algebra of C^∞ functions on U.

2. Functions of class M

Definition 1. A function $f \in \mathcal{E}(U)$ is said to be in the class M if for each compact $K \subset U$, there are $C_K, \varrho_K > 0$ such that for all $x \in K$,

$$|D^\alpha f(x)| \le C_K \varrho_K^{|\alpha|} M(|\alpha|) \quad \text{for } |\alpha| \gg 0.$$

We let $C_U(M)$ be the collection of all C^∞ functions on U which are in the class M.

REMARK 1. Let $M_1(t) = cr^t M(t)$, where $c, r > 0$. We easily see that a function $f \in \mathcal{E}(U)$ is in the class M if and only if f is in the class M_1; hence the class does not change when $m(t)$ is replaced by $m(t) + at + b$, $a, b \in \mathbb{R}$; so we will suppose in the following that $m(0) = 0$.

In the following, if $m : [b, \infty[\to \mathbb{R}$, $m(b) = 0$, $b \geq 0$ and $m, m', m'' > 0$ in the interval $[b, \infty[$, we still denote by m the extension of m to $[0, \infty[$ obtained by setting $m(t) = 0$ if $t \leq b$. We see that this extension is convex.

LEMMA 1. *For all $j \in \mathbb{N}$, $j \gg 0$, there exist $C_j, \varrho_j > 0$ with*

$$M(p + j) \leq C_j \varrho_j^p M(p), \qquad \forall p \in \mathbb{N},\ p \gg 0.$$

Proof. There exists $\theta \in \,]p, p + j[$ such that $m(p + j) - m(p) = jm'(\theta)$. Since $m'' \leq \delta$, there exists $C > 0$ with $m'(t) \leq \delta t + C$. We have $m'(\theta) \leq m'(p+j) \leq \delta p + (C + \delta j)$. Put $\varrho_j = e^{j\delta}$ and $C_j = e^{j(\delta j + C)}$; then $M(p+j) \leq C_j \varrho_j^p M(p)$.

LEMMA 2. *$C_U(M)$ is an algebra, closed under differentiation.*

Proof. Since m is convex and $m(0) = 0$, for $0 \leq j \leq n$ we have

$$m(n - j) \leq \frac{n - j}{n}\, m(n), \qquad m(j) \leq \frac{j}{n}\, m(n),$$

hence $m(j) + m(n - j) \leq m(n)$, i.e. $M(n - j)M(j) \leq M(n)$. Using this inequality and the Leibniz formula, we deduce the first statement of the lemma. The second statement follows immediately from Lemma 1.

The following theorem gives a one-dimensional characterization of functions in the class M and can be considered as an extension of a result in [4].

Let Ω be an open subset of the sphere $S^{n-1} \subset \mathbb{R}^n$ $(n > 1)$ and $f \in \mathcal{E}(U)$. We suppose that the following condition on f is satisfied:

(∗) For each $\xi \in \Omega$ and each compact subset $K \subset U$, there exists a constant $C_{K,\xi} > 0$ such that

$$\left| \frac{d^m}{dt^m} f(x + t\xi)_{|t=0} \right| \leq C_{K,\xi} M(m) \qquad \forall x \in K,\ \forall m \in \mathbb{N}.$$

THEOREM 1. *Let $f \in \mathcal{E}(U)$ and suppose that condition (∗) is satisfied. Then $f \in C_U(M)$.*

Proof. Let $K \subset U$ be a fixed compact set. For each $\xi \in \Omega$, we put

$$\theta_m(\xi) = \sup_{x \in K} \frac{\left| \frac{d^m}{dt^m} f(x + t\xi)_{|t=0} \right|}{M(m)}, \qquad m \in \mathbb{N}, \qquad \theta(\xi) = \sup_{m \in \mathbb{N}} \theta_m(\xi).$$

Then θ is a lower semicontinuous function and by Baire's theorem, there exists an open subset $\Omega_1 \subset \Omega$ and a constant $C_1 > 0$ such that

$$\forall \xi \in \Omega_1, \qquad \theta(\xi) \leq C_1.$$

We have

$$\frac{\partial^m f}{\partial \xi^m}(x) := \frac{d^m}{dt^m} f(x+t\xi)_{|t=0} = \sum_{|\omega|=m} D^\omega f(x)\,\frac{m!}{\omega_1!\ldots\omega_n!}\,\xi_1^{\omega_1}\ldots\xi_n^{\omega_n}.$$

Since Ω_1 is open in S^{n-1}, by a result of [6], there exists a constant $C_2 > 0$ such that

$$\sup_{|\xi|=1}\left|\frac{\partial^m f}{\partial \xi^m}(x)\right| \le C_2^m \sup_{\xi\in\Omega_1}\left|\frac{\partial^m f}{\partial \xi^m}(x)\right|.$$

In view of Bernstein's inequality, there exists a constant $C_3 > 0$ such that

$$C_3^m \sup_{|\omega|=m} |D^\omega f(x)| \le \sup_{|\xi|=1}\left|\frac{\partial^m f}{\partial \xi^m}(x)\right|.$$

Putting $\varrho = C_2/C_3$, we have

$$\sup_m \sup_{\substack{x\in K\\ |\omega|=m}} \frac{|D^\omega f(x)|}{M(m)\varrho^m} < \infty,$$

hence $f \in C_U(M)$.

REMARK 2. If $M(t) = t^t$, i.e. $m(t) = t\log t$, we have the analytic class. In the following we will consider M such that the class $C_U(M)$ strictly contains the analytic class. We therefore take m of the form

$$m(t) = t\log t + t\mu(t),$$

where μ is increasing and $\lim_{t\to\infty}\mu(t) = \infty$. In order to have $m''(t) \le \delta$, we must suppose that $\mu(t) \le at$ for $t \gg 0$ $(a > 0)$. We also suppose that μ is in a Hardy field.

PROPOSITION 1. *$C_U(M)$ is closed under composition.*

Proof. Since $t \mapsto t\mu(t)$ is convex, the proposition follows from [3].

Proposition 1 shows that we can define $C_X(M)$ by means of a local coordinate system when X is a real-analytic manifold.

Let $t \mapsto M(t)$ be as above and for $s \in \mathbb{R}_+$, put

$$\Lambda(s) = \inf_{t\ge t_0} M(t)s^{-t},$$

where t_0 is a fixed positive real. The infimum is reached at a point t where $m'(t) = \log s$, and this point is unique since m' is increasing and $\lim_{t\to\infty} m'(t) = \infty$. We define $s \mapsto \omega(s)$ via $\Lambda(s) = e^{-\omega(s)}$. Then

$$\begin{cases} s = e^{m'(t)}, \\ \omega(s) = tm'(t) - m(t), \end{cases}$$

or

$$\begin{cases} s = ete^{\mu(t)+t\mu'(t)}, \\ \omega(s) = t + t^2\mu'(t). \end{cases}$$

Since $\mu' > 0$, we have $\omega > 0$ and $\lim_{t\to\infty}\omega(s) = \infty$. We can easily invert the last system to obtain

$$\begin{cases} t = s\omega'(s), \\ m(t) = s\omega'(s)\log s - \omega(s). \end{cases}$$

Since $m(t) = t\log t + t\mu(t)$, we have

$$\begin{cases} t = s\omega'(s), \\ \mu(t) = -\log\omega'(s) - \dfrac{\omega(s)}{s\omega'(s)}. \end{cases}$$

We see that ω is increasing and as $t \to \infty$, $s\omega'(s) \to \infty$ and $-\log\omega'(s) - \omega(s)/(s\omega'(s)) \to \infty$, so ω' is decreasing and $\omega'(s) \to 0$ as $s \to \infty$.

For $s > 0$ let

$$\lambda(s) = \inf_{n\in\mathbb{N},\, n\geq t_0} M(n)s^{-n}.$$

LEMMA 3. *For $s \gg 0$, we have*

$$e^{-\delta}\lambda(s) \leq \Lambda(s) \leq \lambda(s).$$

Proof. Put $\alpha(t) = m(t) - t\log s$; we have $\Lambda(s) = e^{\alpha(t_0)}$ where $\alpha'(t_0) = 0$; then $\lambda(s) = e^{\alpha(n_0)}$ with $|n_0 - t_0| < 1$. Note that $\alpha(n_0) - \alpha(t_0) = \alpha'((1-\theta)n_0 + \theta t_0)$, $0 < \theta < 1$. Since $m'' \leq \delta$ and $|\alpha'((1-\theta)n_0 + \theta t_0) - \alpha(t_0)| \leq \delta$, we have $e^{-\delta}\lambda(s) \leq \Lambda(s)$. The second inequality is trivial.

PROPOSITION 2. *The following three statements are equivalent*:

(i) $\displaystyle\sum_n \frac{M(n)}{M(n+1)} = \infty$,

(ii) $\displaystyle\int_{s_0}^{\infty} \frac{\omega(s)}{s^2}\,ds = \infty$ *for some* $s_0 > 0$,

(iii) $\displaystyle\int_{s_0}^{\infty} \frac{\log\lambda(s)}{s^2}\,ds = -\infty$ *for some* $s_0 > 0$,

Proof. We have $m'(n) \leq m(n+1) - m(n) \leq m'(n+1)$, hence

$$\sum_n \frac{M(n)}{M(n+1)} = \infty \Leftrightarrow \int_{t_0}^{\infty} e^{-m'(t)}\,dt = \infty.$$

Recall that by the above,

$$\int_{t_0}^{\infty} e^{-m'(t)}\,dt = \int_{s_0}^{\infty} \frac{d(s\omega'(s))}{s}\,ds.$$

Since $\omega'(s) \to 0$ as $s \to \infty$ and it is decreasing, we have

$$\int\limits_{s_0}^{\infty} \frac{d(s\omega'(s))}{s}\, ds = \infty \ \Leftrightarrow \ \int\limits_{s_0}^{\infty} \frac{\omega(s)}{s^2}\, ds = \infty,$$

which proves (i)⇔(ii). By Lemma 3, we have $-\omega(s)/s^2 \le (\log \lambda(s))/s^2$, hence (ii)⇔(iii).

