STUDIA MATHEMATICA 187 (1) (2008)

Classes of distribution semigroups

by

P. C. Kunstmann (Karlsruhe), M. Mijatović (Novi Sad) and S. Pilipović (Novi Sad)

Abstract. We introduce various classes of distribution semigroups distinguished by their behavior at the origin. We relate them to quasi-distribution semigroups and integrated semigroups. A class of such semigroups, called strong distribution semigroups, is characterized through the value at the origin in the sense of Łojasiewicz. It contains smooth distribution semigroups as a subclass. Moreover, the analysis of the behavior at the origin involves intrinsic structural results for semigroups. To this purpose, new test function spaces and distribution semigroups over these spaces are introduced. We give applications to Schrödinger type equations in the spaces C_{b}, L^∞, and BMO with elliptic non-densely defined operators.

0. Introduction. After Arendt's paper [2], n-times integrated semigroups as well as distribution semigroups, which were introduced much earlier by Lions [16], have been studied by many authors (see, e.g., [4], [5], [3], [9], [11]). We refer in particular to the papers of Wang [22] and the first named author [15] as well as to the references therein.

In this paper we discuss regularity properties of distribution semigroups, our main interest being in their behavior at the origin. The analysis of once integrated semigroups as introduced by Kellermann and Hieber [11] can in a certain sense make the impression that results for this class can be easily transferred to the case of n-times integrated semigroups with $n > 1$. In the language of distributions this would mean that, for distribution semigroups, the local order n at the origin is of no significant importance. Actually, it follows from our analysis that the case $n = 1$ is specific and structurally simpler.

We introduce several classes of distribution semigroups distinguished by their behavior at the origin. We start from the properties in Lions' original definition [16] and always require the semigroup property on $(0,\infty)$ ((d.1) in

2000 *Mathematics Subject Classification*: 47D06, 47D62.

Key words and phrases: integrated semigroups, distribution semigroups, smooth distribution semigroups, Schrödinger type equations, pseudo-differential equations.

Section 1) as well as the usual non-degeneracy condition ((d.2) in Section 1) that is needed to define the generator as a closed linear operator. These two properties are required for a *weak distribution semigroup* (or weak DS). We drop Lions' denseness assumption ((d.3) in Section 1) and we investigate the condition that prescribes the behavior at the origin ((d.4) in Section 1). Replacing (d.4) by other conditions we introduce *strong distribution semigroups* and *distribution semigroups*. It turns out that distribution semigroups are the same as *quasi-distribution semigroups* introduced in [22], [15] whereas strong distribution semigroups are characterized via the value 0 at the origin for their primitive where the value is understood in the sense of Łojasiewicz.

Concerning the relations between those classes we find that a strong distribution semigroup is always a distribution semigroup and that a distribution semigroup is always a weak distribution semigroup, but the converse implications are false in general. For distributions of local order 1, however, we show that all notions coincide.

Still, there is an open problem whether the class of strong distribution semigroups is a proper subclass of those quasi-distribution semigroups whose stationary dense infinitesimal generator A satisfies $n(A) \leq 1$. Here $n(A)$ denotes the "density" index introduced in [14] (cf. Section 2 for the definition). In Example 1 we give a quasi-distribution semigroup which has non-densely defined infinitesimal generator A satisfying $n(A) = 2$.

A characterization of a semigroup through its value in the sense of Łojasiewicz is given in Theorem 5 and Proposition 3, and we are led to an intrinsic characterization of some classes of distribution semigroups. This is accomplished with the analysis of test function spaces with appropriate integrability conditions at the origin and the corresponding distribution semigroups (cf. Proposition 4 and Theorems 6 and 7).

The structural properties of strong distribution semigroups are given by considering such semigroups on test function spaces consisting of functions with appropriate integrability properties at the origin. In this way, it is shown that the class of strong distribution semigroups contains properly the class of smooth distribution semigroups introduced by Balabane and Emamirad [6]–[8] whose infinitesimal generators are always densely defined. Roughly speaking, the relation of strong distribution semigroups to smooth distribution semigroups is very similar to the relation of once integrated semigroups to C_0-semigroups (cf. Examples 2–4 and Remark 2 in Section 3). We show in Example 5 that results analogous to those in [8] can be obtained for Schrödinger type evolution equations in Banach spaces like $C_{\mathrm{b}}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$. In those spaces the domain of an operator $A = iP(D)$ (where P is an elliptic differential operator with real coefficients) is not dense. By our approach we obtain sup-norm estimates for corresponding solutions while in [8] such estimates were obtained for L^p-norms with $1 < p < \infty$.

We can summarize the classification results of this paper as follows:

1. A distribution semigroup $\mathcal{G}$ with densely defined infinitesimal generator A is not "too singular" at the origin: it has the local form

$$\mathcal{G}(\cdot,x)=(t^{k-1}F)^{(k)}(\cdot,x),\qquad x\in E_0,$$

where $k\in\mathbb{N}$, $t\mapsto F(t,x)$ is continuous, $F(t,x)=0$ for $t\le 0$, $x\in E_0$, and E_0 is a dense set in E. Smooth semigroups are included in this class.

Remark 5 shows that dense distribution semigroups cannot be of the form $\mathcal{G}(\cdot,x)=(t^kF)^{(k)}(\cdot,x)$ for $x\in E$, where F is continuous on $(-\infty,a)$ $(a>0)$ and $F(t,\cdot)=0$ for $t\le 0$.

2. In classifying non-dense distribution semigroups, we have DS as well as strong and smooth DS. Again, the regularity at the origin cannot be higher than for smooth DS. Introducing *strong r-type distribution semigroups* at the end of Section 2, we show that on a dense set E_0, $\mathcal{G}^{(-r)}$ has the value zero at the origin in the sense of Łojasiewicz, i.e. it is of the local form $(t^{k-1-r}F)^{(k)}$, where $F(\cdot,x)$ is continuous and $F(t,x)=0$ for $t\le 0$, $x\in E_0$. Here $r\in\{1,\dots,k-1\}$. In this way we have a whole scale of semigroups. Their better description leads to appropriate distribution semigroups for test function spaces defined on $(0,\infty)$.

1. Integrated, distribution and quasi-distribution semigroups. We denote by E a Banach space with norm $\|\cdot\|$ and by $L(E)=L(E;E)$ the space of bounded linear operators from E into E. We refer to [2], [11] and [4] for the notions of global and local n-times integrated semigroups.

For a linear operator A, its domain, range and null space are denoted by $D(A),R(A)$ and $N(A)$, respectively. We will always assume that A is a closed operator.

The well-known Schwartz spaces of test functions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}=C_0^\infty$, $\mathcal{E}=C^\infty$ and $\mathcal{S}$ ([19]). Their strong duals are $\mathcal{D}'$, $\mathcal{E}'$, and $\mathcal{S}'$, respectively. By $\mathcal{D}_0$ we denote the subspace of $\mathcal{D}$ of elements with support in $[0,\infty)$. Further, $\mathcal{D}'(L(E))=L(\mathcal{D};L(E))$, $\mathcal{E}'(L(E))=L(\mathcal{E};L(E))$ and $\mathcal{S}'(L(E))=L(\mathcal{S};L(E))$ are the spaces of continuous linear functions $\mathcal{D}\to L(E)$, $\mathcal{E}\to L(E)$ and $\mathcal{S}\to L(E)$, respectively, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{S}$, respectively; $\mathcal{D}'_+(L(E)),\mathcal{E}'_+(L(E))$ and $\mathcal{S}'_+(L(E))$ are the subspaces of $\mathcal{D}'(L(E)),\mathcal{E}'(L(E))$ and $\mathcal{S}'(L(E))$, respectively, consisting of the elements supported in $[0,\infty)$ (for $E=\mathbb{C}$ we drop $(L(E))$ in notation). Note that a distribution $F\in\mathcal{D}'(L(E))$ is also a bilinear continuous mapping $f:\mathcal{D}\times E\to E$.

Let $\alpha\in C_0^\infty$ with $\int_{\mathbb{R}}\alpha(x)\,dx=1$. We will use the following nets of smooth functions:

$$(1)\quad \phi_\varepsilon(t) = \frac{1}{\varepsilon}\,\alpha\Big(\frac{t}{\varepsilon}\Big), \quad \theta_\varepsilon(t) = \int_{-\infty}^{t} \phi_\varepsilon(s)\,ds = \int_{-\infty}^{t/\varepsilon} \alpha(s)\,ds, \quad t\in\mathbb{R},\ \varepsilon\in(0,1).$$

Note that $(\phi_\varepsilon)_\varepsilon$ is a delta net and $(\theta_\varepsilon)_\varepsilon$ is a net converging to the characteristic function of $[0,\infty)$ in the sense of $\mathcal{D}'(\mathbb{R})$.

