STUDIA MATHEMATICA 187 (1) (2008)

Ascent spectrum and essential ascent spectrum

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Abstract. We study the essential ascent and the related essential ascent spectrum of an operator on a Banach space. We show that a Banach space X has finite dimension if and only if the essential ascent of every operator on X is finite. We also focus on the stability of the essential ascent spectrum under perturbations, and we prove that an operator F on X has some finite rank power if and only if $\sigma_{\mathrm{asc}}^{\mathrm{e}}(T+F)=\sigma_{\mathrm{asc}}^{\mathrm{e}}(T)$ for every operator T commuting with F. The quasi-nilpotent part, the analytic core and the single-valued extension property are also analyzed for operators with finite essential ascent.

1. Introduction. Throughout this paper X will be an infinite-dimensional complex Banach space. We will denote by $\mathscr{L}(X)$ the algebra of all operators on X, and by $\mathscr{F}(X)$ and $\mathscr{K}(X)$ its ideals of finite rank and compact operators on X, respectively. For an operator $T\in\mathscr{L}(X)$, write T^* for its adjoint, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. Also, denote by $\sigma(T)$, $\sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{su}}(T)$ its spectrum, approximate point spectrum and surjective spectrum, respectively. An operator $T\in\mathscr{L}(X)$ is *upper semi-Fredholm* (respectively *lower semi-Fredholm*) if $\mathrm{R}(T)$ is closed and $\dim\mathrm{N}(T)$ (respectively $\operatorname{codim}\mathrm{R}(T)$) is finite. If T is upper or lower semi-Fredholm, then T is called *semi-Fredholm*. The *index* of such an operator is given by $\operatorname{ind}(T)=\dim\mathrm{N}(T)-\operatorname{codim}\mathrm{R}(T)$, and when it is finite we say that T is *Fredholm*. Recall that for $T\in\mathscr{L}(X)$, the *ascent*, $\mathrm{a}(T)$, and the *descent*, $\mathrm{d}(T)$, are defined by

$$\mathrm{a}(T)=\inf\{n\geq 0\colon \mathrm{N}(T^n)=\mathrm{N}(T^{n+1})\},$$
$$\mathrm{d}(T)=\inf\{n\geq 0\colon \mathrm{R}(T^n)=\mathrm{R}(T^{n+1})\},$$

where the infimum over the empty set is taken to be infinite. From [12,

2000 *Mathematics Subject Classification*: 47A53, 47A55.

Key words and phrases: spectrum, ascent, essential ascent, perturbation, semi-Fredholm.

Lemma 1.1], given an operator $T \in \mathscr{L}(X)$ and a positive integer d,

$$\text{(1.1)} \quad \mathrm{d}(T) \le d \;\Leftrightarrow\; \mathrm{R}(T^n) + \mathrm{N}(T^d) = X \text{ for some (equivalently all) } n \in \mathbb{N},$$

$$\text{(1.2)} \quad \mathrm{a}(T) \le d \;\Leftrightarrow\; \mathrm{R}(T^d) \cap \mathrm{N}(T^n) = \{0\} \text{ for some (equivalently all) } n \in \mathbb{N}.$$

Associated to an operator T on X we consider the non-increasing sequence ([11])

$$c_n(T) = \dim \mathrm{N}(T^{n+1})/\mathrm{N}(T^n).$$

Clearly, $\mathrm{N}(T^n)$ has finite codimension in $\mathrm{N}(T^{n+1})$ if and only if it has finite codimension in $\mathrm{N}(T^{n+k})$ for some (equivalently all) $k \in \mathbb{N}$, which is equivalent to $\dim \mathrm{R}(T^n) \cap \mathrm{N}(T^k) < \infty$ for some (equivalently all) $k \in \mathbb{N}$ (see [12]). Consequently, $c_n(T) = \dim \mathrm{N}(T) \cap \mathrm{R}(T^n)$ for all $n \in \mathbb{N}$.

Following [17], the *essential ascent* of T is defined as

$$\mathrm{a_e}(T) = \inf\{n \ge 0\colon c_n(T) \text{ is finite}\}.$$

If $\mathrm{a_e}(T)$ is finite, let $\mathrm{p}(T)$ be the smallest positive integer k with $c_n(T) = c_k(T)$ for all $n \ge k$. Trivially, $\mathrm{a_e}(T) \le \mathrm{p}(T)$, and if $\mathrm{a}(T)$ is finite, then $\mathrm{a}(T) = \mathrm{p}(T)$. Operators with finite essential ascent play a significant role in [11], [12] and [17]. In [12], it was established that if $T \in \mathscr{L}(X)$ has finite essential ascent, then

$$\text{(1.3)} \quad \mathrm{R}(T^n) \text{ is closed for some } n > \mathrm{a_e}(T) \;\Leftrightarrow\; \mathrm{R}(T^n) \text{ is closed for all } n \ge \mathrm{a_e}(T).$$