Definition 2. We say that $C_U(M)$ is *quasianalytic* if for any $f \in C_U(M)$ and any $x \in U$ the Taylor series $T_x f$ of f at x uniquely determines f around x.

By a well known result of Denjoy–Carleman, $C_U(M)$ is quasianalytic if and only if

$$\sum_n \frac{M(n)}{M(n+1)} = \infty.$$

If the class is quasianalytic, Proposition 2 tells us that the function $\omega(s)$ tends to ∞ as $s \to \infty$ as rapidly as s^q, for all $q < 1$. Probably the converse of this statement is also true.

In the case of the analytic class ($m(t) = t \log t$), we have $\omega(s) = s\omega'(s)$, hence $\omega(s) = Cs$. The converse is also true:

Proposition 3. *If $\omega(s) \simeq s$ as $s \to \infty$, then any $f \in C_U(M)$ is analytic.*

Proof. By hypothesis, there exist $C > 0$ and $A > 0$ such that $\omega(s) \ge Cs$ for all $s \ge A$; then

$$\forall m \in \mathbb{N},\ \forall s \ge A, \qquad e^{-\omega(s)} \le \frac{c^{-m}}{s^m}\, m!.$$

Since $m'(t) \to \infty$ as $t \to \infty$, there exists $N_0 \in \mathbb{N}$ such that $e^{m'(t)} \ge A$ for all $t > N_0$ (we can suppose $N_0 > t_0$). Let $r > N_0$ and put $s = e^{m'(r)}$; then $s \ge A$ and $M(r)/s^r \le \inf_{n \ge t_0} M(n)/s^n$. By Lemma 3, for all $m > N_0$ we have

$$\frac{M(n)}{s^n} \le e^{\delta} e^{-\omega(s)} \le e^{\delta}\, \frac{C^{-m}}{s^m}\, m!,$$

hence $M(m) \le (e^{\delta}/C^m) m!$. This proves the result.

Proposition 4. *Let $\mu(t) = \log\log t$, i.e. $m(t) = t \log t + t \log\log t$. Then the class $C_U(M)$ is quasianalytic (recall that $M(t) = e^{m(t)}$).*

Proof. We will show that $\int_{s_0}^{\infty} (\omega(s)/s^2)\, ds = \infty$. We have $s = e^{m'(t)} = et \log t e^{1/\log t} \sim et \log t$, and

$$\omega(s) = tm'(t) - m(t) = t + \frac{t}{\log t} \sim t \sim \frac{s}{e \log s}.$$

Hence

$$\frac{\omega(s)}{s^2} \sim \frac{1}{es(\log s)} \qquad \text{as } s \to \infty,$$

which proves the proposition.

From now on we take $m(t) = t \log t + t\mu(t)$, μ increasing, $\mu(t) \le at$ for $t \gg 0$, $a > 0$, and $\lim_{t\to\infty} \mu(t) = \infty$. We also suppose that μ is in a Hardy field. Then the class $C_U(M)$ is an algebra, closed under differentiation and composition. We also take μ such that $C_U(M)$ is quasianalytic; for example, $\mu(t) = \log\log t$.

2.1. *The ring of germs of quasianalytic functions.* Let $r > 0$. We use the notation $\Delta_n(r) = \{x \in \mathbb{R}^n \mid |x_i| < r \text{ for } 1 \le i \le n\}$; if $x \in \mathbb{R}^n$, then $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$; and we put $C_{n,r}(M) = C_{\Delta_n(r)}(M)$. If $f \in \mathcal{E}(\Delta_n(r))$, we define, for $\varrho > 0$,

$$\|f\|_{\varrho,r,M} = \sup_m \sup_{\substack{|\alpha|=m \\ x\in\Delta_n(r)}} \frac{|D^\alpha f(x)|}{M(|\alpha|)\varrho^{|\alpha|}} \in [0,\infty]$$

and we set $C_{n,\varrho,r}(M) = \{f \in C_{n,r}(M) \mid \|f\|_{\varrho,r,M} < \infty\}$. Clearly $C_{n,\varrho,r}(M)$ is a Banach space. Let $C_n(M)$ be the inductive limit of $C_{n,\varrho,r}(M)$ as $r \to 0$, $\varrho \to \infty$. We have an injection

$$C_n(M) \to \mathbb{R}[[X_1,\dots,X_n]]$$

defined by $f \mapsto T_0 f$.

In general, we will not distinguish notationally between the germ of a function and a representative of the germ.

Lemma 4. *The algebra $C_n(M)$ is local and its maximal ideal is generated by $(x_1,\dots,x_n)$.*

Proof. Let $f \in C_n(M)$ be such that $f(0) = a_0 \ne 0$; put $\varrho = |a_0| > 0$ and $\varphi(\xi) = 1/(\xi + a_0)$. The function φ is analytic in $\{\xi \in \mathbb{R} \mid |\xi| < \varrho\}$. Put $g = f - a_0$; then $g \in C_n(M)$ and $g(0) = 0$. There exists $\eta > 0$ such that $g([-\eta,\eta]^n) \subset \{\xi \in \mathbb{R} \mid |\xi| < \varrho\}$. By Proposition 1, $\varphi \circ g \in C_n(M)$, hence $1/f \in C_n(M)$. The algebra is then local and its maximal ideal is $\mathcal{M} = \{f \in C_n(M) \mid f(0) = 0\}$.

Let $f \in \mathcal{M}$. Then $f(x) = \sum_{j=1}^n x_j g_j(x)$, where $g_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)\,dt$; we easily see that $g_j \in C_n(M)$ for $j = 1,\dots,n$.

Corollary 1. *If $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$ divides $f \in C_n(M)$ in the ring of formal power series at $0 \in \mathbb{R}^n$, then x^α divides f in $C_n(M)$.*

Proof. This is an immediate consequence of the previous lemma.

Proposition 5. *Let $f \in C_{n,\varrho_1,r_1}(M) - \{0\}$ be such that $f(0) = 0$. For each $\varepsilon > 0$, there exist $r', \varrho' > 0$ with $r' < r_1$ and $\varrho' > \varrho_1$ such that $\|f\|_{\varrho,r,M} \le \varepsilon$ for all $r < r'$ and $\varrho > \varrho'$.*

Proof. By hypothesis, we have

$$\sup_m \sup_{\substack{|\omega|=m \\ x\in\Delta_n(r_1)}} \frac{|D^\omega f(x)|}{M(|\omega|)\varrho_1^m} < \infty.$$

Put

$$R = \sup_{m\neq 0} \sup_{\substack{|\omega|=m \\ x\in\Delta_n(r_1)}} \frac{|D^\omega f(x)|}{M(|\omega|)\varrho_1^m}.$$

Since $f(0)=0$ and $f\neq 0$, we have $R\neq 0$. Let $\varepsilon>0$ (we can suppose $\varepsilon<1$). There exists $\varrho'>\varrho_1$ such that $(\varrho_1/\varrho)^m \le \varepsilon/R$ for all $\varrho>\varrho'$ and $m\in\mathbb{N}^*$. We then have

$$\sup_{m\neq 0} \sup_{\substack{|\omega|=m \\ x\in\Delta_n(r_1)}} \frac{|D^\omega f(x)|}{M(|\omega|)\varrho^m} \le \varepsilon.$$

Since $f(0)=0$, there exists $r'<r_1$ such that $|f(x)|\le\varepsilon$ for all $r\le r'$ and $x\in\Delta_n(r)$. Hence $\|f\|_{\varrho,r,M}\le\varepsilon$.

3. The implicit function theorem. It was proved in [7] that if the sequence $M(n)=M_n$ satisfies the conditions

$$\left(\frac{M_q}{q!}\right)^{1/(q-1)} \le C\left(\frac{M_p}{p!}\right)^{1/(p-1)}, \quad 2\le q\le p, \tag{1}$$

$$M_0 = M_1 = 1, \tag{2}$$

where $C>0$ is a constant, then the implicit function theorem holds in the ring $C_n(M)$.

Recall that $M(t)=e^{m(t)}$, $m(t)=t\log t + t\mu(t)$. We put $g(t)=t\mu(t)$. By Remark 1, we can suppose $M(1)=1$; we see that the condition (1) is satisfied if

$$\forall p\ge q\ge 2, \quad (p-1)g(q)\le C(q-1)g(p) \tag{$*$}$$

for a constant $C>0$.

We remark that (μ is increasing)

$$\forall p\ge 1, \quad pg(p-1)\le (p-1)g(p).$$

By repeating the process, we prove $(*)$. Hence the implicit function theorem holds in $C_n(M)$.

4. Algebraic properties. It is well known that the Weierstrass preparation theorem does not hold in $C_n(M)$ (see [3]). We do not know if $C_n(M)$ is a noetherian ring ($n>1$). In this section we will show that $C_n(M)$ has a weak noetherian property which we call topological noetherianity. This property will be enough for us to extend some well known properties of semianalytic germs to the case where the germs are defined by equations and inequalities for elements in $C_n(M)$.