Recall that a (scalar-valued) distribution f has a *value* $C\in\mathbb{C}$ *at the origin in the sense of Łojasiewicz* if $f(\varepsilon\cdot)\to C$ in $\mathcal{D}'$ as $\varepsilon\to 0^+$. This is equivalent to any one of the following two assertions:

(a) $f(\phi_\varepsilon)\to C$ as $\varepsilon\to 0^+$ for all ϕ_ε of the form (1),
(b) there exist $k\in\mathbb{N}$ and a function $F\in C(V_0)$ such that $f(t)=F^{(k)}(t)$ for all $t\in V_0$ and $F(t)/t^k\to C/k!$ as $t\to 0$,

where the kth derivative is understood in the sense of distributions and $C(V_0)=C(V_0;\mathbb{C})$ is the space of continuous complex-valued functions in a neighborhood V_0 of the origin. (For functions that are in addition bounded we use the notation C_{b}.)

In this paper, we need the corresponding assertion for vector-valued distributions $f\in L(\mathcal{D};L(E))$ supported in $[0,\infty)$. The proof will be given in the Appendix.

THEOREM A. *Let* $f\in L(\mathcal{D};L(E))$ *with* $\operatorname{supp} f\subset[0,\infty)$ *and let* E_0 *be a subset of* E. *For all* ϕ_ε *of the form* (1) *we have*

$$f(\phi_\varepsilon,x)\to 0, \qquad x\in E_0\subset E,$$

if and only if

$$(*)\qquad (\exists F\in C(V_0\times E;E),\ \operatorname{supp} F(\cdot,x)\subset[0,\infty),\ x\in E)(\exists k\in\mathbb{N})$$
$$f(t,x)=F^{(k)}(t,x), \qquad t\in V_0,\ x\in E,$$
$$\|F(t,x)/t^k\|_E\to 0 \qquad \textit{as } t\to 0,\ x\in E_0.$$

In this case we say that f *has the value* 0 *at the origin on* E_0. (The kth derivative is understood in the sense of distributions.)

J.-L. Lions ([16]) introduced the notion of a distribution semigroup, which we shall call here a *distribution semigroup in the sense of Lions* or a DS-L for short: a $\mathcal{G}\in\mathcal{D}'_+(L(E))$ is a DS-L if it has the properties (d.1)–(d.4), where:

$$(\text{d.1})\qquad \mathcal{G}(\phi*\psi,\cdot)=\mathcal{G}(\phi,\mathcal{G}(\psi,\cdot)), \qquad \phi,\psi\in\mathcal{D}_0,$$

where $\phi*\psi=\int_{\mathbb{R}}\phi(\cdot-t)\psi(t)\,dt$ is the usual convolution;

(d.2) $\bigcap_{\phi\in\mathcal{D}_0} N(\mathcal{G}(\phi,\cdot))=\{0\}$;
(d.3) the linear hull $\mathcal{R}$ of $\bigcup_{\phi\in\mathcal{D}_0} R(\mathcal{G}(\phi,\cdot))$ is dense in E;
(d.4) for all $x\in\mathcal{R}$ there is a continuous function $u:[0,\infty)\to E$ satisfying $u(0)=x$ and $\mathcal{G}(\phi,x)=\int_0^\infty \phi(t)u(t)\,dt$ for all $\phi\in\mathcal{D}$.

Remark 1. (a) Note that property (d.1) reflects the *semigroup property* on $(0,\infty)$ and (d.2) means a kind of *non-degeneracy* which is needed to define the generator A (see below). If (d.1) and (d.2) hold then property (d.3) is related to *denseness* of the domain of A.

(b) Clearly, (d.4) implies

$$\mathcal{G}(\phi\theta_\varepsilon, x) \to \mathcal{G}(\phi, x) \quad \text{as } \varepsilon \to 0^+, \text{ for all } \phi \in \mathcal{D},\ x \in \mathcal{R}, \tag{2}$$

for every $(\theta_\varepsilon)_\varepsilon$ of the form (1).

In this paper we are interested in dropping the assumption (d.3) and in replacing the assumption (d.4) which expresses a *regularity condition* at the origin. To this end we introduce

Definition 1. Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$. If (d.1) and (d.2) hold then $\mathcal{G}$ is called a *distribution semigroup on* $(0,\infty)$. If (d.1), (d.2) and (d.4) hold then $\mathcal{G}$ is called a *representable distribution semigroup*. If $\mathcal{G}$ is a distribution semigroup on $(0,\infty)$ then it is called a *strong distribution semigroup* if the following property $(\mathrm{d}.5)^{\mathrm{s}}$ holds:

$(\mathrm{d}.5)^{\mathrm{s}}$ There is a dense subspace E_0 of E such that $\mathcal{R} \subset E_0$ and

$$\mathcal{G}(\phi\theta_\varepsilon, x) \to \mathcal{G}(\phi, x) \quad \text{as } \varepsilon \to 0^+, \text{ for all } \phi \in \mathcal{D},\ x \in E_0$$

and for every $(\theta_\varepsilon)_\varepsilon$ of the form (1).

Remark 2. Any DS-L is a strong distribution semigroup. Indeed, by Remark 1(b) we know that $(\mathrm{d}.5)^s$ holds with $E_0 = \mathcal{R}$, which is dense in E by (d.3).

Now assume that $\mathcal{G}$ is a distribution semigroup on $(0,\infty)$. The following construction goes back to the paper of Lions [16] (we also refer to [15]):

Let $T \in \mathcal{E}'_+$, i.e., T is a scalar-valued distribution with compact support in $[0,\infty)$. Define the operator $\mathcal{G}(T)$ in E by

$$x \in D(\mathcal{G}(T)),\ y = \mathcal{G}(T)x \ \Leftrightarrow\ \forall \phi \in \mathcal{D}_0 : \mathcal{G}(T * \phi, x) = \mathcal{G}(\phi, y). \tag{3}$$

We also write $\mathcal{G}(T,x)$ for $\mathcal{G}(T)x$. By (d.2) we find that $y_1 = \mathcal{G}(T)x$ and $y_2 = \mathcal{G}(T)x$ imply $y_1 = y_2$ (because $\mathcal{G}(\varphi, y_1 - y_2) = 0$ for every $\varphi \in \mathcal{D}_0$), hence $\mathcal{G}(T)$ is a well-defined single-valued operator. Linearity and closedness of $\mathcal{G}(T) : D(\mathcal{G}(T)) \to E$ are obvious. By (d.1), the definition is consistent for $T \in \mathcal{D}_0$. Properties of the operators $\mathcal{G}(T)$ are collected in the next proposition (cf. [15, Lemma 3.6] and its proof).

Proposition 1. *Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$ be a distribution semigroup on $(0,\infty)$ and let $T, S \in \mathcal{E}'_+$. Then*:

(a) $\mathcal{G}(\delta) = \mathrm{Id}_E$.

(b) $\mathcal{G}(\phi)$ *commutes with* $\mathcal{G}(T)$ *for all* $\phi \in \mathcal{D}_0$.

(c) $\mathcal{G}(S)\mathcal{G}(T) \subset \mathcal{G}(S * T)$ *with* $D(\mathcal{G}(S)\mathcal{G}(T)) = D(\mathcal{G}(S*T)) \cap D(\mathcal{G}(T))$.

(d) $\mathcal{G}(S) + \mathcal{G}(T) \subset \mathcal{G}(S+T)$.
(e) $\mathcal{R} \subset D(\mathcal{G}(T))$.

In particular, if $\mathcal{G}$ is a distribution semigroup on $(0,\infty)$, then we can define the *generator* $A := \mathcal{G}(-\delta')$ of $\mathcal{G}$; it is a closed linear operator in E.

We are also interested in the following case. Let $\psi \in \mathcal{D}$ and $\psi_+ := \psi 1_{[0,\infty)}$. Then $\psi_+ \in \mathcal{E}'_+$ and, by (3), the operator $\mathcal{G}(\psi_+,\cdot)$ with domain $D(\mathcal{G}(\psi_+))$ is given by

$$x \in D(\mathcal{G}(\psi_+)),\ y = \mathcal{G}(\psi_+, x) \ \Leftrightarrow\ \forall \phi \in \mathcal{D}_0 : \mathcal{G}(\phi, \mathcal{G}(\psi_+, x)) = \mathcal{G}(\phi * \psi_+, y).$$

We consider the following condition:

$$\text{(d.5)} \qquad \mathcal{G}(\psi, x) = \mathcal{G}(\psi_+, x) \quad \text{for all } \psi \in \mathcal{D},\ x \in E.$$

Definition 2. If $\mathcal{G}$ is a distribution semigroup on $(0,\infty)$ then it is called a *distribution semigroup* if (d.5) holds.

We shall show in Theorem 3 below that (d.1), (d.2) and (d.5)^s together imply (d.5), i.e., any strong distribution semigroup is a distribution semigroup. Clearly, (d.5) implies $\mathcal{G} \in \mathcal{D}'_+(L(E))$ if this is not assumed from the beginning.

A *quasi-distribution semigroup* ([22]) or *pre-distribution semigroup* ([15]) (we shall call it QDS for short) on a Banach space E is defined as an element $\mathcal{G} \in \mathcal{D}'(L(E))$ satisfying

$$\text{(Q.D.1)} \qquad \mathcal{G}(\phi *_0 \psi, \cdot) = \mathcal{G}(\phi, \mathcal{G}(\psi, \cdot)), \qquad \phi, \psi \in \mathcal{D},$$

where $\phi *_0 \psi(t) = \int_0^t \phi(t-u)\psi(u)\,du$ for $t \in \mathbb{R}$, and such that

(Q.D.2) (d.2) holds.