The *ascent resolvent set* and *essential ascent resolvent set* of an operator $T \in \mathscr{L}(X)$ are respectively defined by

$$\varrho_{\mathrm{asc}}(T) = \{\lambda \in \mathbb{C}\colon \mathrm{a}(T-\lambda) \text{ is finite and } \mathrm{R}(T^{\mathrm{a}(T-\lambda)+1}) \text{ is closed}\},$$
$$\varrho^{\mathrm{e}}_{\mathrm{asc}}(T) = \{\lambda \in \mathbb{C}\colon \mathrm{a_e}(T-\lambda) \text{ is finite and } \mathrm{R}(T^{\mathrm{a_e}(T-\lambda)+1}) \text{ is closed}\}.$$

The complementary sets $\sigma_{\mathrm{asc}}(T) = \mathbb{C} \setminus \varrho_{\mathrm{asc}}(T)$ and $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T) = \mathbb{C} \setminus \varrho^{\mathrm{e}}_{\mathrm{asc}}(T)$ are the *ascent spectrum* and *essential ascent spectrum* of T, respectively. It is clear that

$$\sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \subseteq \sigma_{\mathrm{asc}}(T) \subseteq \sigma(T).$$

This paper is organized as follows. The second section is devoted to the study of the ascent spectrum and essential ascent spectrum. We show that they are compact subsets of the spectrum, and that for $T \in \mathscr{L}(X)$, $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ is empty precisely when T is algebraic. Furthermore, we establish that the ascent (respectively essential ascent) of every operator acting on X is finite if and only if X has finite dimension. In Section 3, we are concerned with the stability of the essential ascent spectrum under finite rank perturbations. We prove that $F^n \in \mathscr{F}(X)$ for some $n \in \mathbb{N}$ if and only if $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T+F) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ (equivalently, $\sigma_{\mathrm{asc}}(T+F) = \sigma_{\mathrm{asc}}(T)$) for every operator T in the commutant of F. The results appearing in these sections cover the essential ascent counterpart of the descent case studied in [8], which partially motivated this study. Finally, in terms of the quasi-nilpotent part and the analytic core, we derive in Section 4 several necessary and sufficient conditions for an operator $T \in \mathscr{L}(X)$, with $0 \notin \sigma_{\mathrm{asc}}^{\mathrm{e}}(T)$, and its adjoint to satisfy a localized version of the single-valued extension property.

2. Ascent and essential ascent spectra. Given an operator $T \in \mathscr{L}(X)$, the *generalized kernel* and *generalized range* of T are the subspaces of X defined by $\mathscr{N}^{\infty}(T) = \bigcup_n \mathrm{N}(T^n)$ and $\mathscr{R}^{\infty}(T) = \bigcap_n \mathrm{R}(T^n)$, respectively. Recall that T is *semiregular* if $\mathrm{R}(T)$ is closed and $\mathrm{N}(T) \subseteq \mathscr{R}^{\infty}(T)$. The *semiregular spectrum* of T, $\sigma_{\mathrm{s}}(T)$, is defined as those complex numbers λ for which $T-\lambda$ is not semiregular. It is well known that $\sigma_{\mathrm{s}}(T)$ is a closed subset of $\sigma(T)$ and that $\partial\sigma(T) \subset \sigma_{\mathrm{s}}(T)$ (see [18]). In the next theorem, we show that if $0 \notin \sigma_{\mathrm{asc}}^{\mathrm{e}}(T)$ then either T is semiregular or 0 is an isolated point of its semiregular spectrum. The proof requires the following technical lemmas.

LEMMA 2.1. *If $T \in \mathscr{L}(X)$ is such that $\mathrm{a_e}(T)$ is finite and $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed, then the operator induced by T on $X/\mathrm{N}(T^{\mathrm{p}(T)})$ is both semiregular and upper semi-Fredholm.*

Proof. Let $p = \mathrm{p}(T)$ and let $\widetilde{T}$ be the operator induced by T on $X/\mathrm{N}(T^p)$. Then $\mathrm{N}(\widetilde{T})$ has finite dimension, and $\mathrm{R}(\widetilde{T}) = (\mathrm{R}(T) + \mathrm{N}(T^p))/\mathrm{N}(T^p) = (T^{-p}(\mathrm{R}(T^{p+1}))/\mathrm{N}(T^p)$ is closed. Thus $\widetilde{T}$ is upper semi-Fredholm. Now, as $\mathrm{N}(T) \cap \mathrm{R}(T^p) = \mathrm{N}(T) \cap \mathrm{R}(T^{p+n})$ for all $n \in \mathbb{N}$, it is straightforward to show that $\mathrm{N}(T^{p+1}) \subseteq \mathrm{R}(T^n) + \mathrm{N}(T^p)$ for all $n \in \mathbb{N}$. Hence, $\mathrm{N}(\widetilde{T}) \subseteq \mathscr{R}^{\infty}(T)$, which proves that $\widetilde{T}$ is semiregular. ∎

LEMMA 2.2. *Let $T \in \mathscr{L}(X)$ be semiregular with finite-dimensional kernel. Then* $\dim \mathrm{N}(T^n) = n \dim \mathrm{N}(T)$ *for all* $n \in \mathbb{N}$.