We shall use a very elementary version of resolution of singularities consisting of blowings-up of a neighborhood of $0 \in \mathbb{R}^n$, $n > 1$, say V, either with center an open subset, $W \subset \mathbb{R}^{n-p}$, $p < n$, such that $\{0\} \times W \subset V$, or with center $\{0\} \subset \mathbb{R}^n$.

4.1. *Blowings-up.* For each positive integer r, let $\mathbb{P}^{r-1}(\mathbb{R})$ denote the $(r-1)$-dimensional real projective space of lines through the origin in $\mathbb{R}^r$. Let $\sigma : \mathbb{R}^r - \{0\} \to \mathbb{P}^{r-1}(\mathbb{R})$ be the canonical surjection which associates to each $t \in \mathbb{R}^r - \{0\}$ the line, say $\sigma(t)$, in $\mathbb{R}^r$ passing through 0 and t. For each $i = 1, \dots, r$, let $V_i = \{x = (x_1, \dots, x_r) \mid x_i \neq 0\}$ and $U_i = \sigma(V_i)$; U_i is a coordinate chart of $\mathbb{P}^{r-1}(\mathbb{R})$ with coordinates $\varphi_i : U_i \to \mathbb{R}^{r-1}$ given by

$$\varphi_i(\sigma(t)) = \left(\frac{t_1}{t_i}, \dots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \dots, \frac{t_r}{t_i}\right).$$

Definition 3. Let V be an open neighborhood of 0 in $\mathbb{R}^r$. Put

$$Z = \{(x, \sigma(t)) \in V \times \mathbb{P}^{r-1}(\mathbb{R}) \mid x \in \sigma(t)\}$$

and let $\pi : Z \to V$ denote the mapping $\pi(x, \sigma(t)) = x$. The mapping π is called the *blowing-up* of V with center 0.

The mapping π is proper, it restricts to a homeomorphism on $V - \{0\}$ and $\pi^{-1}(0) = \mathbb{P}^{r-1}(\mathbb{R})$.

We can cover Z with coordinate charts

$$Z_i = Z \cap V \times \sigma(U_i)$$

with coordinates $\psi_i : Z_i \to \mathbb{R}^r$ given by

$$\psi_i(x, \sigma(t)) = \left(\frac{t_1}{t_i}, \dots, \frac{t_{i-1}}{t_i}, x_i, \frac{t_{i+1}}{t_i}, \dots, \frac{t_r}{t_i}\right).$$

In these local coordinates, π is given by

$$\pi(y_1, \dots, y_r) = (y_1 y_i, \dots, y_{i-1} y_i, y_i, y_{i+1} y_i, \dots, y_r y_i).$$

Let $n > r$ be an integer and W an open subset of $\mathbb{R}^{n-r} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = \dots = x_r = 0\}$. Let $w = (w_1, \dots, w_{n-r})$ be the coordinates of a point in $\mathbb{R}^{n-r}$. The mapping $\widetilde{\pi} = \pi \times \mathrm{id}_W : \widetilde{Z} = Z \times W \to V \times W$ is called the *blowing-up* of $V \times W$ with center $\{0\} \times W$. We can cover $\widetilde{Z}$ with coordinate charts

$$\widetilde{Z}_i = \widetilde{Z} \cap V \times \sigma(U_i) \times W$$

with coordinates $\widetilde{\varphi}_i : \widetilde{Z}_i \to V \times W$ given by

$$\widetilde{\varphi}_i(x, \sigma(t), w) = \left(\frac{t_1}{t_i}, \dots, \frac{t_{i-1}}{t_i}, x_i, \frac{t_{i+1}}{t_i}, \dots, \frac{t_r}{t_i}, w\right).$$

We put $\widetilde{\varphi}_i = (y_1, \dots, y_r, w')$.

Recall that $\mathcal{E}_n$ is the ring of germs at $0 \in \mathbb{R}^n$ of C^∞ functions. Let $a \in \widetilde{\pi}^{-1}(0) \cap \widetilde{Z}_i$ and $f \in \mathcal{E}_n$; then the Taylor expansion of $f \circ \widetilde{\pi}$ at a is given by formal substitution of $w = w'$, $X_i = y_i$, and $X_l = y_i(y_l(a) + y_l)$, $l \neq i$, in the Taylor expansion of f at 0. In particular if $\widehat{f} \in \mathbb{R}[[X, W]]$ is a formal series, we will denote by $\widehat{f} \circ \widehat{\pi}_a$ the formal series obtained by formal substitution of $w = w'$, $X_i = y_i$, and $X_l = y_i(y_l(a) + y_l)$, $l \neq i$, in the formal series $\widehat{f}$.

We need the following lemma proved in [10]; for completeness we will give the proof.

Lemma 5. *Let $\Omega \subset \mathbb{N}^n$, $n > 1$, be a finite set and put $\mathcal{F} = \{X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \Omega\}$. Let V be an open neighborhood of* 0 *in $\mathbb{R}^n$. Then there exists a real-analytic manifold Z and a proper real-analytic surjective mapping $\pi : Z \to V$ such that*:

(a) *For all $a \in \pi^{-1}(0)$ there is a chart U with $a \in U$ and with coordinates $y = (y_1, \dots, y_n)$ such that the set $\{\mu_\alpha \in \mathbb{N}^n \mid X^\alpha \circ \widehat{\pi}_a = y^{\mu_\alpha}\}$ is totally ordered by the product order on $\mathbb{N}^n$.*

(b) *$\pi_{|U} : U \to V$ is the composition of a finite sequence of blowings-up.*

Proof. We proceed by induction on $n \geq 2$. We can suppose that the cardinality of Ω is 2. After making a finite number of blowings-up of V with center the origin of $\mathbb{R}^2$, we can easily see that the lemma is true for $n = 2$. Suppose $n \geq 3$. After dividing each monomial by the common factors, we can also suppose that there is $r \in \mathbb{N}$, $r \leq n$, such that the monomials are of the form $X_1^{\alpha_1} \dots X_r^{\alpha_r}$ or $X_{r+1}^{\alpha_{r+1}} \dots X_n^{\alpha_n}$ with $\alpha_n = \min_{i=1,\dots,n} \alpha_i$ (after making a permutation of $(X_1, \dots, X_n)$).

We proceed by induction on α_n. If $\alpha_n = 0$ we are done by the inductive hypothesis on n. Suppose $\alpha_n > 0$ and consider the two monomials $A = X_1^{\alpha_1} \dots X_r^{\alpha_r}$ and $B = X_{r+1}^{\alpha_{r+1}} \dots X_{n-1}^{\alpha_{n-1}}$; by the induction hypothesis on n, if V' is a neighborhood of $0 \in \mathbb{R}^{n-1}$, there exist a real-analytic manifold M and a proper real-analytic surjective mapping $\pi : M \to V'$ such that conditions (a) and (b) of the lemma are satisfied. Let $a \in \pi^{-1}(0)$. There is a chart U' with $a \in U'$ and with coordinates $y = (y_1, \dots, y_{n-1})$ such that $A \circ \widehat{\pi}_a = y_1^{\beta_1} \dots y_{n-1}^{\beta_{n-1}}$ and $B \circ \widehat{\pi}_a = y_1^{\beta'_1} \dots y_{n-1}^{\beta'_{n-1}}$ with $(\beta_1, \dots, \beta_{n-1}) \leq (\beta'_1, \dots, \beta'_{n-1})$ or $(\beta'_1, \dots, \beta'_{n-1}) \leq (\beta_1, \dots, \beta_{n-1})$. Consider the two monomials $y_1^{\beta_1} \dots y_{n-1}^{\beta_{n-1}}$ and $y_1^{\beta'_1} \dots y_{n-1}^{\beta'_{n-1}} X_n^{\alpha_n}$ on $U' \times \mathbb{R}$. If $(\beta_1, \dots, \beta_{n-1}) \leq (\beta'_1, \dots, \beta'_{n-1})$ we are done. Suppose $(\beta'_1, \dots, \beta'_{n-1}) < (\beta_1, \dots, \beta_{n-1})$; after dividing by common factors, we are in the situation of $y_1^{\gamma_1} \dots y_{n-1}^{\gamma_{n-1}}$ and $X_n^{\alpha_n}$. If $\gamma_i < \alpha_n$ for some i, then we use the second induction (on α_n). Suppose $\gamma_i \geq \alpha_n$ for all $i = 1, \dots, n-1$. We will blow up $U' \times \mathbb{R}$ with center $y_1 = X_n = 0$. Let $\widetilde{\pi} : \widetilde{U} \to U' \times \mathbb{R}$ be this blowing-up. We can cover $\widetilde{U}$ by two coordinate charts: $\widetilde{U}_1$ and $\widetilde{U}_2$. With respect to these charts, $\widetilde{\pi}$ is given, respectively, by

$$\widetilde{\pi}(y_1, \dots, y_n) = (y_1, y_2, \dots, y_{n-1}, y_n y_1)$$

and

$$\widetilde{\pi}(y_1,\dots,y_n)=(y_ny_1,y_2,\dots,y_{n-1},y_n).$$

In the chart $\widetilde{U}_1$ our monomials are of the form

$$y_1^{\gamma_1-\alpha_n}y_2^{\gamma_2}\dots y_{n-1}^{\gamma_{n-1}},\qquad y_n^{\alpha_n}.$$

By continuing, we will have $\gamma_1-\alpha_n<\alpha_n$ and the inductive hypothesis on $\inf\gamma_i$ will prove the lemma. In the second chart $\widetilde{U}_2$, the result is true since $(\gamma_1,\dots,\gamma_{n-1},\gamma_1,)\geq(0,\dots,0,\alpha_n)$.