Remark 3. Conditions (Q.D.1) and (Q.D.2) imply that $\mathcal{G} \in \mathcal{D}'_+(L(E))$. Indeed, for every $\psi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\psi \subset (-\infty,0)$ and any $\phi \in \mathcal{D}_0$ we have $\phi *_0 \psi = \phi * \psi_+ = 0$. Hence by (Q.D.1), for any $x \in E$,

$$\mathcal{G}(\phi)\mathcal{G}(\psi)x = \mathcal{G}(\phi *_0 \psi)x = 0.$$

Since $\phi \in \mathcal{D}_0$ was arbitrary, we obtain $\mathcal{G}(\psi) = 0$ by (Q.D.2). (We refer here also to [22, Remark 3.4], where (Q.D.1) is not assumed, without arguments.) In a certain sense (Q.D.1) is related to the semigroup property on $[0,\infty)$, not just on $(0,\infty)$ as (d.1).

Remark 4. It was shown by an example at the end of Section 3 in [15] that there exists an element of $\mathcal{D}'_+(L(E))$ satisfying conditions (d.1), (d.2), (d.4) but not (Q.D.1). Hence, in general, a representable DS is not a QDS. In particular, a distribution semigroup on $(0,\infty)$ is, in general, not a QDS. Moreover, the same example shows that a representable DS is not a strong DS, in general. It was shown by Wang [22] and the first named author [13], [15] that a QDS is a representable DS.

Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$. Then $\mathcal{G}$ is called *of finite order* $n \in \mathbb{N}$, resp., of *local finite order* n, if there exists a strongly continuous function $S \in C([0,\infty); L(E))$ with $S(0) = 0$, resp., $S \in C([0,a); L(E))$ with $a > 0$ and $S(0) = 0$ (so we can put $S(t,\cdot) = 0$ for $t \le 0$), such that

$$\mathcal{G} = S^{(n)} \text{ in } \mathbb{R} \quad (\text{resp., } \mathcal{G} = S^{(n)} \text{ in } (-\infty, a)). \tag{4}$$

If $\mathcal{G}$ is of finite order, then we add this to the name of the corresponding distribution semigroup (for example, weak DS of finite order).

2. Characterizations of distribution semigroups. In this section we study further relations between the classes of distribution semigroups defined in the previous section. The following serves as a preparation for the next proposition.

PROPOSITION 2. *Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$ be a distribution semigroup on $(0,\infty)$.*

(a) *For all $x \in \mathcal{R}$ there is a continuous function $u : [0,\infty) \to E$ such that $u(0) = x$ and*

$$\mathcal{G}(\phi, x) = \int_0^\infty \phi(t) u(t)\, dt, \qquad \phi \in \mathcal{D}_0,$$

i.e., the property in (d.4) *holds for all $\phi \in \mathcal{D}_0$.*

(b) *Let A be the generator of $\mathcal{G}$. Let $\psi \in \mathcal{D}$ and $\psi_+ := 1_{[0,\infty)}\psi$. Then*

$$\begin{aligned}
&A\mathcal{G}(\psi_+) \subset \mathcal{G}(-(\psi')_+) - \psi(0)\,\mathrm{Id}_E,\\
&D(A\mathcal{G}(\psi_+)) = D(\mathcal{G}(-(\psi')_+)) \cap D(\mathcal{G}(\psi_+)),\\
&\mathcal{G}(\psi_+)A \subset \mathcal{G}(-(\psi')_+) - \psi(0)\,\mathrm{Id}_E,\\
&D(\mathcal{G}(\psi_+)A) = D(\mathcal{G}(-(\psi')_+)) \cap D(A).
\end{aligned}$$

Proof. (a) Let $\phi, \psi \in \mathcal{D}_0$, $z \in E$ and $x = \mathcal{G}(\psi, z)$. Condition (d.1) and the continuity of $\mathcal{G}$ on $\mathcal{D}$ imply

$$\mathcal{G}(\phi, \mathcal{G}(\psi, z)) = \mathcal{G}(\phi * \psi, z) = \mathcal{G}\Big(\int_0^\infty \phi(t)\tau_t\psi\, dt, z\Big) = \int_0^\infty \phi(t)\mathcal{G}(\tau_t\psi, z)\, dt,$$

where $\tau_t\psi(s) := \psi(s-t)$. Hence,

$$u(t,\cdot) := \mathcal{G}(\tau_t\psi, \cdot), \qquad t \ge 0,$$

defines $u \in C([0,\infty), E)$ with the desired properties.

(b) Proposition 1 implies

$$\begin{aligned}
A\mathcal{G}(\psi_+) &= \mathcal{G}(-\delta')\mathcal{G}(\psi_+) \subset \mathcal{G}(-\delta' * \psi_+) = \mathcal{G}(-(\psi')_+ - \psi(0)\delta)\\
&= \mathcal{G}(-(\psi')_+) - \psi(0)\,\mathrm{Id}_E\,.
\end{aligned}$$

Clearly, $D(A\mathcal{G}(\psi_+)) = D(\mathcal{G}(-(\psi')_+)) \cap D(\mathcal{G}(\psi_+))$. The second part is similar. ∎

THEOREM 1. *Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$. Then $\mathcal{G}$ is a distribution semigroup if and only if it is a quasi-distribution semigroup. In particular, a distribution semigroup is a representable distribution semigroup.*

Proof. Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$ be a distribtion semigroup on $(0,\infty)$. If (d.5) holds then Proposition 2(b) implies that $\mathcal{G} \in \mathcal{D}'_+(L(E;D(A)))$ is a fundamental solution for the operator

$$P_A = \delta' \otimes I - \delta \otimes A \in \mathcal{D}'_+(L(D(A);E))$$

(cf. [15] for this notion). This is equivalent to $\mathcal{G}$ being a QDS ([15], [22]).

On the other hand, if $\mathcal{G}$ is a QDS and $\psi \in \mathcal{D}$ then for $\phi \in \mathcal{D}$ we have $\psi *_0 \phi = \psi_+ * \phi$, and (Q.D.1) yields

$$\mathcal{G}(\psi_+ * \phi) = \mathcal{G}(\psi *_0 \phi) = \mathcal{G}(\phi)\mathcal{G}(\psi),$$

from which (d.5) follows. By Remark 4 we know that a QDS satisfies (d.4). ∎

Hence we obtain the following diagram:

$$\text{DS on } (0,\infty) \ \underset{\Leftarrow}{\not\Rightarrow}\ \text{representable DS} \ \underset{\Leftarrow}{\not\Rightarrow}\ \text{DS} \ \Leftrightarrow\ \text{QDS}.$$

In order to investigate how strong distribution semigroups fit into the picture we recall the relation between local integrated semigroups and quasi-distribution semigroups.

THEOREM 2 ([13], [22], [15]). *Let $\mathcal{G}$ be a QDS. For every $a > 0$ there is $n \in \mathbb{N}$ and a local n-times integrated non-degenerate semigroup $(S(t))_{t\in[0,a)}$ such that $\mathcal{G} = S^{(n)}$ on $(-a,a)$. Conversely, if $a > 0$, $n \in \mathbb{N}$, and $(S(t))_{t\in[0,a)}$ is a local n-times integrated non-degenerate semigroup on $[0,a)$, then the nth distributional derivative $S^{(n)}$ coincides on $(-a,a)$ with a QDS $\mathcal{G}$.*

We also recall the definition of the "density index" $n(A)$ for a closed linear operator A in E (cf. [14]):

$$n(A) := \inf\{k \in \mathbb{N}_0 : \forall m \geq k : \overline{D(A^m)} = \overline{D(A^k)}\}.$$

The operator A is called *stationary dense* if $n(A) < \infty$. Any QDS has a stationary dense generator A and all indices $n(A) \in \mathbb{N}_0$ actually occur (cf. [14]).