Proof. Let n be a positive integer. Since $\mathrm{N}(T^{n-1}) \subseteq \mathrm{R}(T)$, T is a surjection from $\mathrm{N}(T^n)$ to $\mathrm{N}(T^{n-1})$, and consequently $\dim \mathrm{N}(T^n) = \dim \mathrm{N}(T) + \dim \mathrm{N}(T^{n-1})$. By induction, $\dim \mathrm{N}(T^n) = n \dim \mathrm{N}(T)$ for all $n \in \mathbb{N}$. ∎

THEOREM 2.3. *Let $T \in \mathscr{L}(X)$ be such that $\mathrm{a_e}(T)$ is finite and $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed. There exists $\delta > 0$ such that for every λ with $0 < |\lambda| < \delta$ the following assertions hold*:

(i) $T - \lambda$ *is semiregular*,
(ii) $\dim \mathrm{N}(T-\lambda)^n = n \dim \mathrm{N}(T^{\mathrm{p}(T)+1})/\mathrm{N}(T^{\mathrm{p}(T)})$ *for all* $n \in \mathbb{N}$,
(iii) $\operatorname{codim} \mathrm{R}(T-\lambda)^n = n \dim \mathrm{R}(T^{\mathrm{p}(T)})/\mathrm{R}(T^{\mathrm{p}(T)+1})$ *for all* $n \in \mathbb{N}$.

Proof. Let $p = \mathrm{p}(T)$ and let $\widetilde{T}$ be the operator induced by T on $X/\mathrm{N}(T^p)$. By Lemma 2.1 and [16, Proposition 2.1] there exists $\delta > 0$ such that $\widetilde{T} - \lambda$ is semiregular and semi-Fredholm with $\dim \mathrm{N}(\widetilde{T} - \lambda) = \dim \mathrm{N}(\widetilde{T})$ for $|\lambda| < \delta$.

Fix $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \delta$. We have

$$\mathrm{N}(\widetilde{T}-\lambda)^n = \mathrm{N}((T-\lambda)^n T^p)/\mathrm{N}(T^p) = (\mathrm{N}(T-\lambda)^n \oplus \mathrm{N}(T^p))/\mathrm{N}(T^p), \tag{2.1}$$

$$\mathrm{R}(\widetilde{T}-\lambda) = (\mathrm{R}(T-\lambda) + \mathrm{N}(T^p))/\mathrm{N}(T^p) = \mathrm{R}(T-\lambda)/\mathrm{N}(T^p). \tag{2.2}$$

Consequently, $\mathrm{R}(T-\lambda)$ is closed and contains the finite-dimensional subspace $\mathrm{N}(T-\lambda)^n$ for all $n \in \mathbb{N}$. This implies that $T-\lambda$ is semiregular and upper semi-Fredholm. Moreover, by (2.1) and the previous lemma,

$$\begin{aligned}\dim \mathrm{N}(T-\lambda)^n &= \dim \mathrm{N}(\widetilde{T}-\lambda)^n = n \dim \mathrm{N}(\widetilde{T}-\lambda) = n \dim \mathrm{N}(\widetilde{T}) \\ &= n \dim \mathrm{N}(T^{p+1})/\mathrm{N}(T^p).\end{aligned}$$

Now by the continuity of the index we get

$$\begin{aligned}\operatorname{codim} \mathrm{R}(T-\lambda)^n &= \operatorname{codim} \mathrm{R}(T-\lambda)^n/\mathrm{N}(T^p) = \operatorname{codim} \mathrm{R}(\widetilde{T}-\lambda)^n \\ &= \dim \mathrm{N}(\widetilde{T}-\lambda)^n - \operatorname{ind}(\widetilde{T}-\lambda)^n = n \dim \mathrm{N}(\widetilde{T}) - n \operatorname{ind}(\widetilde{T}-\lambda) \\ &= n \dim \mathrm{N}(\widetilde{T}) - n \operatorname{ind}(\widetilde{T}) = n \operatorname{codim} \mathrm{R}(\widetilde{T}) \\ &= n \dim X/(\mathrm{R}(T) + \mathrm{N}(T^p)) = n \dim \mathrm{R}(T^p)/\mathrm{R}(T^{p+1}). \ \blacksquare\end{aligned}$$

As a direct consequence of Theorem 2.3 we obtain the following result for operators with finite ascent.