Proposition 6. *Let $\widehat{f}\in\mathbb{R}[[X_1,\dots,X_n]]$ and let $V\subset\mathbb{R}^n$ be an open neighborhood of* 0. *There exists a real-analytic manifold Z and a proper real-analytic surjective mapping $\pi:Z\to V$ such that each $a\in\pi^{-1}(0)$ admits a coordinate neighborhood U with coordinates $y=(y_1,\dots,y_n)$ such that $\widehat{f}\circ\widehat{\pi}_a=y_1^{\alpha_1}\dots y_n^{\alpha_n}\widehat{h}$, where $\widehat{h}\in\mathbb{R}[[Y_1,\dots,Y_n]]$ is a unit.*

Proof. Let us remark that we can write $\widehat{f}$ in the form

$$\widehat{f}=\sum_{\omega\in\Omega\subset\mathbb{N}^n}\widehat{f}_\omega X^\omega,$$

where $\Omega\subset\mathbb{N}^n$ is finite and $\widehat{f}_\omega\in\mathbb{R}[[X_1,\dots,X_n]]$ is a unit for each $\omega\in\Omega$. By Lemma 5 there exists a real-analytic manifold Z and a real-analytic proper surjective mapping $\pi:Z\to V$ such that each $a\in\pi^{-1}(0)$ admits a coordinate neighborhood U with coordinates $y=(y_1,\dots,y_n)$ and the set $\{\mu_\omega\mid X^\omega\circ\widehat{\pi}_a=y^{\mu_\omega}\}$ is totally ordered. Let μ_{ω_0} be the least element. We have

$$\widehat{f}\circ\widehat{\pi}_a=\sum_{\omega\in\Omega\subset\mathbb{N}^n}\widehat{f}_\omega\circ\widehat{\pi}_aX^\omega\circ\widehat{\pi}_a=y^{\mu_{\omega_0}}\sum_{\omega\in\Omega\subset\mathbb{N}^n}\widehat{f}_\omega\circ\widehat{\pi}_ay^{\mu_\omega-\mu_{\omega_0}}.$$

This proves the result.

Proposition 7. *Let $f\in C_n(M)$. Then there exists an open neighborhood V of $0\in\mathbb{R}^n$, a real-analytic manifold Z and a proper real-analytic surjective mapping $\pi:Z\to V$ such that each $a\in\pi^{-1}(0)$ admits a coordinate neighborhood U with coordinates $y=(y_1,\dots,y_n)$ such that $f\circ\pi_{|U}(y)=y^\mu\varphi(y)$, where $\mu\in\mathbb{N}^n$, $\varphi\in C_U(M)$ and $\varphi(y)\neq0$ for all $y\in U$.*

Proof. Choose an open neighborhood V of $0\in\mathbb{R}^n$ where f is defined and Proposition 6 can be applied. Then there exists a real-analytic manifold Z and a proper real-analytic surjective mapping $\pi:Z\to V$ such that each $a\in\pi^{-1}(0)$ admits a coordinate neighborhood U with coordinates $y=(y_1,\dots,y_n)$ in which $T_0f\circ\widehat{\pi}_a=y^\mu\widehat{h}$, where $\widehat{h}\in\mathbb{R}[[y_1,\dots,y_n]]$ is a unit. Since $f\circ\pi_{|U}\in C_U(M)$, Corollary 1 implies that $f\circ\pi_{|U}=y^\mu\varphi(y)$, $\varphi(0)\neq0$, which proves the proposition.

5. Topological noetherianity

Lemma 6. *Every decreasing sequence of germs $f_1^{-1}(0)\supset f_2^{-1}(0)\supset\dots$ with $f_j\in C_n(M)$ is stationary.*

Proof. By induction on n; the lemma is trivially true for $n = 1$. Suppose $n > 1$ and the result holds for $n-1$. According to Proposition 7, there exists an open neighborhood V of $0 \in \mathbb{R}^n$, a real-analytic manifold Z and a proper real-analytic surjective mapping $\pi : Z \to V$ such that each $a \in \pi^{-1}(0)$ admits a coordinate neighborhood U with coordinates $y = (y_1, \ldots, y_n)$ in which $f_1 \circ \pi_{|U}(y) = y^\mu \varphi(y)$ and $\varphi(y) \neq 0$ for all $y \in U$. It is enough to prove that the sequence $(f_j \circ \pi)^{-1}(0)$ is stationary in a neighborhood of every point $a \in \pi^{-1}(0)$. We can suppose that $f_1(y) = y_1^{\mu_1} \ldots y_n^{\mu_n} \varphi(y)$ and $\varphi(y) \neq 0$ for all $y \in U$. Let $J = \{j = 1, \ldots, n \mid \mu_j \neq 0\}$. For each $j \in J$ the sequence $(f_l^{-1}(0) \cap \{y \in U \mid y_j = 0\})_l$ is stationary by the inductive hypothesis; so our sequence is stationary near a, which proves the lemma.

5.1. *M-manifolds*

Definition 4. An n-dimensional manifold is a Hausdorff space with countable basis in which each point has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. An *M-structure* on a manifold Z is a family $\mathcal{F} = \{(U_i, \varphi_i) \mid i \in I\}$ of homeomorphisms φ_i, called local coordinate systems, of an open set $U_i \subset Z$ onto an open set $\widetilde{U}_i \subset \mathbb{R}^n$ such that:

(a) If $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{F}$, then each cartesian component of the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subset \mathbb{R}^n \to \varphi_j(U_i \cap U_j) \subset \mathbb{R}^n$ is in $C_{\varphi_i(U_i \cap U_j)}(M)$.

(b) $Z = \bigcup_{i \in I} U_i$.

A manifold with an M-structure is called an *M-manifold.*

Let Z be an M-manifold and $U \subset Z$ an open set. A function φ defined in U will be said to be in $C_U(M)$ if for every coordinate system (U_i, φ_i), the composite $\varphi \circ \varphi_i^{-1}$ is in $C_{\varphi_i(U_i \cap U)}(M)$. We shall sometimes denote $\varphi \circ \varphi_i^{-1}$ by $\varphi_{|U_i \cap U}$.

Let us remark that every real-analytic manifold is an M-manifold.

Let $Y \subset Z$. We say that Y is a *smooth M-submanifold* if Y is covered by coordinate charts U of M, each of which has local coordinates $z = (x, y)$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_p)$, in which $Y \cap U = \{y_1 = \ldots = y_p = 0\}$.

Let Z be an M-manifold and Y a closed M-submanifold of Z. We define the blowing-up $\pi : Z' \to Z$ with center Y as follows: Z' is an M-manifold and π is a proper map in the class M such that:

(1) π restricts to an isomorphism $Z' - \pi^{-1}(Y) \to Z - Y$ in the class M.

(2) Let $U \subset Z$ be a coordinate chart with local coordinates in U defined by $\varphi : U \to V \times W$, where U, W are open neighborhoods of the origin in $\mathbb{R}^p$, $\mathbb{R}^{n-p}$, respectively, and $\varphi(U \cap Y) = \{0\} \times W$. Let $\pi_0 : V' \to V$ be the blowing-up of V with center $\{0\}$. Then there is an isomorphism $\varphi' : \pi^{-1}(U) \to V' \times W$ in the class M such that

$$\pi_0 \times \mathrm{id}_W \circ \varphi' = \varphi \circ \pi_{|\pi^{-1}(U)}.$$

Definition 5. Let Z be an M-manifold. Let U be an open subset of Z and let Y be a closed M-submanifold of U. Let $\pi : Z' \to Z$ denote the composition of the blowing-up $Z' \to U$ of U with center Y and the inclusion $U \to Z$. We call π a *local blowing-up* of Z with center Y.

We will consider mappings $\pi : \ Z' \to Z$ obtained as the composition of a finite sequence of local blowings-up; i.e. $\pi = \pi_1 \circ \ldots \circ \pi_k$, where, for each $i = 1, \ldots, k$, $\pi_i : Z_{i+1} \to Z_i$ is a local blowing-up of Z_i, and $Z_1 = Z$, $Z_{k+1} = Z'$.

6. Łojasiewicz's inequality. In the following, Z will be an M-manifold with $\dim Z = n$ and W an open subset of Z. As an immediate consequence of Proposition 7, we have:

Proposition 8. *Let $f \in C_W(M)$. Then each $a \in W$ admits an open neighborhood V for which there exists an M-manifold Z' and a proper surjective mapping $\pi : Z' \to V$ in the class M such that*:

(i) *each $b \in \pi^{-1}(a)$ admits a coordinate neighborhood U with coordinates $y = (y_1, \ldots, y_n)$ in which $f \circ \pi(y) = y^\mu \varphi(y)$ for all $y \in U$, where $\varphi \in C_U(M)$ and $\varphi(y) \neq 0$ for all $y \in U$.*

(ii) *$\pi_{|U} : U \to V$ is a finite composition of local blowings-up.*

Remark 3. We require that the mapping $\pi : Z' \to V$ satisfy the following additional condition: each $b \in \pi^{-1}(a)$ admits a coordinate neighborhood U_b for which there exists $q \in \mathbb{N}$ and an isomorphism $\varphi' : U_b \to \varphi'(U_b) \subset V \times \mathbb{P}^q(\mathbb{R})$ in the class M such that $\varphi'(U_b)$ is an M-submanifold defined by homogeneous polynomial equations (in homogeneous coordinates of $\mathbb{P}^q(\mathbb{R})$) whose coefficients are in $C_V(M)$.