THEOREM 3. *If $\mathcal{G}$ is a strong distribution semigroup then it is a distribution semigroup with stationary dense infinitesimal generator A satisfying $n(A) \leq 1$.*

Proof. Assume $(\text{d.5})^{\text{s}}$. Let $\psi \in \mathcal{D}$ and $x \in E_0$. Let $\phi_{\widetilde{\varepsilon}}$ be a net of the form (1) and θ_ε be a net of the form (1) constructed with an $\alpha \geq 0$ and with the additional property $\operatorname{supp}\alpha \subset [1,2]$. Fix $\widetilde{\varepsilon} < 1$. Since $\psi_+\theta_\varepsilon \to \psi_+$ in L^1,

$$\psi_+\theta_\varepsilon * \phi_{\widetilde{\varepsilon}} \to \psi_+ * \phi_{\widetilde{\varepsilon}} \quad \text{as } \varepsilon \to 0, \text{ in } \mathcal{D}.$$

Since $\psi_+\theta_\varepsilon = \psi\theta_\varepsilon$, (d.1) and (d.5)$^{\rm s}$ imply

$$\begin{aligned}\mathcal{G}(\phi_{\widetilde{\varepsilon}} * \psi_+, x) &= \lim_{\varepsilon\to 0} \mathcal{G}(\psi_+\theta_\varepsilon * \phi_{\widetilde{\varepsilon}}, x) = \lim_{\varepsilon\to 0} \mathcal{G}(\psi_+\theta_\varepsilon, \mathcal{G}(\phi_{\widetilde{\varepsilon}}, x)) \\ &= \lim_{\varepsilon\to 0} \mathcal{G}(\psi\theta_\varepsilon, \mathcal{G}(\phi_{\widetilde{\varepsilon}}, x)) = \mathcal{G}(\psi * \phi_{\widetilde{\varepsilon}}, x).\end{aligned}$$

Letting $\widetilde{\varepsilon} \to 0$, we obtain

$$\lim_{\widetilde{\varepsilon}\to 0} \mathcal{G}(\phi_{\widetilde{\varepsilon}} * \psi_+, x) = \mathcal{G}(\psi, x).$$

For $\eta \in \mathcal{D}_0$ the continuity of $\mathcal{G}(\eta, \cdot)$ and (d.1) yield

$$\mathcal{G}(\eta, \mathcal{G}(\psi, x)) = \lim_{\widetilde{\varepsilon}\to 0} \mathcal{G}(\eta, \mathcal{G}(\phi_{\widetilde{\varepsilon}} * \psi_+, x)) = \lim_{\widetilde{\varepsilon}\to 0} \mathcal{G}(\eta * \phi_{\widetilde{\varepsilon}} * \psi_+, x) = \mathcal{G}(\eta * \psi_+, x).$$

Hence $\mathcal{G}(\psi, x) = \mathcal{G}(\psi_+, x)$ for all $x \in E_0$. Now (d.5) holds on E since $\overline{E}_0 = E$ and $\mathcal{G}(\psi_+, \cdot)$ is a closed operator.

Moreover, (d.5)$^{\rm s}$ implies $\overline{\mathcal{R}} \supset D(A)$. This is a consequence of Lemma 1 below. Now, $\overline{\mathcal{R}} \supset D(A)$ implies that A is stationary dense with $n(A) \le 1$ (cf. [14]). ∎

In the proof we used the following lemma ([13, Lemma 2.31]), which we prove here for convenience.

Lemma 1. *Let $\mathcal{G}$ be a distribution semigroup with generator A and let $k \in \mathbb{N}$. For any $a > 0$ we have*

$$\begin{aligned}D(A^k) = \operatorname{span}\{\mathcal{G}(\psi)y : y \in E,\ \psi \in \mathcal{D},\ \operatorname{supp}\psi \subset [-a, a], \\ \forall j \in \{0, \dots, k-2\} : \psi^{(j)}(0) = 0\}.\end{aligned}$$

Proof. Let Y denote the set on the right hand side and fix $a > 0$. If $y \in E$ and $\psi \in \mathcal{D}$ with $\operatorname{supp}\psi \subset [-a, a]$ and $\psi^{(j)}(0) = 0$ for $j = 0, \dots, k-2$, then Proposition 2(b) and (d.5) imply $\mathcal{G}(\psi)y \in D(A)$ and $A\mathcal{G}(\psi)y = \mathcal{G}(-\psi')y - \psi(0)y$, which equals $\mathcal{G}(-\psi')y \in D(A)$ in case $k \ge 2$. Iterating this argument yields $\mathcal{G}(\psi)y \in D(A^k)$, and the inclusion $Y \subset D(A^k)$ is proved.

To prove $D(A^k) \subset Y$ we first observe that Proposition 2(b) and (d.5) imply by induction on $m \in \mathbb{N}$ that for any $m \in \mathbb{N}$, $x \in D(A^m)$, and $\psi \in \mathcal{D}$,

$$\mathcal{G}(\psi)A^m x = \mathcal{G}((-1)^m \psi^{(m)})x - \sum_{j=0}^{m-1} (-1)j\psi^{(j)}(0)A^{m-1-j}x. \tag{5}$$

For $x \in D(A^k)$ we choose $\psi \in \mathcal{D}$ with $\operatorname{supp}\psi \subset [-a, a]$ such that $\psi^{(k-1)}(0) = 1$ and $\psi^{(j)}(0) = 0$ for all $j \in \{0, \dots, k-2\} \cup \{k, \dots, 2k-2\}$. Then (5) yields $x = \mathcal{G}((-1)^k\psi^{(k)})x - \mathcal{G}(\psi)A^k x \in Y$. ∎

By Theorem 3 we have obtained the following picture:

$$\text{QDS} \Leftrightarrow \text{DS} \underset{\not\Rightarrow}{\Leftarrow} \text{strong DS},$$

since there are quasi-distribution semigroups whose generators A have

$n(A) \geq 2$ (cf. also the examples given below). A distribution semigroup whose generator satisfies $n(A) = 0$ is a DS-L, hence a strong distribution semigroup. However, we have the following

Open Problem. If $\mathcal{G}$ is a distribution semigroup and $n(A) = 1$, is it true that $\mathcal{G}$ is a strong distribution semigroup?

The next theorem can be viewed as a partial answer: observe that combination with Theorem 3 shows that local order 1 of a distribution semigroup on $(0,\infty)$ implies $n(A) \leq 1$ for its generator A. The theorem also shows that the case of local order 1 is simpler than the general case.

Theorem 4. *Let* $\mathcal{G} \in \mathcal{D}'_+(L(E))$.

(a) *If* $\mathcal{G}$ *is of local order* $n = 1$ *with corresponding* S *as in* (4) *on* $(-\infty, a)$ *for some* $a > 0$, *then* (d.5)$^{\mathrm{s}}$ *holds for* $\mathcal{G}$ *with* $E_0 = E$.

(b) *If* $\mathcal{G}$ *is of local order* $n = 1$ *then the following are equivalent*:

- (i) $\mathcal{G}$ *is a representable DS*;
- (ii) $\mathcal{G}$ *is a strong DS*;
- (iii) *there exists* $a > 0$ *such that* $(S(t))_{t\in[0,a)}$ *is a once local integrated non-degenerate semigroup.*

Proof. (a) Let $x \in E$, $\mathcal{G} = S'$ in $(-\infty, a)$ and $\psi \in \mathcal{D}((-a, a))$. Assuming that $\operatorname{supp} \alpha \subset (-\infty, a)$, we have

$$\begin{aligned}\mathcal{G}(\psi\theta_\varepsilon, x) &= \langle S'(u,x), (\psi\theta_\varepsilon)(u)\rangle \\ &= -\langle S(u,x), \psi'(u)\theta_\varepsilon(u)\rangle - \int_0^\infty S(u,x)\psi(u)\,\frac{1}{\varepsilon}\,\alpha\Big(\frac{u}{\varepsilon}\Big)\,du.\end{aligned}$$

By Lebesgue's theorem,

$$\int_0^\infty S(u,x)\psi(u)\,\frac{1}{\varepsilon}\,\alpha\Big(\frac{u}{\varepsilon}\Big)\,du = \int_0^\infty S(\varepsilon u, x)\psi(\varepsilon u)\alpha(u)\,du \to 0 \qquad \text{as } \varepsilon \to 0.$$

This implies $\mathcal{G}(\psi\theta_\varepsilon, x) \to \mathcal{G}(\psi, x)$ as $\varepsilon \to 0$. Thus we have (d.5)$^{\mathrm{s}}$.

(b) follows from (a) and Theorems 1–3. ∎

We now characterize strong distribution semigroups via the value 0 at the origin in the sense of Łojasiewicz for their primitive. To this end we define the *primitive* $\mathcal{G}^{(-1)}$ of a distribution $\mathcal{G} \in \mathcal{D}'_+(L(E))$ by $(\mathcal{G}^{(-1)})' = \mathcal{G}$, $\operatorname{supp} \mathcal{G}^{(-1)} \subset [0,\infty)$, i.e.,

$$\langle \mathcal{G}^{-1}(t,\cdot), \varphi(t)\rangle = \Big\langle \mathcal{G}(t,\cdot), \eta(t)\int_t^\infty \varphi(s)\,ds\Big\rangle, \qquad \varphi \in \mathcal{D}, \tag{6}$$

where η is a smooth function that equals 1 on $[0,\infty)$ and 0 on $(-\infty, -a]$, for some $a > 0$.

THEOREM 5. *Let* $\mathcal{G} \in \mathcal{D}'_+(L(E))$. *Then*:

(a) *Condition* $(\mathrm{d}.5)^{\mathrm{s}}$ *implies that* $\mathcal{G}^{(-1)}$ *has the value* 0 *at the origin in the sense of Łojasiewicz on the set* E_0.
(b) *Let* E_0 *denote the set of all* $x \in E$ *such that there exist* $n \in \mathbb{N}$, $a > 0$, *and* $F_x \in C((-a,a);E)$ *with* $F_x(t) = 0$ *for* $t \le 0$ *such that* $\mathcal{G}(\cdot, x) = F_x^{(n)}$ *on* $(-a,a)$ *and* $\|F_x(t)\| = o(t^{n-1})$ *as* $t \to 0$. *If* E_0 *is dense in* E, *then* $\mathcal{G}$ *satisfies* $(\mathrm{d}.5)^{\mathrm{s}}$.