Corollary 2.4. *Let $T \in \mathscr{L}(X)$ be such that* $\mathrm{a}(T)$ *is finite and* $\mathrm{R}(T^{\mathrm{a}(T)+1})$ *is closed. There exists $\delta > 0$ such that for every λ with $0 < |\lambda| < \delta$ the following assertions hold*:

(i) $T-\lambda$ *is bounded below*,
(ii) $\operatorname{codim} \mathrm{R}(T-\lambda)^n = n \dim \mathrm{R}(T^{\mathrm{a}(T)})/\mathrm{R}(T^{\mathrm{a}(T)+1})$.

Remark 2.5. As a consequence of Theorem 2.3, Corollary 2.4, and the continuity of the index, a semi-Fredholm operator $T \in \mathscr{L}(X)$ has finite essential ascent (respectively finite ascent) if and only if T is upper semi-Fredholm (respectively 0 is an isolated point of $\sigma_{\mathrm{ap}}(T)$).

Corollary 2.6. *Let $T \in \mathscr{L}(X)$. Then $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ and $\sigma_{\mathrm{asc}}(T)$ are compact subsets of $\sigma(T)$. Moreover, $\sigma_{\mathrm{asc}}(T) \setminus \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ is an open set.*

Proof. The first assertion follows directly from Theorem 2.3 and Corollary 2.4. Let $\lambda \in \sigma_{\mathrm{asc}}(T) \setminus \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ and set $p = \mathrm{p}(T-\lambda)$. By Theorem 2.3, there exists a punctured open neighbourhood V of λ such that $V \subseteq \varrho^{\mathrm{e}}_{\mathrm{asc}}(T)$ and

$$\dim \mathrm{N}(T-\mu)^n = n \dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p) \quad \text{for all } n \in \mathbb{N} \text{ and } \mu \in V.$$

Since $T-\lambda$ has infinite ascent, $\dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p$ is non-zero, and consequently $\{\dim \mathrm{N}(T-\mu)^n\}_n$ is a strictly increasing sequence for each $\mu \in V$. Thus $V \subseteq \sigma_{\mathrm{asc}}(T)$, as desired. $\blacksquare$

It is clear that the ascent spectrum, and therefore the essential ascent spectrum, of an operator can be empty. In the next theorem we show that this occurs precisely for algebraic operators. For $T \in \mathscr{L}(X)$, denote by $\mathrm{E}(T)$ the set of all poles of the resolvent of T.

THEOREM 2.7. *Let $T \in \mathscr{L}(X)$. Then*

$$\rho_{\mathrm{asc}}^{\mathrm{e}}(T) \cap \partial\sigma(T) = \rho_{\mathrm{asc}}(T) \cap \partial\sigma(T) = \mathrm{E}(T).$$

Moreover, the following assertions are equivalent:

(1) $\sigma_{\mathrm{asc}}(T) = \emptyset$,
(2) $\sigma_{\mathrm{asc}}^{\mathrm{e}}(T) = \emptyset$,
(3) $\partial\sigma(T) \subseteq \rho_{\mathrm{asc}}(T)$,
(4) $\partial\sigma(T) \subseteq \rho_{\mathrm{asc}}^{\mathrm{e}}(T)$,
(5) *T is algebraic.*

Proof. Since $\mathrm{E}(T) \subseteq \rho_{\mathrm{asc}}(T) \cap \partial\sigma(T) \subseteq \rho_{\mathrm{asc}}^{\mathrm{e}}(T) \cap \partial\sigma(T)$, it suffices to show that $\rho_{\mathrm{asc}}^{\mathrm{e}}(T) \cap \partial\sigma(T) \subseteq \mathrm{E}(T)$. Let $\lambda \in \rho_{\mathrm{asc}}^{\mathrm{e}}(T) \cap \partial\sigma(T)$ and $p = \mathrm{p}(T-\lambda)$. By Theorem 2.3, there exists a punctured neighbourhood U of λ such that $\dim \mathrm{N}(T-\mu) = \dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p$ and $\operatorname{codim} \mathrm{R}(T-\mu) = \dim \mathrm{R}(T-\lambda)^p/\mathrm{R}(T-\lambda)^{p+1}$ for all $\mu \in U$. As $U \setminus \sigma(T)$ is non-empty,

$$\dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p = \dim \mathrm{R}(T-\lambda)^p/\mathrm{R}(T-\lambda)^{p+1} = 0.$$

Hence $T-\lambda$ has finite ascent and descent, that is, λ is a pole of the resolvent of T.