A local blowing-up has this property. We can easily see that the composition of two local blowings-up also has this property. By condition (ii) of the last proposition, we see that π can be chosen as in the remark.

Theorem 2. *Let $f \in C_W(M)$ and set $V_W(f) = \{x \in W \mid f(x) = 0\}$. Let g be any C^∞ function on W such that $g(x) = 0$ for all $x \in V_W(f)$. Then, for every compact subset $K \subset W$, there exist $N, C > 0$ such that*

$$|g(x)|^N \leq C|f(x)|, \quad \forall x \in K.$$

Proof. We can suppose that $|g(x)| \leq 1$ for all $x \in K$. The question is local in W, so we will prove that each $a \in W$ admits a coordinate neighborhood V_a for which there exist $N_a, C_a > 0$ such that

$$|g(x)|^{N_a} \leq C_a|f(x)|, \quad \forall x \in V_a.$$

Then we can cover K by finite V_{a_i}, $i = 1, \ldots, l$, and take $N = \max_i N_{a_i}$, $C = \max_i C_{a_i}$.

Let $a \in W$. By Proposition 8, there exists a coordinate neighborhood V_a of a with coordinates $x = (x_1, \ldots, x_n)$ centered at a, i.e. $x_i(a) = 0$ for all $i = 1, \ldots, n$, an M-manifold Z' and a proper surjective mapping $\pi : Z' \to V_a$ in the class M such that

(∗) each $b \in \pi^{-1}(a)$ admits a coordinate neighborhood U_b with coordinates $y = (y_1, \ldots, y_n)$ centered at b in which $f \circ \pi(y) = y^\mu \varphi(y)$ for all $y \in U_b$, where $\mu \in \mathbb{N}^n$, $\varphi \in C_{U_b}(M)$ and $\varphi(y) \neq 0$ for all $y \in U_b$. Since π is proper, there exists a finite set $\Lambda \subset \mathbb{N}$ such that $\bigcup_{\alpha \in \Lambda} U_{b_\alpha}$ is an open covering of $\pi^{-1}(a)$, $b_\alpha \in \pi^{-1}(a)$ for all $\alpha \in \Lambda$ and $f \circ \pi(y) = y^{\mu_\alpha} \varphi_\alpha(y)$ for all $y \in U_{b_\alpha}$, $\varphi_\alpha(y) \neq 0$, for all $y \in U_{b_\alpha}$ and $\varphi_\alpha \in C_{U_{b_\alpha}}(M)$.

Write $\mu_\alpha = (\mu_{\alpha 1}, \ldots, \mu_{\alpha n})$ and let Δ_α be the set of those i where $\mu_{\alpha i} > 0$ (Δ_α may be empty for some α). The assumption on g implies that $g \circ \pi$ vanishes identically on each of the hyperplanes $y_i = 0$ in U_{b_α} with $i \in \Delta_\alpha$. Hence $g \circ \pi$ is divisible by the product of those y_i with $i \in \Delta_\alpha$. Then $g \circ \pi(y) = y^{\beta_\alpha} h_\alpha(y)$ for all $y \in U_{b_\alpha}$, $\beta_\alpha = (\beta_{\alpha 1}, \ldots, \beta_{\alpha n})$ and h_α is a C^∞ function on U_{b_α}. Recall that $\beta_{\alpha j} > 0$ if $j \in \Delta_\alpha$.

Let $\Delta'_\alpha = \{j \in \Delta_\alpha \mid \beta_{\alpha j} < \mu_{\alpha j}\}$ and put $q_\alpha = \max_{j \in \Delta'_\alpha} \mu_{\alpha j}/\beta_{\alpha j}$. We see that $(g \circ \pi(y))^{q_\alpha} = \psi_\alpha(y)(f \circ \pi)(y)$ for all $y \in U_{b_\alpha}$, where ψ_α is a C^∞ function on U_{b_α}.

If $r > 0$, we write $U_{b_\alpha}(r) := \{y \in U_{b_\alpha} \mid \sum_{i=1}^n y_i^2 \leq r\}$; since π is proper, there exists $\varrho > 0$ such that

$$V_a(\varrho) = \Big\{x \in V_a \,\Big|\, \sum_{i=1}^n x_i^2 \leq \varrho\Big\} \subset \bigcup_{\alpha \in \Lambda} \pi(U_{b_\alpha}(\varrho)).$$

Let $C_\alpha = \sup_{y \in U_{b_\alpha}(\varrho)} |\psi_\alpha(y)|$, $C = \max C_\alpha$ and $N = \max q_\alpha$. Then for all $x \in V_a(\varrho)$, we have $|g(x)|^N \leq |f(x)|$, which proves the theorem.

Let us remark that by the previous proof the infimum of $\lambda > 0$ such that there exists $C > 0$ with $|g(x)|^\lambda \leq C|f(x)|$ for all $x \in K$ is a rational number.

Theorem 3. *Suppose that $W \subset \mathbb{R}^n$ is an open set and $f \in C_W(M)$. Then for each compact subset $K \subset W$, we can find $N, C > 0$ such that*

$$C|f(x)| \geq d(x, V_W(f))^N, \qquad \forall x \in K.$$

Proof. We will prove that each $a \in W$ admits a neighborhood V_a and constants $N_a, C_a > 0$ such that

$$C_a|f(x)| \geq d(x, V_{V_a}(f))^{N_a}, \qquad \forall x \in V_a.$$

Let $a \in W$. There exists $\pi : Z' \to V_a$ as in the proof of the previous theorem. We then have a finite covering of $\pi^{-1}(a) \subset \bigcup_{\alpha \in \Lambda} U_{b_\alpha}$ and for all $\alpha \in \Lambda$, $f \circ \pi(y) = y^{\mu_\alpha} \varphi_\alpha(y)$ for all $y \in U_{b_\alpha}$. Then $V_{U_{b_\alpha}}(f \circ \pi)$ is equal to the union of those coordinate hyperplanes $H_{\alpha i}$ defined by y_i with $i \in \Delta_\alpha$. Define $\psi_{\alpha i}(y) = d(\pi(y), \pi\gamma_{\alpha i}(y))^2$ for $y \in U_{b_\alpha}$, where $\gamma_{\alpha i}(y)$ denotes the orthogonal projection from $U_{b_\alpha} \simeq \mathbb{R}^n$ to $H_{\alpha i}$. We see that $\psi_{\alpha i}$ is a C^∞ function on U_{b_α}. Let $\psi_\alpha = \prod_{i \in \Delta_\alpha} \psi_{\alpha i}$. Then:

- ψ_α is a C^∞ function on U_{b_α},
- $\psi_\alpha(y) \geq d(\pi(y), V_{V_a}(f))^{2n_\alpha}$, where n_α is the number of elements of Δ_α.

We have $V_{U_{b_\alpha}}(f \circ \pi) \subset V_{U_{b_\alpha}}(\psi_\alpha)$; by the previous theorem, there exist $\varrho, N_\alpha, C_\alpha > 0$ such that for all $y \in U_{b_\alpha}(\varrho)$,

$$C_\alpha |(f \circ \pi)(y)| \geq |\psi_\alpha(y)|^{N_\alpha}.$$

Let $N = \max_{\alpha \in \Lambda} N_\alpha/(2n_\alpha)$ with $n_\alpha \neq 0$. Then for all $x \in V_a(\varrho)$ we have $C|f(x)| \geq d(x, V_{V_a(\varrho)})^N$, where $C = \max_{\alpha \in \Lambda} C_\alpha$.

7. Quasi semianalytic sets

Definition 6. Let A be a subset of an M-manifold Z. Then A is said to be *quasi semianalytic at* $a \in Z$ if there exists an open neighborhood V of a in Z and a finite number of elements of $C_V(M)$, g_i and f_{ij}, such that

$$A \cap V = \bigcup_i \{x \in V \mid g_i(x) = 0,\ f_{ij}(x) > 0,\ \forall j\}.$$

If A is quasi semianalytic at every point of Z, we say that A is *quasi semianalytic in* Z.

Remark 4. (i) The property of being quasi semianalytic is preserved under locally finite unions, locally finite intersections and complements.

(ii) If $A \subset Z$ is a quasi semianalytic set it is easy to see that for all $a \in Z$, there exists an open neighborhood V of a in Z such that $A \cap V$ is a finite disjoint union of sets of the form

$$\{x \in V \mid \varphi_0(x) = 0,\ \varphi_1(x) > 0, \ldots, \varphi_r(x) > 0\},$$

where $\varphi_0, \varphi_1, \ldots, \varphi_r$ are in $C_V(M)$.

Theorem 4. *Let A be a quasi semianalytic set in Z. Then each $x \in Z$ admits a neighborhood V such that $A \cap V$ has only a finite number of connected components.*

Proof. We will use the notation of Theorem 2 with $f = \prod_{i,j} g_i f_{ij}$. It is enough to prove that for each $\alpha \in \Lambda$, the number of connected components of $U_{b_\alpha} \cap \pi^{-1}(A)$ is finite. Since $f \circ \pi(y) = y^{\mu_\alpha}\varphi_\alpha(y)$ and $\varphi_\alpha(y) \neq 0$ for all $y \in U_{b_\alpha}$, we can easily see that

$$g_i \circ \pi(y) = y^{\mu_{\alpha_i}}\varphi_{\alpha_i}(y), \qquad f_{ij} \circ \pi(y) = y^{\mu_{\alpha_{ij}}}\varphi_{\alpha_{ij}}(y), \qquad \forall y \in U_{b_\alpha},$$

where $\varphi_{\alpha_i}(y) \neq 0$ and $\varphi_{\alpha_{ij}}(y) \neq 0$ for all $y \in U_{b_\alpha}$ and all i, j. This shows that $U_{b_\alpha} \cap \pi^{-1}(A)$ has only a finite number of connected components, which proves the theorem.