COROLLARY 1. *In particular*, $\mathcal{G} \in \mathcal{D}'_+(L(E))$ *satisfies* $(\mathrm{d}.5)^{\mathrm{s}}$ *if and only if* $\mathcal{G}^{(-1)}$ *has the value* 0 *at the origin on a dense set* $E_0 \subset E$. *Thus a* $\mathcal{G} \in \mathcal{D}'_+(L(E))$ *is a strong distribution semigroup if and only if it is a distribution semigroup on* $(0,\infty)$ *and* $\mathcal{G}^{(-1)}$ *has the value* 0 *at the origin on a dense set* $E_0 \subset E$.

Proof of Theorem 5. (a) Let $(\mathrm{d}.5)^{\mathrm{s}}$ hold. Let θ_ε be of the form (1) and let $x \in E_0$. We have $\mathcal{G}(\theta_\varepsilon \psi, x) \to \mathcal{G}(\psi, x)$ for all $\psi \in \mathcal{D}$, and this implies

$$\mathcal{G}((1-\theta_\varepsilon)\psi, x) \to 0, \qquad \psi \in \mathcal{D}. \tag{7}$$

Since the intersection of the supports of $\mathcal{G}$ and $1-\theta_\varepsilon$ is contained in a compact set, (7) implies that, for every θ_ε of the form (1) and η as in (6),

$$\mathcal{G}(\eta(t)(1-\theta_\varepsilon), x) \to 0.$$

We have

$$\begin{aligned}\eta(t)(1-\theta_\varepsilon(s)) &= \eta(t)\Big(1 - \int_{-\infty}^{s/\varepsilon} \alpha(r)\,dr\Big) = \eta(t)\Big(\int_{s/\varepsilon}^{\infty} \alpha(r)\,dr\Big) \\ &= \eta(t)\int_s^\infty \varepsilon^{-1}\alpha(u/\varepsilon)\,du.\end{aligned}$$

From the definition of ϕ_ε, this implies $\mathcal{G}^{(-1)}(\phi_\varepsilon, x) = \mathcal{G}(\eta(t)(1-\theta_\varepsilon), x) \to 0$ as $\varepsilon \to 0$. By the definition given after Theorem A, we have $\mathcal{G}^{(-1)}(0,x) = 0$ on E_0 in the sense of Łojasiewicz.

(b) Assume $n \ge 2$, since the case $n = 1$ is studied in Theorem 4. By assumption, we have

$$\mathcal{G}^{(-1)}(\cdot, x) = S^{(n-1)}(\cdot, x) \quad \text{on } (-a,a),\ x \in E_0,$$

so that

$$S(t,x) = t^{n-1} g(t,x), \quad \text{and} \quad g(t,x) \to 0 \quad \text{as } t \to 0^+.$$

Let $\psi \in \mathcal{D}((-a,a))$ and $x \in E_0$ (and $\operatorname{supp}\alpha \subset (-a,a)$). We have

$$\begin{aligned}\mathcal{G}(\psi\theta_\varepsilon, x) &= \langle S^{(n)}(u,x), (\psi\theta_\varepsilon)(u)\rangle \\ &= (-1)^n\Big(\sum_{j=1}^{n}\binom{n}{j}\int_0^\infty S(u,x)\psi^{(n-j)}(u)\,\frac{1}{\varepsilon^j}\,\alpha^{(j-1)}\Big(\frac{u}{\varepsilon}\Big)\,du \\ &\qquad + \int_0^\infty S(u,x)\psi^{(n)}(u)\theta_\varepsilon(u)\,du\Big).\end{aligned}$$

Lebesgue's theorem implies that

$$\sum_{j=1}^{n}\binom{n}{j}\int_0^\infty |\varepsilon^{n-j}u^{n-1}g(\varepsilon u,x)\psi^{(n-j)}(\varepsilon u)\alpha^{(j-1)}(u)|\,du \to 0 \quad \text{as } \varepsilon\to 0.$$

Thus (d.5)$^{\mathrm{s}}$ follows from

$$\mathcal{G}(\psi\theta_\varepsilon, x) \to (-1)^n\int_0^\infty S(u,x)\psi^{(n)}(u)\,du = (-1)^n\langle S(\cdot,x),\psi^{(n)}\rangle = \mathcal{G}(\psi,x). \ \blacksquare$$

Remark 5. Note that a dense distribution semigroup cannot be of the form

$$\mathcal{G}(\cdot,x) = (t^kF)^{(k)}(\cdot,x), \qquad x\in D(A),$$

where F is continuous on $(-a,a)$ $(a>0)$ and supported in $[0,a)$. Let us show this. Assume that the above representation holds with $F(0,x)=0$ for $x\in D(A)$. Then the corresponding k-times integrated semigroup $(S(t,\cdot))_{t\ge 0}$ satisfies

$$S(t,x) = (t^k/k!)x + \int_0^t S(u,Ax)\,du, \qquad x\in D(A),\ t\ge 0.$$

Since $\int_0^t S(u,Ax)\,du = \mathcal{O}(t^{k+1})$ as $t\to 0$ ($\mathcal{O}$ is Landau's symbol) it follows that

$$F(t,x) = S(t,x)/t^k = x/k! + \mathcal{O}(t),$$

which implies

$$x = k!F(0,x) = 0, \qquad x\in D(A).$$

This is a contradiction since $\mathcal{G}\ne 0$, and thus $D(A)\ne\{0\}$.

There exist QDS with arbitrary $n(A)\in\mathbb{N}_0$ (cf. [14]), hence in particular with $n(A)>1$. We mention that by Theorem 4.1 of [11] the $([n/2]+2)$th distributional derivative of an $[n/2]+2$-times integrated semigroup which corresponds to $i\Delta$ gives a QDS on $C_{\mathrm{b}}(\mathbb{R}^n)$. One can easily show that $n(i\Delta)>1$.

Here, we will give an example based on the theory of distribution cosine functions [12].

Example 1. Let $A_1 = d/dx$ have a maximal domain in $L^1(\mathbb{R})$. Then A_1 is the generator of a C_0-group of translations $(T(t))_{t\in\mathbb{R}}$. Thus A_1^* and $-A_1^*$ generate once integrated semigroups on $L^\infty(\mathbb{R})$, and $(A_1^*)^2$ generates a once integrated cosine function on $L^\infty(\mathbb{R})$. This is proved in [3, Example 3.15.5]. Moreover, in this example it is shown that $A = (A_1^*)^2$ is not densely defined since $D(A) \subset W^{1,\infty}(\mathbb{R})$, which is not dense in $L^\infty(\mathbb{R})$. Thus, $n(A) \neq 0$. By [12], if A generates a k-times integrated cosine function, then $n(A) \le [(k+1)/2]$. It follows that $n(A) = 1$. This implies that the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

generates a twice integrated semigroup on $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ and $n(\mathcal{A}) = 2$. Thus $\mathcal{A}$ generates a QDS which is not a strong DS.

Now we will use our previous results to obtain a scale of strong distribution semigroups with respect to their behavior at the origin. First, we adapt the notation. Denote

$$\theta_\varepsilon^{-r}(t) = \int_0^t \frac{(t-u)^{r-1}}{(r-1)!}\,\theta_\varepsilon(u)\,du, \qquad t \in \mathbb{R},\ r \in \mathbb{N},$$

and introduce the following condition:

$$(\mathrm{d.5})^{\mathrm{s}}_r \qquad \mathcal{G}(\phi(t)\theta_\varepsilon^{-r}(t), x) \to \mathcal{G}\left(\frac{t^r}{r!}\,\phi(t), x\right) \qquad \varepsilon \to 0^+,\ \phi \in \mathcal{D},\ x \in E_0,$$

where E_0 is a dense subspace of E containing $\mathcal{R}$. If (d.1), (d.2) and $(\mathrm{d.5})^{\mathrm{s}}_r$ hold for $\mathcal{G}$, then we call it a *strong r-type DS.*

Define $\mathcal{G}^{(-r)} = (\mathcal{G}^{(-r+1)})'$, $\operatorname{supp} \mathcal{G}^{(-r+1)} \subset [0,\infty)$, $r = 2, 3, \ldots$. With the same proof as for strong DS we have:

PROPOSITION 3. *Let $r \in \{1, \ldots, k-1\}$ and $\mathcal{G}$ be a strong r-type DS. Then $\mathcal{G}^{(-r-1)}$ has the value 0 at the origin in the sense of Łojasiewicz on a dense set E_0, i.e. it is of the local form $(t^{k-1-r}F)(t,\cdot)^{(k)}$, $t \in (-a,a)$, where F is continuous and $F(0,x) = 0$ for $x \in E_0$.*

This can be seen by inspection of the proofs in this section. We omit the details.

Our Example 1 gives $\mathcal{G} = (tF)'''$, where F has prescribed properties.