All the desired implications are clear from the above paragraph just by pointing out that $\partial\sigma(T) \subseteq \rho_{\mathrm{asc}}^{\mathrm{e}}(T)$ if and only if $\sigma(T) = \partial\sigma(T)$ consists of the poles of the resolvent of T, which is equivalent to T being algebraic (cf. [8, Theorem 1.5]). ∎

COROLLARY 2.8. *Let X be a Banach space. The following assertions are equivalent*:

(1) *X has finite dimension*,
(2) *every $T \in \mathscr{L}(X)$ has finite ascent*,
(3) *every $T \in \mathscr{L}(X)$ has finite essential ascent*,

Proof. The implications (1)⇒(2)⇒(3) are obvious. Suppose that X has infinite dimension. Let $\{e_n\}$ be an infinite sequence of linearly independent vectors, and $\{f_n\}$ be continuous linear forms such that $f_i(e_j) = \delta_{ij}$ for all positive integers i, j. Consider the bounded operator $T = \sum_p \lambda_p f_{2p} \otimes e_p$ where $\{\lambda_p\}$ is a sequence of non-zero elements such that $\sum |\lambda_p| \, \|f_{2p}\| \, \|e_p\|$ is finite. It is easy to verify that the sequence $\{e_{2^{k+1}p+2^k}\}$ consists of linearly independent vectors of $\mathrm{N}(T^{k+1}) \setminus \mathrm{N}(T^k)$. Hence T has infinite essential ascent. ∎

THEOREM 2.9. *Let $T \in \mathscr{L}(X)$ and Ω be a connected component of $\varrho_{\mathrm{asc}}^{\mathrm{e}}(T)$. Then*

$$\Omega \subset \sigma(T) \quad \text{or} \quad \Omega \setminus E_\Omega \subseteq \rho(T),$$

where $E_\Omega = \Omega \cap \mathrm{E}(T)$.

Proof. Let $\Omega_{\mathrm{r}} = \{\lambda \in \Omega\colon T-\lambda$ is both semiregular and semi-Fredholm$\}$. By Theorem 2.3, $\Omega_{\mathrm{o}} = \Omega \setminus \Omega_{\mathrm{r}}$ is at most countable, and hence Ω_{r} is connected. Suppose that $\Omega \cap \rho(T)$ is non-empty. Then so is $\Omega_{\mathrm{r}} \cap \rho(T)$, and since $\dim \mathrm{N}(T-\lambda)$ and $\operatorname{codim} \mathrm{R}(T-\lambda)$ are constant functions on Ω_{r} (see [16, Proposition 2.1]), we obtain $\Omega_{\mathrm{r}} \subseteq \rho(T)$. Consequently, Ω_{o} consists of isolated points of the spectrum with finite essential ascent. That is,

$$\Omega_{\mathrm{o}} \subseteq \partial\sigma(T) \cap \varrho^{\mathrm{e}}_{\mathrm{asc}}(T) = \mathrm{E}(T).$$

Finally, $\Omega \setminus E_\Omega \subseteq \Omega_{\mathrm{r}} \subseteq \rho(T)$, as desired. ∎

Directly from the preceding theorem and Corollary 2.6 we obtain the next result.

Corollary 2.10. *Let $T \in \mathscr{L}(X)$. The following conditions are equivalent*:

(1) $\sigma(T)$ *is at most countable*,
(2) $\sigma_{\mathrm{asc}}(T)$ *is at most countable*,
(3) $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ *is at most countable.*

In this case, $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T) = \sigma_{\mathrm{asc}}(T)$ *and* $\sigma(T) = \sigma_{\mathrm{asc}}(T) \cup \mathrm{E}(T)$.

From this corollary it follows in particular that $T \in \mathscr{L}(X)$ is meromorphic (i.e. $\sigma(T) \setminus \{0\} \subseteq \mathrm{E}(T)$) if and only if $\sigma_{\mathrm{asc}}(T) \subseteq \{0\}$, if and only if $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \subseteq \{0\}$.

It is an important well known fact that

$$\sigma(TS) \setminus \{0\} = \sigma(ST) \setminus \{0\} \tag{2.3}$$

for all $T, S \in \mathscr{L}(X)$. In [4] Barnes studies the common operator properties of the operators TS and ST and shows that the equality (2.3) holds for some parts of the spectrum. In particular, he proves that for all non-zero $\lambda \in \mathbb{C}$, $\lambda - TS$ has closed range if and only if $\lambda - ST$ has closed range, and that $\mathrm{a}(\lambda - TS)$ is finite if and only if so is $\mathrm{a}(\lambda - ST)$.

Theorem 2.11. *For all $T, S \in \mathscr{L}(X)$ the following equalities hold*:

$$\sigma_{\mathrm{asc}}(TS) \setminus \{0\} = \sigma_{\mathrm{asc}}(ST) \setminus \{0\}, \quad \sigma^{\mathrm{e}}_{\mathrm{asc}}(TS) \setminus \{0\} = \sigma^{\mathrm{e}}_{\mathrm{asc}}(ST) \setminus \{0\}.$$