Let us give some notations and definitions. Let U be an open subset of Z, and $A \subset U$. We define $I_U(A) := \{f \in C_U(M) \mid f(x) = 0, \forall x \in A\}$; it is an ideal of $C_U(M)$. Let $F \subset U$; we say that F is a *global quasianalytic set* in U if there exist $h_1, \ldots, h_q \in C_U(M)$ such that $F = \{x \in U \mid h_1(x) = 0, \ldots, h_q(x) = 0\}$. Suppose that U is a chart of Z, $a \in U$, with coordinates $x = (x_1, \ldots, x_n)$ centered at a. If $f \in C_U(M)$ we denote by $\nu_a(f)$ the maximum of $q \in \mathbb{N}$ such that the Taylor expansion of f at a, $T_a f$, is in $\underline{m}^q$ ($\underline{m}$ is the maximal ideal of $\mathbb{R}[[X_1, \ldots, X_n]]$).

Proposition 9. *Let F be a global quasianalytic set in U. Let $k \in \mathbb{N}$ be the maximum of the integers such that there exist $f_1, \ldots, f_k \in I_U(F)$ with jacobian $\Delta = \frac{D(f_1,\ldots,f_k)}{D(x_{i_1},\ldots,x_{i_k})} \notin I_U(F)$. Put $\Gamma = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0, \Delta(x) \neq 0\}$. Then $F - V(\Delta) := \{x \in F \mid \Delta(x) \neq 0\}$ is a submanifold of U, and is quasi semianalytic; moreover $F - V(\Delta)$ is the union of some connected components of Γ.*

Proof. Clearly we have $F - V(\Delta) \subset \Gamma$. In order to prove the proposition, it is enough to prove that for each $x \in F - V(\Delta)$, the germs of Γ and $F - V(\Delta)$ at x are the same. We will prove that the germ of Γ at x, Γ_x, is contained in $(F - V(\Delta))_x$. Suppose, for a contradiction, that $\Gamma_x \not\subset (F - V(\Delta))_x$; then there exists $g \in I_U(F)$ such that $g_{|\Gamma_x} \neq 0$. By Lemma 7 below, there exists $h \in \{1, \ldots, n\} - \{i_1, \ldots, i_k\}$ such that if $g_1 = \frac{D(f_1,\ldots,f_k,g)}{D(x_{i_1},\ldots,x_{i_k},x_h)}_{|\Gamma_x}$, then $\nu_x(g_1) < \nu_x(g_{|\Gamma_x})$. By definition of k, we have $g_1 \in I_U(F)$ and also $g_{1|\Gamma_x} \neq 0$. We continue with g_1 in place of g and so on. At the end we find $g_q \in I_U(F)$ with $g_q(x) \neq 0$, which is a contradiction.

Lemma 7. *Let U be an open neighborhood of 0 in $\mathbb{R}^n$, and put*

$$S = \left\{ x \in U \;\middle|\; f_1(x) = \ldots = f_k(x) = 0,\ \Delta(x) = \frac{D(f_1, \ldots, f_k)}{D(x_1, \ldots, x_k)}(x) \neq 0 \right\},$$

where $f_1, \ldots, f_k \in C_U(M)$. Suppose that $0 \in S$. Let $g \in C_U(M)$ be such that $g_{|S} \neq 0$. Then there exists h with $k < h \leq n$ such that

$$\nu_0(g_{|S}) > \nu_0\left[\frac{D(f_1, \ldots, f_k, g)}{D(x_1, \ldots, x_k, x_h)}_{|S}\right].$$

Proof. Since the mapping

$$x = (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_k(x), x_{k+1}, \ldots, x_n)$$

is a local diffeomorphism near 0, we can suppose that $f_i(x) = x_i$ for all $i = 1, \ldots, k$. The result is then obvious.

In the following we call $\Gamma = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0,\ \Delta(x) \neq 0\}$ a *quasianalytic stratum*. Let $B \subset U$; B is called a *quasi semianalytic stratum* if B is the intersection of a quasi analytic stratum with an open set of the form $\{x \in U \mid \varphi_1(x) > 0, \dots, \varphi_q(x) > 0\}$, where $\varphi_1, \dots, \varphi_q \in C_U(M)$.

Let $U \subset Z$ be a chart of Z with coordinates $y = (y_1, \dots, y_n)$. Let $B \subset U$. We say that B is a *quadrant* if B is defined by a system of some equalities $y_i = 0$ and some inequalities $\varepsilon_j y_j > 0$ with $\varepsilon_j = \pm 1$.

THEOREM 5. *Let $A \subset Z$ be a quasi semianalytic set. Then each $a \in Z$ admits an open neighborhood V such that $A \cap V = \bigcup_{j=1}^{s} \Lambda_j$, where each Λ_j is a submanifold of V, $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \neq j$, and Λ_j is a finite union of connected components of a quasi semianalytic stratum.*

Proof. By Remark 4(ii), one can assume $A = \{x \in U \mid \varphi_0(x) = 0, \varphi_1(x) > 0, \dots, \varphi_q(x) > 0\}$, where $\varphi_0, \dots, \varphi_q \in C_U(M)$ and U is an open neighborhood of a in Z. Let $F = \{x \in U \mid \varphi_0(x) = 0\}$. By Proposition 9, there exists $f_0 \in C_U(M)$, $f_0 \notin I_U(F)$, such that the set $F - V(f_0) = \{x \in F \mid f_0(x) \neq 0\}$ is the union of some connected components of a quasinalytic stratum. Put $F_1 = \{x \in U \mid \varphi_0^2(x) + f_0^2(x) = 0\}$; then $F_1 \subset F$. We repeat the same thing with F_1 in place of F. Thus we construct a decreasing sequence $F \supset F_1 \supset \dots$, where $F_j = V(f_j)$, $f_j \in C_U(M)$, such that for each $j \in \mathbb{N}$, $F_j - F_{j+1}$ is the union of some connected components of a quasianalytic stratum. By Lemma 6, there exists $s \in \mathbb{N}$ and an open neighborhood V of a such that $F_j \cap V = F_{j+1} \cap V$ for all $j > s$. For $j \leq s$, put $\widetilde{\Gamma}_j = F_j - F_{j+1}$; then $V \cap F = \bigcup_{j=1}^{s} \widetilde{\Gamma}_j \cap V$. We then see that

$$A \cap V = \bigcup_{j=1}^{s} \Lambda_j,$$

where $\Lambda_j = \{x \in \widetilde{\Gamma}_j \cap V \mid \varphi_1(x) > 0, \dots, \varphi_q(x) > 0\}$. By shrinking V if necessary, we see that Λ_j has a finite number of connected components (Theorem 4), which proves the theorem.

By the previous theorem, we define the *topological dimension* of A at $a \in Z$, $\dim_a A$, to be the maximum of the dimensions of Λ_j, $j = 1, \dots, s$. This definition is independent of the family Λ_j: $\dim_a A = q$ if and only if A contains an open set homeomorphic to an open ball in $\mathbb{R}^q$, but not an open set homeomorphic to an open ball in $\mathbb{R}^l$, $l > q$.

THEOREM 6. *Let $A \subset Z$ be a quasi semianalytic set. Then each connected component of A is a quasi semianalytic set. The closure of A in Z, $\overline{A}$, is also a quasi semianalytic set.*

Proof. Let $\Gamma \subset A$ be a connected component of A. Let $a \in Z$ be such that the germ of Γ at a is not empty. There exists a neighborhood V_a of a in Z such that $A \cap V_a$ is a finite union of sets of the form

$$\Lambda = \{x \in V_a \mid \varphi_0(x) = 0,\, \varphi_1(x) > 0, \dots, \varphi_q(x) > 0\},$$

where $\varphi_0, \varphi_1, \dots, \varphi_q \in C_{V_a}(M)$.

Clearly we can suppose that $A \cap V = \Lambda$. Let $f = \varphi_0\varphi_1 \dots \varphi_q$. We keep the notation of the proof of Theorem 2. Since $\pi^{-1}(\Gamma) \cap U_{b_\alpha}$ is open and closed in $\pi^{-1}(A) \cap U_{b_\alpha}$, it follows that $\pi^{-1}(\Gamma) \cap U_{b_\alpha}$ is a finite union of quadrants in U_{b_α}. By Remark 3, there exists $q \in \mathbb{N}$ such that U_{b_α} is isomorphic to an M-submanifold of $V_a \times \mathbb{P}^q(\mathbb{R})$ defined by homogeneous polynomials with coefficients in $C_{V_a}(M)$. By Lemma 8 below, $\pi(\pi^{-1}(\Gamma) \cap U_{b_\alpha})$ is a quasi semianalytic set. Since π is proper, there exists a neighborhood $V_a' \subset V_a$ of a such that $\pi^{-1}(V_a') \subset \bigcup_{\alpha \in \Lambda} U_{b_\alpha}$; then $\pi[\bigcup_{\alpha \in \Lambda} U_{b_\alpha}]$ is a neighborhood of a (π is surjective) and $\bigcup_\alpha \pi(U_{b_\alpha}) \cap \Gamma = \bigcup_\alpha \pi(\pi^{-1}(\Gamma) \cap U_{b_\alpha})$, which proves the first statement.