3. Generalization of smooth distribution semigroups. In this section we present generalizations of the classes of smooth distribution semigroups introduced in [6]–[8]. Recall ([6], [8]) that the underlying test function space for smooth distribution semigroups is the space $\mathcal{F}_0$, the completion of $\mathcal{D}((0,\infty))$ under the sequence of seminorms

$$q_j(\psi) = \|t^j \psi^{(j)}\|_{L^1((0,\infty))}, \qquad j \in \mathbb{N}_0.$$

A *smooth distribution semigroup* on a Banach space E is a continuous linear mapping $\mathcal{G} : \mathcal{F}_0 \to L(E)$ satisfying

(d.1-smooth) $\forall \phi, \psi \in \mathcal{F}_0 : \mathcal{G}(\phi * \psi, \cdot) = \mathcal{G}(\phi, \mathcal{G}(\psi, \cdot))$,
(d.4-smooth) there is a dense subspace $D \subset E$ such that, for all $x \in D$, there is a function $u \in C([0,\infty); E)$ satisfying $u(0) = x$ and $\mathcal{G}(\phi, x) = \int_0^\infty \phi(t)u(t)\,dt$ for $\phi \in \mathcal{F}_0$.

It has been shown (cf., e.g., [8, p. 369]) that these conditions imply (d.2) and (d.3). Hence any smooth distribution semigroup is a DS-L. In particular, smooth distribution semigroups are strong distribution semigroups. It is also shown there that $\mathcal{G}(\phi_\varepsilon)x \to x$ as $\varepsilon \to 0$ for any $x \in E$, where (ϕ_ε) is as in (1) and the underlying α belongs to $\mathcal{D}((0,\infty))$. This means that smooth distribution semigroups have, in strong operator topology, the value Id_E at $0+$ in the sense of Łojasiewicz.

Again, we are interested in dropping the density assumption (d.4-smooth). But this assumption is used to show (d.2) which we need for the definition of the generator. Hence we put (d.2-smooth) := (d.2) and define

Definition 3. If (d.1-smooth) and (d.2-smooth) hold for $\mathcal{G} \in \mathcal{F}_0'(L(E))$ we call $\mathcal{G}$ a *distribution semigroup on* $\mathcal{F}_0$.

Remark 6. We remark that $\psi_+ \in \mathcal{F}_0$ for every $\psi \in \mathcal{D}$, and that $\phi * \psi = \phi *_0 \psi$ for $\phi, \psi \in \mathcal{F}_0$. Thus one can see directly via (d.5) that every distribution semigroup on $\mathcal{F}_0$ can be extended to $\mathcal{D}$ to become a distribution semigroup. Moreover, we see from Lemma 1 that the generator A of a distribution semigroup on $\mathcal{F}_0$ satisfies $n(A) \le 1$.

Our next result characterizes distribution semigroups on $\mathcal{F}_0$ in terms of integrated semigroups. Specializing to densely defined operators A we recover the characterization of smooth distribution semigroups that was given in [5, Thm. 4.4] (notice that our proof is different).

Theorem 6. *Let A be closed linear operator in E. Then A generates a distribution semigroup $\mathcal{G}$ on $\mathcal{F}_0$ if and only if A generates a distribution semigroup and there are $n \in \mathbb{N}$ and $C > 0$ such that $\mathcal{G} = S^{(n)}$ for an n-times integrated semigroup $(S(t))_{t\ge 0}$ satisfying, for some $C > 0$,*

$$\|S(t)\| \le Ct^n, \qquad t \ge 0.$$

Proof. Let $\mathcal{G}$ be a distribution semigroup with generator A. We first suppose that $\mathcal{G} = S^{(n)}$ for an n-times integrated semigroup $(S(t))$ satisfying $\|S(t)\| \le Ct^n$ for $t \ge 0$. For $\phi \in \mathcal{D}_0$ and $x \in E$ we then have

$$\begin{aligned}\|\mathcal{G}(\phi)x\| &= \Big\|(-1)^n \int_0^\infty \phi^{(n)}(t)S(t)x\,dt\Big\| \le \int_0^\infty |t^n\phi^{(n)}(t)|\,\|t^{-n}S(t)x\|\,dt\\ &\le Cq_n(\phi)\|x\|.\end{aligned}$$

Hence $\mathcal{G}$ is a distribution semigroup on $\mathcal{F}_0$.

Conversely, let $\mathcal{G}$ be a distribution semigroup on $\mathcal{F}_0$. We choose $k \in \mathbb{N}_0$ and $K > 0$ such that $\|\mathcal{G}(\phi)\|_{L(E)} \le K \sum_{j=0}^{k} q_j(\varphi)$ for $\phi \in \mathcal{F}_0$. By continuity we extend $\mathcal{G}$ to $\mathcal{F}_0^k$, the closure of $\mathcal{F}_0$ for the norm $\|\cdot\|_k := \sum_{j=0}^{k} q_j$. We denote the extension by $\widetilde{\mathcal{G}}$ and let $n := k+1$. By [15, Sect. 4] we know that $\mathcal{G} = S^{(n)}$ for $S(t) = \widetilde{\mathcal{G}}(I_t^n)$, $t \ge 0$, where $I_t^n(s) := \frac{(t-s)^{n-1}}{(n-1)!} 1_{[0,t]}(s)$ for $s, t \ge 0$ (observe that $t \mapsto I_t^n$ is continuous for $\|\cdot\|_k$). Now we estimate, for $j = 0, \ldots, k$ and $t > 0$,

$$q_j(I_t^n) = \int_0^t s^j \left| \left(\frac{d}{ds}\right)^j \frac{(t-s)^{n-1}}{(n-1)!} \right| ds = \int_0^t s^j \, \frac{(t-s)^{n-1-j}}{(n-1-j)!} \, ds$$
$$= t^n \int_0^1 \frac{\sigma^j (1-\sigma)^{n-1-j}}{(n-1-j)!} \, d\sigma = c_{n,j} t^n,$$

which yields $\|S(t)\| \le Ct^n$ for $C := K \sum_{j=0}^{k} c_{n,j}$. ∎

If we combine this theorem with Theorem 5 and Corollary 1, we obtain

Corollary 2. *Let $\mathcal{G}$ be a distribution semigroup on $\mathcal{F}_0$. Then*

(d.5)$^{\mathrm{glob}}$ $\qquad \mathcal{G}(\theta_\varepsilon \psi)x \to \mathcal{G}(\psi)x \quad$ *for all* $\psi \in \mathcal{D}$, $x \in E$,

where θ_ε is given in (1). *In particular*, $\mathcal{G}$ *is a strong distribution semigroup.*

We give a number of examples.

Example 2. If $\mathcal{G}$ is a smooth distribution semigroup in E then the dual operators $\mathcal{G}^*(\cdot)$ in E^* form a distribution semigroup on $\mathcal{F}_0$.

Example 3. Let $T : (0,\infty) \to L(E)$ be a strongly continuous function satisfying $T(s)T(t) = T(t+s)$, $s, t > 0$ and $\bigcap_{t>0} N(T(t)) = \{0\}$. If we let $T(0) := I$, then $(T(t))_{t\ge0}$ is a semigroup which is strongly continuous on $(0,\infty)$, but not necessarily a C_0-semigroup. Notice also that we do not require $\bigcup_{t>0} \mathcal{R}(T(t))$ to be dense in E. Assume that

$$\|T(t)\| \le Mt^{-\alpha}, \quad t > 0, \text{ for some } M > 0,\ \alpha \in (0,1).$$

Then the formula $\mathcal{G}(\psi, x) := \int_0^\infty \psi(t)T(t)x\,dt$ defines a DS in E. Letting $S(t)x := \int_0^t T(s)x\,ds$ we obtain a norm continuous function S satisfying

$$\|S(t)\| \le Mt^{1-\alpha}/(1-\alpha), \quad t > 0.$$

In particular, $\|S(t)\| = o(1)$ as $t \to 0$. Since $\mathcal{G} = S'$ in the distribution sense, $\mathcal{G}$ satisfies (d.5)$^{\mathrm{glob}}$ by Theorem 5. But, in general, $\mathcal{G}$ is not a distribution semigroup on $\mathcal{F}_0$ (cf. Theorem 6). In this example $\|R(\lambda, A)\| \le M\lambda^{\alpha-1}$ for $\lambda > 0$, and this implies $n(A) \le 1$ directly by an argument similar to [14, Lemma 1.5].

For the sake of completeness, we construct a family of operators $(T(t))_{t>0}$ which satisfies the above conditions. Take $\beta > 1$ and $E = l^2 \times l^2$, whose elements will be denoted as sequences of pairs (x_n, y_n), and define the operator A as a diagonal operator, mapping the nth pair (x_n, y_n) to $(-nx_n + n^\beta y_n, -ny_n)$. Then the semigroup operators $T(t)$ map (x_n, y_n) to $(e^{-nt}x_n + tn^\beta e^{-nt}y_n, e^{-nt}x_n)$. An easy calculation shows that $\|T(t)\| = O(t^{1-\beta})$ as $t \to 0$ and that this is optimal. Hence one gets an example for $\beta = 1 + \alpha$.