Proof. Arguing as in the proof of [4, Proposition 10] we get, for $n \in \mathbb{N}$,

$$(I-TS)^{n+1} = I - TU_n, \quad (I-ST)^{n+1} = I - U_nT,$$

where $U_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} S(TS)^{k-1}$. Therefore, $\mathrm{R}(I-TS)^{n+1}$ is closed if and only if $\mathrm{R}(I-ST)^{n+1}$ is closed. To complete the proof, it suffices to show that $\mathrm{a_e}(I-TS) = \mathrm{a_e}(I-ST)$. Since $S\mathrm{N}(I-U_nS) = \mathrm{N}(I-SU_n)$ for all $n \in \mathbb{N}$, the operator $\widetilde{S}$ induced by S from $\mathrm{N}(I-U_{n+1}S)/\mathrm{N}(I-U_nS)$ to $\mathrm{N}(I-SU_{n+1})/\mathrm{N}(I-SU_n)$ is surjective. Moreover, because $\mathrm{N}(S) \cap \mathrm{N}(I-U_{n+1}) = \{0\}$, we easily see that $\widetilde{S}$ is an isomorphism, as desired. ∎

For $T \in \mathscr{L}(X)$, $L_T : \mathscr{L}(X) \to \mathscr{L}(X)$ denotes the left multiplication operator by T, given by $L_T(S) = TS$ for all $S \in \mathscr{L}(X)$.

PROPOSITION 2.12. *Let $T \in \mathscr{L}(X)$. The following are equivalent*:

(1) $\mathrm{a}(T)$ *is finite*,
(2) $\mathrm{a}(L_T)$ *is finite*,
(3) $\mathrm{a_e}(L_T)$ *is finite.*

Moreover, *if X is a Hilbert space*, *then $\sigma_{\mathrm{asc}}(L_T) = \sigma_{\mathrm{asc}}(T)$.*

Proof. The implications (1)⇒(2)⇒(3) are clear. Assume that T has infinite ascent. Let n be a positive integer and choose $x \in \mathrm{N}(T^{n+1}) \setminus \mathrm{N}(T^n)$. Consider an infinite sequence $\{f_k\}$ of linearly independent continuous linear forms. It follows that $\{f_k \otimes x\} \subseteq \mathrm{N}(L_{T^{n+1}})$ is a sequence of linearly independent vectors of $\mathrm{N}(L_{T^{n+1}}) \setminus \mathrm{N}(L_{T^n})$. Thus L_T has infinite essential ascent. This shows (3)⇒(1).

It is well known that if $\mathrm{R}(L_T)$ is closed, then so is $\mathrm{R}(T)$. If in addition X is a Hilbert space, then the converse is also true (see [9]). From this, the last assertion follows easily. ∎

COROLLARY 2.13. *Let $T \in \mathscr{L}(X)$. The following are equivalent*:

(i) $\mathrm{a}(T)$ *and* $\mathrm{d}(T)$ *are finite*,
(ii) $\mathrm{a}(L_T)$ *and* $\mathrm{d}(L_T)$ *are finite.*

Proof. Taking into account Proposition 2.12 and [8], it suffices to show that, if $\mathrm{a}(T) = \mathrm{d}(T) = d$ is finite, then so is $\mathrm{d}(L_T)$. This follows from [5, Theorem 17], since $X = \mathrm{R}(T^{d+1}) \oplus \mathrm{N}(T^{d+1})$. ∎

We conclude this section by considering the ascent and essential ascent of an element in the more general setting of unital complex Banach algebras. Let $\mathscr{A}$ be a unital complex (infinite-dimensional) Banach algebra. Denote by $\mathrm{Rad}(\mathscr{A})$ its (Jacobson) radical and by $\mathscr{N}(\mathscr{A})$ its set of nilpotent elements. The *ascent* and *essential ascent* of an element $x \in \mathscr{A}$ are respectively defined by $\mathrm{a}(x) = \mathrm{a}(L_x)$ and $\mathrm{a_e}(x) = \mathrm{a_e}(L_x)$. The associated spectra, $\sigma_{\mathrm{asc}}(x) = \sigma_{\mathrm{asc}}(L_x)$ and $\sigma^{\mathrm{e}}_{\mathrm{asc}}(x) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(L_x)$, are called the *ascent spectrum* and *essential ascent spectrum* of x, respectively. For the Banach algebra $\mathscr{L}(X)$, Proposition 2.12 ensures that the ascent of T as an operator coincides with the ascent (even with the essential ascent) of T as an element of the algebra $\mathscr{L}(X)$. Nevertheless, the essential ascent of T as an operator is not equal to the essential ascent of T as an element of the algebra $\mathscr{L}(X)$. Indeed, by Proposition 2.12, it suffices to consider an operator T with finite essential ascent and infinite ascent.

Having in mind Theorem 2.7 and [3, Theorem 5.4.2], the equivalence of the following conditions follows easily (cf. [8, Theorem 2.2]):

(i) $\dim(\mathscr{A}/\mathrm{Rad}(\mathscr{A}))$ is finite and $\mathrm{Rad}(\mathscr{A}) \subseteq \mathscr{N}(\mathscr{A})$,
(ii) $\sigma_{\mathrm{asc}}(x) = \emptyset$ for all $x \in \mathscr{A}$,
(iii) $\sigma^{\mathrm{e}}_{\mathrm{asc}}(x) = \emptyset$ for all $x \in \mathscr{A}$,
(iv) $\sigma_{\mathrm{asc}}(x) = \emptyset$ for all x in a non-empty open subset of $\mathscr{A}$,
(v) $\sigma^{\mathrm{e}}_{\mathrm{asc}}(x) = \emptyset$ for all x in a non-empty open subset of $\mathscr{A}$,
(vi) $\mathscr{A}$ is algebraic.