We can choose, for each $\alpha \in \Lambda$, a closed neighborhood $U_{b_\alpha}' \subset U_{b_\alpha}$ of a such that $\pi^{-1}(a) \subset \bigcup_\alpha U_{b_\alpha}'$. Let

$$A_1 = \bigcup_\alpha \pi(U_{b_\alpha}' \cap \overline{\pi^{-1}(A)}).$$

We have $A_1 \subset V_a \cap \overline{A}$ and $V_a' \cap \overline{A} \subset A_1$.

Now since $U_{b_\alpha}' \cap \overline{\pi^{-1}(A)} = \overline{\pi^{-1}(A) \cap U_{b_\alpha}'}$, and $\pi^{-1}(A) \cap U_{b_\alpha}'$ is a finite union of quadrants, by Lemma 8, $\pi(U_{b_\alpha}' \cap \overline{\pi^{-1}(A)})$ is a quasi semianalytic set, hence so is $V_a \cap \overline{A}$ since it coincides with A_1 in a neighborhood of a (namely, in V_a').

It remains to recall Łojasiewicz's version of the Tarski–Seidenberg theorem.

LEMMA 8 ([8]). *Let $U \subset Z$ be an open set. Put*

$$A = \bigcup_{i=1}^{s} \{(x, t_1, \dots, t_q) \in U \times \mathbb{R}^q \mid g_i(x, t_1, \dots, t_q) = 0,$$
$$f_{i,1}(x, t_1, \dots, t_q) > 0, \dots, f_{i,r}(x, t_1, \dots, t_q) > 0\},$$

where $g_i, f_{i,j} \in C_U(M)[t_1, \dots, t_q]$ for all i, j. If $\pi : U \times \mathbb{R}^q \to U$ denotes the projection, then $\pi(A)$ is a quasi semianalytic set.

8. Quasi subanalytic sets. Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin and $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ a mapping with components $\varphi_1, \varphi_2, \varphi_3 \in C_U(M)$. We suppose that there are no nontrivial formal relations between the Taylor series $T_0\varphi_1, T_0\varphi_2, T_0\varphi_3$ of $\varphi_1, \varphi_2, \varphi_3$ at the origin. Let $r > 0$ be such that $W = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le r\} \subset U$. Then $A = \varphi(W) \subset \mathbb{R}^3$ is not quasi semianalytic at the origin in $\mathbb{R}^3$, whereas A is the projection of the set $\{(x, y, t_1, t_2, t_3) \in U \times \mathbb{R}^3 \mid x^2 + y^2 \le r,\, t_i = \varphi_i(x, y),\, i = 1, 2, 3\}$ which is a relatively compact quasi semianalytic set.

Thus the Tarski–Seidenberg theorem is false for quasi semianalytic sets.

Definition 7. Let Z be an M-manifold and $A \subset Z$. We say that A is *quasi subanalytic* in Z if for each $a \in Z$, there exists an open neighborhood U of a in Z, an M-manifold Z' and a relatively compact quasi semianalytic set $A \subset Z \times Z'$ in $Z \times Z'$ such that $\pi(A) = A \cap U$, where $\pi : Z \times Z' \to Z$ is the projection.

From the properties of quasi semianalytic sets, we can easily see that a locally finite union and intersection of quasi subanalytic sets is quasi subanalytic. The closure and each connected component of a quasi subanalytic set are quasi subnalytic; a projection of a relatively compact quasi subanalytic set is quasi subanalytic.

We will prove that the complement (and thus the interior) of a quasi subanalytic set is quasi subanalytic. First, we establish some measure properties of a quasi subanalytic set. By the work of Charbonnel [2] and Wilkie [12], we will first show that we have a uniform bound on the number of connected components of the fibers of a projection restricted to a relatively compact quasi subanalytic set; more precisely:

Theorem 7. *Let Z and Z' be two M-manifolds and A be a relatively compact quasi subanalytic set in $Z \times Z'$. Let $\pi : Z \times Z' \to Z$ be the projection. Then the number of connected components of a fiber $\pi^{-1}(x) \cap A$ is bounded uniformly in $x \in Z$.*

Proof. We proceed by induction on $\dim Z$. If $\dim Z = 0$, the result is true, since A is relatively compact. Suppose that $n := \dim Z \geq 1$ and the result is true for $n - 1$. We can assume that $Z = \mathbb{R}^n$, $Z' = \mathbb{R}^p$ and A is relatively compact and quasi semianalytic in $\mathbb{R}^n \times \mathbb{R}^p$. We argue by induction on the maximum dimension of the fibers $A_x = \pi^{-1}(x) \cap A$ for $x \in \mathbb{R}^n$. By Lemma 6, it is enough to find a quasianalytic set $F \subset \mathbb{R}^n \times \mathbb{R}^p$ such that the assertion is true for $A - F$. By Theorem 5, we can suppose that A is a connected component of a quasi semianalytic stratum

$$S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid f_1(x, y) = \ldots = f_k(x, y) = 0,$$
$$\delta(x, y) \neq 0,\ g_1(x, y) > 0, \ldots, g_q(x, y) > 0\},$$

where $\delta(x, y)$ is the jacobian of $(f_1, \ldots, f_k)$. Let $n - \beta$, $0 \leq \beta \leq n$, be the maximum rank of $\pi_{|S}$. Then there exists a jacobian

$$\delta_1(x, y) = \frac{D(f_1, \ldots, f_k)}{D(x_{i_1}, \ldots, x_{i_\beta}, y_{j_1}, \ldots, y_{j_\alpha})}$$

with $\alpha + \beta = k$ such that $\delta_1 \notin I(S)$. We take $F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid \delta_1(x, y) = 0\}$ and put $S' = S - F$. The rank of $\pi_{|S'} : S' \to \mathbb{R}^n$ is constant and equal to $n - \beta$. For all $x \in \mathbb{R}^n$, $S'_x = \pi^{-1}(x) \cap S'$ is a submanifold of dimension $p - \alpha$.

We can suppose, for the proof, that $p-\alpha=0$. Indeed, if $p-\alpha\geq 1$, then each connected component, say C, of $\pi^{-1}(x)\cap S'$ satisfies $\overline{C}-C\neq\emptyset$ (the projection $\pi^{-1}(x)\cap S'\to\{y\in\mathbb{R}^p \mid y_{j_1}=\ldots=y_{j_\alpha}=0\}$ is open). Let $\psi(x,y)=\sum_{j=1}^{q} g_j(x,y)+\delta(x,y)^2+\delta_1(x,y)^2$; then $\psi(x,y)>0$ on C and $\psi(x,y)=0$ if $(x,y)\in\overline{C}-C$. Put $S''=\{(x,y)\mid \operatorname{grad}(\psi_{|\pi^{-1}(x)\cap S'})(x,y)=0\}$. Then S'' is a quasi semianalytic set. Since ψ is not constant on any connected component of $\pi^{-1}(x)\cap S'$, we have $\dim S''_x<\dim S'_x$ for all $x\in\mathbb{R}^n$ ($S''_x=S''\cap\pi^{-1}(x)$). We remark that ψ has a positive maximum on each connected component of $\pi^{-1}(x)\cap S'$, hence $S''_x\neq\emptyset$. By the inductive hypothesis on the dimension of the fibers, the theorem is true for S'', which implies the result for S'.

Suppose $p-\alpha=0$. Then S'_x is a finite set for all $x\in\mathbb{R}^n$. We consider two cases:

Case 1: $n-\beta<n$. Let $\pi_1:\mathbb{R}^n\to\mathbb{R}^{n-\beta}=\{x\in\mathbb{R}^n\mid x_{i_1}=\ldots=x_{i_\beta}=0\}$ be the projection. The inductive hypothesis on n implies that the assertion is true for the mapping $\pi_1\circ\pi_{|S'}$, and hence for $\pi_{|S'}:S'\to\mathbb{R}^n$.

Case 2: $n-\beta=n$. Let $\pi':\mathbb{R}^n\to\mathbb{R}^{n-1}$ be the projection onto $x_n=0$ and put $\widetilde{\pi}=\pi'\circ\pi$. Then $\widetilde{\pi}_{|S'}:S'\to\mathbb{R}^{n-1}$ is a submersion. For all $x'\in\mathbb{R}^{n-1}$, $\widetilde{\pi}^{-1}(x')\cap S'$ is the disjoint union of a finite number of connected curves of class M; by the inductive hypothesis on n, the number of these curves is bounded when $x'\in\mathbb{R}^{n-1}$. In order to prove that the number of points in $\pi^{-1}(x)\cap S'$ is bounded ($x=(x',x_n)$), we will prove that no connected component of $\widetilde{\pi}^{-1}(x')\cap S'$ contains two points of $\pi^{-1}(x)\cap S'$, which proves our result, since the number of connected components of $\widetilde{\pi}^{-1}(x')\cap S'$ is bounded when $x'\in\mathbb{R}^{n-1}$.

Suppose, for a contradiction, that there exists a connected component C of $\widetilde{\pi}^{-1}(x')\cap S'$ which contains $a,b\in\pi^{-1}(x)\cap S'$, $a\neq b$. The curve C intersects $\pi^{-1}(x)$ in two points a,b. By the generalized Rolle lemma [5], there exists $\xi\in C$ such that the tangent space to C at ξ contains a parallel vector to $\pi^{-1}(x)=\mathbb{R}^n$. Hence the tangent space $T_\xi S'$ contains a vector parallel to $\pi^{-1}(x)=\mathbb{R}^n$, which is a contradiction since $T_\xi S'$ is transverse to $\mathbb{R}^n$.