Example 4. Let A be a Hille–Yosida operator in E, i.e.

$$\sup_{\lambda>0,\, n\in\mathbb{N}} \|\lambda^n R(\lambda, A)^n\| < \infty.$$

By [2], A generates a once integrated semigroup $(S(t))_{t\geq 0}$ satisfying $\|S(t)\| \leq Mt$ for $t \geq 0$. Hence $\mathcal{G} := S'$ is a distribution semigroup on $\mathcal{F}_0$, in particular, $\mathcal{G}$ satisfies $(\mathrm{d}.5)^{\mathrm{glob}}$.

Remark 7. Let $\mathcal{G}$ be a distribution semigroup in E with generator A and assume that $n(A) = 1$. Define $F := \overline{D(A)}$ $[= \overline{\mathcal{R}}]$. Then F is a closed subspace of E which is invariant under each operator $\mathcal{G}(\psi, \cdot)$, $\psi \in \mathcal{D}$, and letting $H(\psi, \cdot) := \mathcal{G}(\psi, \cdot)|_F$ we obtain a dense DS H in F with generator B (which is the part of A in F, i.e. $x \in D(B)$ and $Bx = y$ if and only if $x \in D(A) \cap F$, $Ax = y$ and $y \in F$). Suppose now that H is a smooth distribution semigroup. We show that $\mathcal{G}$ satisfies $\mathcal{G}^{(-1)}(\phi_\varepsilon)x \to 0$ for $x \in E$.

Choosing $\lambda_0 \in \varrho(A)$ we have the representation $\mathcal{G} = (\lambda_0 - A)H(\lambda_0 - A)^{-1}$. We take $\lambda_0 = 0$ without loss of generality. We have $\mathcal{G}' - A\mathcal{G} = \delta \otimes \mathrm{Id}_E$ and obtain by integration $\mathcal{G} - A\mathcal{G}^{(-1)} = I \otimes \mathrm{Id}_E$, i.e.

$$\mathcal{G}^{(-1)}(\phi_\varepsilon, x) = A^{-1}\mathcal{G}(\phi_\varepsilon, x) - I(\phi_\varepsilon)A^{-1}x = H(\phi_\varepsilon)A^{-1}x - A^{-1}x.$$

By smoothness of H we have $H(\phi_\varepsilon)A^{-1}x \to A^{-1}x$, and the claim is proved.

In fact, Example 4 is related to this remark: If $n(A) = 1$ in Example 4, then the operator B defined above generates a C_0-semigroup in F.

Again, we summarize in a picture:

$$\text{DS on } \mathcal{F}_0 \ \underset{\Rightarrow}{\not\Leftarrow}\ \text{DS with } (\mathrm{d}.5)^{\mathrm{glob}} \ \Rightarrow\ \text{strong DS}.$$

Open Problem. Let $\mathcal{G}$ be a distribution semigroup on $(0, \infty)$. Does $(\mathrm{d}.5)^{\mathrm{s}}$ imply $(\mathrm{d}.5)^{\mathrm{glob}}$?

This seems unlikely to hold, but we do not have a counterexample.

The next result for non-densely defined infinitesimal generators is an extension of Theorem 4 in [7].

Proposition 4. *Let $\mathcal{G}$ be a DS on $\mathcal{F}_0$ of the form $\mathcal{G} = S^{(k)}$, where S is continuous and supported by $[0, \infty)$, and let A be the infinitesimal generator of $\mathcal{G}$. Then for any $x \in D(A^k)$, $\mathcal{G}(\cdot, x)$ is a continuous function on $\mathbb{R}$ supported by $[0, \infty)$ satisfying*

$$\|\mathcal{G}(t, x)\| \leq (\|x\| + \|A^k x\|)(1 + t^k), \qquad t \in \mathbb{R}. \tag{8}$$

Proof. It is well known that for $x \in D(A^k)$,

$$\frac{d^k}{dt^k} S(t,x) = S(t, A^k x) + \frac{t^{k-1}}{(k-1)!} A^{k-1}x + \cdots + x, \quad t \in \mathbb{R}.$$

This implies that, for $x \in D(A^k)$,

$$\mathcal{G}(\phi, x) = \langle S^{(k)}(t,x), \phi(t)\rangle, \quad \phi \in \mathcal{F}_0,$$

where $t \mapsto S^{(k)}(\cdot, x) = \mathcal{G}(\cdot, x)$ is a continuous bounded function with the value zero at the origin.

The proof of (8) is the same as the proof of Theorem 4 in [7]. In fact, we have to observe that A generates a stationary dense DS with $n(A) \le 1$, hence $D(A^{k+2})$ is dense in $D(A^k)$. In this way, using Lemmas 1–4 of [7] we have the same proof as in that paper. ∎

Example 5 (Schrödinger type evolution equations in $C_{\mathrm{b}}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$, and $\mathrm{BMO}(\mathbb{R}^n)$). Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is the space of functions of bounded mean oscillation (modulo constants) (cf. [21]); it is the dual of $H^1(\mathbb{R}^n)$. Consider

$$\partial U/\partial t = iP(D)U, \quad t > 0,\ x \in \mathbb{R}^n,$$

where $P(\xi)$ is a real-valued elliptic polynomial on $\mathbb{R}^n$, homogeneous of degree $2m$ (with $m \in \mathbb{N}$), i.e., in the usual multi-index notation,

$$P(\xi) = \sum_{|\alpha|=2m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n,$$

with $a_\alpha \in \mathbb{R}$. Here we set $D := (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$, and by ellipticity we mean $P(\xi) \ge \delta|\xi|^{2m}$ for some $\delta > 0$ and all $\xi \in \mathbb{R}^n$.

Problems of this type have been studied in [6] and, more generally, in [8] within the framework of smooth distribution (semi)groups. We recall the fact (proved in [20]) that

$$\xi \mapsto \int_0^1 (1-s)^k e^{isP(\xi)}\, ds \in \mathcal{F}L^1(\mathbb{R}^n)$$

if $k > n/2$ ($\mathcal{F}$ denotes Fourier transform). Arguing as in [7, p. 27] (we have $N = 1$ in our case), we see that the operator $A := iP(D)$ generates a smooth distribution semigroup G of order k, given by

$$G(\phi)f = \mathcal{F}^{-1}\Big(\xi \mapsto \int_0^\infty \phi(t) e^{itP(\xi)}\, dt\Big) * f,$$

on any homogeneous Banach space E on $\mathbb{R}^n$ (homogeneous in the sense of [10]), hence in particular on $L^1(\mathbb{R}^n)$ and on the Hardy space $H^1(\mathbb{R}^n)$ (defined, e.g., in terms of atoms, cf. [21]). Recalling Example 2 we obtain a DS satisfying $(\mathrm{d}.5)^{\mathrm{glob}}$ in $L^\infty(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ whose generator is not densely defined. Since the DS is given by convolution it is clear that we also have a DS satisfying (d.5)$^{\text{glob}}$ on every translation invariant closed subspace of $L^\infty(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$, e.g. on $C_{\mathrm{b}}(\mathbb{R}^n)$, $\mathrm{BUC}(\mathbb{R}^n)$ (the bounded uniformly continuous functions), or $\mathrm{VMO}(\mathbb{R}^n)$ (the functions of vanishing mean oscillation, cf. [21]). Observe that the generator is densely defined in those spaces if and only if the translation group is strongly continuous. Since the same arguments apply to $-P(D)$ we have in fact not only a semigroup but a group.

We may even adapt arguments from Section 5 in [8] to study the equation for the operator $iP(D)+V(x)$ in place of $iP(D)$, where V is a suitable potential. To this end we recall some estimates from [8]. By Remark 1 in [8] we have

$$\|e^{itP(D)}\|_{L^1\to L^\infty} \le Ct^{-n/2m}, \qquad t>0.$$

For simplicity we restrict ourselves to the case $2m>n$. Then integration leads to the bound

$$\|(\lambda - iP(D))^{-1}\|_{L^1\to L^\infty} \le C'|\mathrm{Re}\,\lambda|^{n/2m-1}, \qquad \mathrm{Re}\,\lambda \ne 0.$$

This means that, for $V\in L^1(\mathbb{R}^n)$ and $c\ge 0$ sufficiently large,

$$\|V(\lambda - iP(D))^{-1}\|_{L^1\to L^1} \le 1/2, \qquad |\mathrm{Re}\,\lambda| > c.$$

As usual, we thus can write, for $|\mathrm{Re}\,\lambda|>c$,

$$(\lambda - iP(D) - V)^{-1} = (\lambda - iP(D))^{-1}\sum_{k=0}^{\infty}(V(\lambda - iP(D))^{-1})^k,$$

which means that $\{|\mathrm{Re}\,\lambda|>c\}\subset \varrho(iP(D)+V)$ in $L^1(\mathbb{R}^n)$. Moreover, this representation of the resolvent shows that

$$\|(\lambda - iP(D) - V)^{-1}\|_{L^1\to L^1} \le 2\|(\lambda - iP(D))^{-1}\|_{L^1\to L^1} \qquad \text{for } |\mathrm{Re}\,\lambda|>c.$$

By [8, Propositions 1, 2], we now infer that $iP(D)+V$ generates a smooth distribution group in $L^1(\mathbb{R}^n)$ of order $k+2$ (recall $k>n/2$) and some exponential growth $\delta\ge 0$ (where δ depends on c). By Example 2 the operator $iP(D)+V$ generates a DS satisfying (d.5)$^{\text{glob}}$ in $L^\infty(\mathbb{R}^n)$.