3. Ascent, essential ascent and perturbation. In [13, Theorem 2.2], Kaashoek and Lay established that if F is a bounded operator for which there exists some positive integer n such that $F^n \in \mathscr{F}(X)$, then for every $T \in \mathscr{L}(X)$ commuting with F, T has finite ascent if and only if $T+F$ does. We generalize this result to the essential ascent, and characterize those operators having some finite rank power as the class of operators that leave invariant the ascent spectrum and essential ascent spectrum of the operators in their commutants.

Proposition 3.1. *Let $F \in \mathscr{L}(X)$ with $F^n \in \mathscr{F}(X)$ for some $n \in \mathbb{N}$. If $T \in \mathscr{L}(X)$ commutes with F, then $\mathrm{a_e}(T)$ is finite if and only if $\mathrm{a_e}(T+F)$ is finite. If this is the case, then $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed if and only if $\mathrm{R}((T+F)^{\mathrm{a_e}(T)+1})$ is closed.*

Proof. Clearly it suffices to prove only one direction. Assume that $\mathrm{a_e}(T)$ is finite and set $p = \mathrm{p}(T)$. Given $k \geq n+p$, as T^{n+k} maps $\mathrm{N}(T+F)^k$ into $\mathrm{R}(F^n)$, it is clear that

$$\dim \mathrm{N}(T+F)^k/(\mathrm{N}(T+F)^k \cap \mathrm{N}(T^{n+k})) < \infty.$$

Moreover, since $\dim \mathrm{N}(T^{n+k})/\mathrm{N}(T^p) < \infty$, it follows that $\dim \mathrm{N}(T+F)^k/(\mathrm{N}(T+F)^k \cap \mathrm{N}(T^p)) < \infty$. Furthermore, $\mathrm{N}(F^n) \cap \mathrm{N}(T^p) \subseteq \mathrm{N}(T+F)^k \cap \mathrm{N}(T^p) \subseteq \mathrm{N}(T^p)$ and as $F^n \in \mathscr{F}(X)$, $\dim \mathrm{N}(T^p)/(\mathrm{N}(F^n) \cap \mathrm{N}(T^p)) < \infty$. Therefore, we obtain

$$\dim \mathrm{N}(T+F)^k/(\mathrm{N}(F^n) \cap \mathrm{N}(T^p)) < \infty,$$

which implies that $\mathrm{a_e}(T+F) \leq n+p$.

Now, suppose that $\mathrm{a_e}(T)$ is finite and that $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed. By (2.2) we only have to see that $\mathrm{R}(T+F)^k)$ is closed for some $k > n+p$. Denote by $\widetilde{T}$ and $\widetilde{F}$ the operators induced by T and F on $X/\mathrm{N}(T^p)$, respectively. By Lemma 2.1, $\widetilde{T}$ is upper semi-Fredholm. As $\widetilde{F}$ is Riesz and commutes with $\widetilde{T}$, $\widetilde{T}+\widetilde{F}$ is upper semi-Fredholm [22]. This shows that $\mathrm{R}(T+F)^k + \mathrm{N}(T^p)$ is closed. By [20, Lemma 20.3], in order to conclude the proof, it suffices to prove that $\mathrm{R}(T+F)^k \cap \mathrm{N}(T^p)$ is closed. To this end, note that as $(T+F)^k$ maps $\mathrm{N}(T^p)$ into $\mathrm{R}(F^n)$, $\dim \mathrm{N}(T^p)/(\mathrm{N}(T+F)^k \cap \mathrm{N}(T^p))$ is finite. Hence,

$$\dim(\mathrm{R}(T+F)^k \cap \mathrm{N}(T^p))/(\mathrm{R}(T+F)^k \cap \mathrm{N}(T+F)^k \cap \mathrm{N}(T^p)) < \infty.$$

Moreover, since $\mathrm{a_e}(T+F)$ is finite, we know that $\dim \mathrm{R}(T+F)^k \cap \mathrm{N}(T+F)^k$

is finite. We conclude that $\mathrm{R}(T+F)^k \cap \mathrm{N}(T^p)$ has finite dimension, and in particular, it is closed. ∎

We mention that the conclusion of the previous proposition remains true for all finite rank operators F not necessarily commuting with T (see [17]).