Definition 8. Let Z be an M-manifold and $A\subset Z$. We say that A is *Lebesgue measurable* [resp. *A has measure zero*] if for any coordinate chart U with coordinates $\varphi=(x_1,\ldots,x_n)$, $\varphi(U\cap A)$ is Lebesgue measurable in $\mathbb{R}^n$ [resp. $\varphi(U\cap A)$ has measure zero].

Using the last theorem and properties of the class of quasi subanalytic sets cited above, we prove:

Theorem 8. *Let A be a quasi subanalytic set. The following conditions are equivalent*:

(1) *A has no interior point.*

(2) $\overline{A}$ *has no interior point.*
(3) A *has measure zero.*
(4) $\overline{A}$ *has measure zero.*

Proof. The proof uses Theorem 7 and is the same as in [9].

DEFINITION 9. Let Z' be an M-manifold. A mapping $f : A \subset Z \to Z'$ is *quasi subanalytic* if its graph Γ_f is quasi subanalytic in $Z \times Z'$.

We will use the following result:

PROPOSITION 10 ([9]). *If* $f : A \subset Z \to Z'$ *is a quasi subanalytic mapping, then the set of points in* A *where* f *is not continuous has no interior points.*

In the following we will show that the dimension of a quasi semianalytic set is well behaved.

LEMMA 9. *If* $A \subset Z$ *is a nonempty quasi semianalytic set, then we have* $\dim(\overline{A} - A) < \dim A$.

Proof. Recall that, by Theorem 6, $\overline{A} - A$ is quasi semianalytic. Suppose, for a contradiction, that $\dim(\overline{A} - A) =: n - k \geq \dim A =: n - l$. We can suppose that $Z = \mathbb{R}^n$ and A is relatively compact. Let Λ be a connected component of a quasi semianalytic stratum $S \subset \mathbb{R}^n$ such that $\Lambda \subset \overline{A} - A$ and $\dim \Lambda = \dim(\overline{A} - A)$. We have

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; f_1(x) = \ldots = f_k(x) = 0, \right.$$
$$\left. \delta(x) = \frac{D(f_1, \ldots, f_k)}{D(x_{i_1}, \ldots, x_{i_k})}(x) \neq 0,\; g_1(x) > 0, \ldots, g_q(x) > 0 \right\};$$

note that $k \leq l$ by hypothesis.

Let $\pi_{n-k} : \mathbb{R}^n \to \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n \mid x_{i_1} = \ldots = x_{i_k} = 0\}$ be the projection; then $\pi_{n-k|\Lambda} : \Lambda \to \mathbb{R}^{n-k}$ is a local diffeomorphism. Let $a \in \Lambda$ and put $a' = \pi_{n-k}(a)$. There exist balls $B_n(a, r)$ and $B_{n-k}(a', r)$ in $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ respectively such that $\pi_{n-k|\Lambda \cap B_n(a,r)} : \Lambda \cap B_n(a, r) \to B_{n-k}(a', r)$ is a diffeomorphism; let $g : B_{n-k}(a', r) \to \Lambda \cap B_n(a, r)$ be the inverse.

Let $B = \{x' \in B_{n-k}(a', r) \mid \exists x \in A \cap B_n(a, r),\, \pi_{n-k}(x) = x'\}$. Then B is a quasi subanalytic set. Clearly, we have $B_{n-k}(a', r/2) \subset \overline{B}$; hence, by Theorem 8, $\operatorname{int}(B) \neq \emptyset$; this implies that $k = l$. Put $\pi_k : \mathbb{R}^n \to \mathbb{R}^k = \{x \in \mathbb{R}^n \mid x_j = 0, \forall j \notin \{i_1, \ldots, i_k\}\}$. For each $p = 1, 2, \ldots$, let

$$B_p = \{x' \in B \mid \exists y_1, \ldots, y_p \in \mathbb{R}^k,\, y_i \neq y_j \text{ if } i \neq j,\, y_i \in \pi_k[A \cap \pi_{n-k}^{-1}(x')]\}.$$

We have

$$\ldots \subset B_{\nu+1} \subset B_\nu \subset \ldots \subset B_2 \subset B_1 = B.$$

By Theorem 7, there exists $\mu \in \mathbb{N}^*$ such that $\operatorname{int}(B_\mu) \neq \emptyset$ and $\operatorname{int}(B_{\mu+1}) = \emptyset$. We then have $\operatorname{int}(\overline{B}_{\mu+1}) = \emptyset$, hence $\operatorname{int}(B_\mu) \cap B - \overline{B}_{\mu+1} \neq \emptyset$. Thus there exists a ball $B' \subset B_\mu - B_{\mu+1}$. For each $x' \in B'$, $\pi_k^{-1}(x') \cap A$ contains exactly

μ elements, so we can construct μ functions $h_1, \ldots, h_\mu : B' \subset \mathbb{R}^{n-k} \to \mathbb{R}^k$ such that Γ_{h_j} is quasi subanalytic for all $j = 1, \ldots, \mu$, and $\pi_k[A \cap \pi_{n-k}^{-1}(x')] = \{h_1(x'), \ldots, h_\mu(x')\}$ for all $x' \in B'$. By construction, $h_j(x') \neq \pi_k \circ g(x')$ for all $j = 1, \ldots, \mu$ and $x' \in B'$.

By Proposition 10, there is a ball $B'' \subset B$ such that the restrictions of all $h_1, \ldots, h_\mu$ are continuous on B'' and there exists $c > 0$ such that $|h_j(x') - \pi_k(g(x'))| > c$ for all $x' \in B''$ and $j = 1, \ldots, \mu$; but this contradicts the fact that $g(x') \in \Lambda \subset \overline{A} - A$ for all $x' \in B''$; hence the lemma.

9. Theorem of the complement

THEOREM 9. *Let Z be an M-manifold and let $B \subset Z$ be a quasi subanalytic set. Then $Z - B$ is quasi subanalytic.*

Proof. We can assume that $Z = \mathbb{R}^n$ and B is relatively compact. We argue by induction on n. There exists a relatively compact semianalytic set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ such that $\pi(A) = B$, where $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the projection. By Theorem 5, we can assume that A is a connected component of a quasi semianalytic stratum

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \mid f_1(x,y) = \ldots = f_k(x,y) = 0,\\ \delta(x,y) \neq 0,\ g_1(x,y) > 0, \ldots, g_q(x,y) > 0\}.$$

As in the proof of Theorem 7 (and with the same notations), it is enough to find a quasianalytic set $F \subset \mathbb{R}^n \times \mathbb{R}^p$ such that $A - F \neq \emptyset$ and the assertion is true for $\pi(A - F)$. We take F as in the proof of Theorem 7 and put $A' = A - F \subset S' = S - F$. We proceed by induction on the maximum dimension of the fibers $\pi^{-1}(x) \cap A'$. Recall that $\dim(\pi^{-1}(x) \cap S') = p - \alpha$ for all $x \in \mathbb{R}^n$.

Suppose that $p - \alpha = 0$; then $\dim S' = n - \beta \leq n$. We consider two cases:

CASE 1: $\beta > 0$. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{n-\beta} = \{x \in \mathbb{R}^n \mid x_{i_1} = \ldots = x_{i_\beta} = 0\}$ be the projection. The inductive hypothesis shows that the theorem is true in $\mathbb{R}^{n-\beta}$. Put $\pi' = \pi_1 \circ \pi$. The number of points in $S' \cap \pi'^{-1}(u)$ is bounded when $u \in \mathbb{R}^{n-\beta}$. Therefore the number of points in $\pi(A') \cap \pi_1^{-1}(u)$ is bounded. By Lemma 10 below, the complement of $\pi(A')$ in $\mathbb{R}^n$ is quasi subanalytic.

CASE 2: $\beta = 0$. We then have $\dim S' = n$. Let $Q = \overline{A'} - A'$; by Lemma 9, $\dim Q < n$, hence, by the first case, $\mathbb{R}^n - \pi(Q)$ is quasi subanalytic. We have $\mathbb{R}^n - \pi(A') = (\mathbb{R}^n - \pi(\overline{A'}) \cup (\pi(Q) - \pi(A') \cap \pi(Q))$. By case 1, $\mathbb{R}^n - \pi(A') \cap \pi(Q)$ is quasi subanalytic, hence $\mathbb{R}^n - \pi(A')$ is quasi subanalytic.

If $p - \alpha > 0$, we see that there exists $S'' \subset S'$ such that $\dim S'' < \dim S'$, S'' is quasi semianalytic and $\pi(S'') = \pi(S')$. By using the inductive hypothesis on the maximum dimension of the fibers $\pi^{-1}(x) \cap A'$, we deduce that $\mathbb{R}^n - \pi(A')$ is quasi subanalytic.

LEMMA 10. *Suppose that, in* $\mathbb{R}^n$, *the complement of every quasi subanalytic set is quasi analytic. Let* $A \subset \mathbb{R}^n \times \mathbb{R}^p$ *be a relatively compact quasi subanalytic set. Suppose that the number of points in the fibers* $A \cap \pi^{-1}(x)$, $x \in \mathbb{R}^n$, *is bounded, where* $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ *is the projection. Then* $\mathbb{R}^n \times \mathbb{R}^p - A$ *is quasi subanalytic.*

Proof. The proof is the same as in [1, Lemma 3.9].

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Faculty of Sciences
Department of Mathematics
B.P. 133, Kénitra, 14000, Morocco
E-mail: kabdelhafed@hotmail.com

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