Thus we can apply (8) in Proposition 4 to these operators.

4. Distribution semigroups on $\mathcal{F}_\alpha$. In this section we define further test function spaces $\mathcal{F}_\alpha$, $\alpha\in(0,1]$, generalizing the case $\alpha=0$ we studied in the previous section. For $\alpha\in(0,1]$ the space $\mathcal{F}_\alpha$ is the completion of $\mathcal{D}((0,\infty))$ under the sequence of seminorms

$$p_{\alpha,j}(\psi) = \|t^j(\psi(t)/t^\alpha)^{(j)}\|_{L^1((0,\infty))}, \qquad j\in\mathbb{N}_0.$$

An equivalent sequence of seminorms is given by

$$q_{\alpha,j}(\psi) = \|t^{j-\alpha}\psi^{(j)}\|_{L^1((0,\infty))}, \qquad j\in\mathbb{N}_0.$$

If $\psi\in\mathcal{F}_\alpha$ has a bounded support, then $\psi\in\mathcal{F}_\beta$ for all $0\le\beta<\alpha$. The space $\mathcal{D}((0,\infty))$ is dense in each $\mathcal{F}_\alpha$. Topological properties are not important for our analysis and therefore not discussed. Consider the condition

$$\text{(d.1-}\alpha\text{)} \qquad \forall \phi, \psi \in \mathcal{F}_\alpha : \quad \mathcal{G}(\phi * \psi, \cdot) = \mathcal{G}(\phi, \mathcal{G}(\psi, \cdot)).$$

Definition 4. If $\alpha \in (0,1]$ and (d.1-α) and (d.2) hold for $\mathcal{G} \in \mathcal{F}'_\alpha(L(E))$, then we call $\mathcal{G}$ a *distribution semigroup on* $\mathcal{F}_\alpha$.

Remark 8. We remark that, for $\alpha \in (0,1)$, we have $\psi_+ \in \mathcal{F}_\alpha$ for every $\psi \in \mathcal{D}$. As in Remark 6 we thus see that, for $\alpha \in (0,1)$, every distribution semigroup on $\mathcal{F}_\alpha$ can be extended to $\mathcal{D}$ to become a distribution semigroup and that its generator A satisfies $n(A) \le 1$.

It is not clear whether the assertions of this remark hold in case $\alpha = 1$. However, we have the following. Suppose that $\mathcal{G} \in \mathcal{D}'_+(L(E))$ is such that its restriction to $(0,\infty)$ can be extended to a distribution semigroup on $\mathcal{F}_1$ with generator A. Since $\mathcal{F}_1$ includes all functions ψ_+ with $\psi \in \mathcal{D}$ and $\psi(0) = 0$, Lemma 1 shows that $n(A) \le 2$.

We can characterize distribution semigroups on $\mathcal{F}_\alpha$ by modifying the proof of Theorem 6.

Theorem 7. *Let A be a closed linear operator in E and $\alpha \in (0,1)$. Then A generates a distribution semigroup $\mathcal{G}$ on $\mathcal{F}_\alpha$ if and only if A generates a distribution semigroup and there are $n \in \mathbb{N}$ and $C > 0$ such that $\mathcal{G} = S^{(n)}$ for an n-times integrated semigroup $(S(t))_{t\ge 0}$ satisfying*, *for some $C > 0$*,

$$\|S(t)\| \le C t^{n-\alpha}, \quad t \ge 0. \tag{9}$$

Proof. We only indicate the modifications in the proof of Theorem 6. If $\mathcal{G} = S^{(n)}$ is a distribution semigroup and (9) holds then

$$\begin{aligned}\|G(\varphi)x\| &= \Big\|(-1)^n \int_0^\infty \varphi^{(n)}(t) S(t)x\,dt\Big\| \le \int_0^\infty |t^{n-\alpha}\varphi^{(n)}(t)|\,\|t^{\alpha-n}S(t)x\|\,dt \\ &\le C q_{\alpha,n}(\varphi)\|x\|\end{aligned}$$

for all $\varphi \in \mathcal{D}_0$ and $x \in E$. If, on the other hand, $\mathcal{G}$ is a distribution semigroup on $\mathcal{F}_\alpha$ with generator A then

$$\begin{aligned} q_{\alpha,j}(I_t^n) &= \int_0^t s^{j-\alpha}\bigg|\Big(\frac{d}{ds}\Big)^j \frac{(t-s)^{n-1}}{(n-1)!}\bigg|\,ds = \int_0^t s^{j-\alpha}\,\frac{(t-s)^{n-1-j}}{(n-1-j)!}\,ds \\ &= t^{n-\alpha}\int_0^1 \frac{\sigma^{j-\alpha}(1-\sigma)^{n-1-j}}{(n-1-j)!}\,d\sigma = c_{n,\alpha,j}t^{n-\alpha},\end{aligned}$$

which yields (9). ∎

The second part of the proof may be applied to the case $\alpha = 1$ if we assume in addition that $\mathcal{G}$ extends to a distribution semigroup. We can combine Theorem 7 with Theorem 5 and Corollary 1 to obtain

Corollary 3. *Let* $\alpha \in (0,1)$ *and let* $\mathcal{G}$ *be a distribution semigroup on* $\mathcal{F}_\alpha$. *Then* $\mathcal{G}$ *satisfies condition* (d.5)$^{\text{glob}}$. *In particular*, $\mathcal{G}$ *is a strong distribution semigroup.*

For an example of a distribution semigroup on $\mathcal{F}_\alpha$ for $\alpha \in (0,1)$ and with $n = 1$ we refer to Example 3.

Remark 9. It is not clear whether the assertion of Corollary 3 holds for distribution semigroups on $\mathcal{F}_1$ that extend to distribution semigroups, since Theorem 5 would need $\|S(t)x\| = o(t^{n-1})$ for x in a dense subset, and the variant of Theorem 7 for $\alpha = 1$ only yields $\|S(t)\| = O(t^{n-1})$ as $t \to 0+$.

Open Problem. Let $\mathcal{G}$ be a distribution semigroup on $\mathcal{F}_1$ that extends to a distribution semigroup. Is $\mathcal{G}$ a strong distribution semigroup?

We consider this more likely to be false.

Appendix

Proof of Theorem A. We will prove that both conditions of Theorem A are in fact equivalent to:

$$(**)\qquad (\forall x \in E_0)(\exists F_x \in C(V_{x,0}; E),\ \operatorname{supp} F_x \subset [0,\infty))(\exists k_x \in \mathbb{N})$$
$$f(t,x) = F_x^{(k_x)}(t) \text{ for } t \in V_{x,0} \quad\text{and}\quad \|F_x(t)/t^{k_x}\|_E \to 0 \text{ as } t \to 0.$$

It is easy to see that $(**)$ implies $f(\phi_\varepsilon, x) \to 0$ as $\varepsilon \to 0$, for every $x \in E_0$. The proof of the converse (for fixed x) is the same as in the classical theory (see for example [1, Section 3.5]).

We will prove that $(**)\Rightarrow(*)$. We know that in some neighborhood of the origin,

$$(***)\qquad f(t,x) = \Phi^{(k)}(t,x), \qquad t \in V_0,\ x \in E,$$

where $\Phi \in C(V_0 \times E; E)$ with $\operatorname{supp}\Phi(\cdot,x) \subset [0,\infty)$ for $x \in E$.

Now by integration over $[0,t]$ of F_x in $(**)$ or Φ in $(***)$, for every fixed $x \in E_0$, $k - k_x$ times (for $k - k_x > 0$) or $k_x - k$ times (for $k_x - k > 0$), and by using the uniquenesss of the representation of f in the form $(***)$, one finds that $(**)\Rightarrow(*)$. Actually, for $k > k_x$ this is clear while for $k < k_x$ we will show it explicitly.

So assume $k_x = k + 1$. Then

$$\int_0^t \Phi(s,x)\,ds = t^k \widetilde{F}_x(t), \qquad t \in V_{0,x}, \qquad \widetilde{F}_x(0) = 0.$$

Since $t\widetilde{F}_x(t)$ has a derivative in $V_{0,x}$ (equal to 0 at $t = 0$), we see that the equality

$$\Phi(t,x) = kt^{k-1}\widetilde{F}_x(t) + t^{k-1}\frac{d}{dt}(t\widetilde{F}_x(t)), \qquad t \in V_{0,x},$$

shows that we can take $k_x = k$.

If $k_x - k > 1$, then we can use the same argument $k_x - k$ times to obtain the assertion. ■

Acknowledgements. The second and third authors are supported by the Project 144016 of the Serbian Ministry of Science.

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Mathematisches Institut I
Universität Karlsruhe
Englerstr. 2
D-76128 Karlsruhe, Germany
E-mail: peer.kunstmann@math.uni-karlsruhe.de

Department of Mathematics
and Informatics
University of Novi Sad
Trg D. Obradovića 4
21 000 Novi Sad, SCG
E-mail: pilipovic@im.ns.ac.yu

Received May 30, 2005
Revised version April 8, 2008 (5659)