THEOREM 3.2. *Let $F \in \mathscr{L}(X)$. The following conditions are equivalent*:

(i) *there exists a positive integer n such that F^n has finite rank*,
(ii) $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T+F) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ *for all $T \in \mathscr{L}(X)$ commuting with F*,
(iii) $\sigma_{\mathrm{asc}}(T+F) = \sigma_{\mathrm{asc}}(T)$ *for all $T \in \mathscr{L}(X)$ commuting with F*.

Proof. The implication (i)⇒(ii) follows from the preceding proposition, and (i)⇒(iii) is a consequence of the preceding proposition and [13, Theorem 2.2]. For the reverse implications let Γ denote the ascent spectrum or essential ascent spectrum. By considering $T = 0$ and Proposition 2.7, it is clear that F is an algebraic operator. The same argument used in [8, Theorem 3.1(i)⇒(ii)] allows us to conclude that F has some power of finite rank. ∎

4. Quasi-nilpotent part, analytic core and the SVEP. Recall that the *algebraic core*, $\mathrm{Co}(T)$, of $T \in \mathscr{L}(X)$ is the largest subspace M of X such that $T(M) = M$. It is clear that $\mathrm{Co}(T) \subseteq \mathscr{R}^\infty(T)$. This inclusion becomes an equality when the decreasing sequence $\{\mathrm{N}(T) \cap \mathrm{R}(T^n))\}_n$ is stationary (see [14]). So in particular, if T has finite essential ascent, $\mathrm{Co}(T) = \mathscr{R}^\infty(T)$. Let us also recall the definition and the main properties of the analytic core and quasi-nilpotent part of $T \in \mathscr{L}(X)$. These subspaces have been introduced in [14] and deeply studied by Mbekhta in [14], [18] and [19].

The *analytic core* of T is the set $\mathrm{K}(T)$ given by

$$\begin{aligned}\mathrm{K}(T) = \{x \in X : \exists\{x_n\}_{n\geq 0} \subseteq X \text{ and } \exists c > 0 \text{ such that } x = x_0,\\ Tx_{n+1} = x_n \text{ and } \|x_n\| \leq c^n\|x\| \text{ for every } n \in \mathbb{N}\}.\end{aligned}$$

It is well known that $T(\mathrm{K}(T)) = \mathrm{K}(T) \subseteq \mathrm{Co}(T)$ and that neither $\mathrm{K}(T)$ nor $\mathrm{Co}(T)$ has to be closed. If $\mathrm{Co}(T)$ is closed, then $\mathrm{K}(T) = \mathrm{Co}(T)$.

The *quasi-nilpotent part* of T is defined as

$$\mathrm{H_o}(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Notice that $\mathscr{N}^\infty(T) \subseteq \mathrm{H_o}(T)$ and that $x \in \mathrm{H_o}(T)$ if and only if $Tx \in \mathrm{H_o}(T)$. In general, $\mathrm{H_o}(T)$ is not closed. In fact, if $\mathrm{H_o}(T)$ is closed then $T|_{\mathrm{H_o}(T)}$ is quasi-nilpotent (see [19]). Moreover, it is easy to verify that $\mathscr{N}^\infty(T) \subseteq \mathrm{K}(T-\lambda)$ and $\mathrm{N}(T-\lambda) \cap \mathrm{H_o}(T) = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

In [19], it was shown that for a semiregular operator $T \in \mathscr{L}(X)$, $\mathrm{K}(T) = \mathrm{Co}(T) = \mathscr{R}^\infty(T)$ is closed, $T(\mathrm{H_o}(T)) = \mathrm{H_o}(T)$ and $\overline{\mathrm{H_o}(T)} = \overline{\mathscr{N}^\infty(T)} \subseteq \mathscr{R}^\infty(T)$. Moreover, by [15], if T is semiregular, then $\mathrm{H_o}(T)$ is closed if and only if $\mathrm{H_o}(T) = \{0\}$. In the next proposition we focus our attention on the analytic core and the quasi-nilpotent part of those $T \in \mathscr{L}(X)$ such that $0 \notin \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$.

PROPOSITION 4.1. *Let $T \in \mathscr{L}(X)$ be such that $\mathrm{a_e}(T)$ is finite and $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed. The following conditions hold*:

(i) $\overline{\mathrm{H_o}(T)} = \overline{\mathscr{N}^\infty(T)}$,
(ii) $\mathrm{K}(T) = \mathscr{R}^\infty(T)$ *is closed*,
(iii) $\mathscr{N}^\infty(T) \subseteq \mathscr{R}^\infty(T) + \mathrm{N}(T^{\mathrm{p}(T)})$,
(iv) $\mathrm{H_o}(T)$ *is closed if and only if* $\mathrm{H_o}(T) = \mathrm{N}(T^{\mathrm{p}(T)})$
(v) $\overline{\mathrm{H_o}(T)} = T(\overline{\mathrm{H_o}(T)}) + \mathrm{N}(T^{\mathrm{p}(T)})$.

Proof. Let $p = \mathrm{p}(T)$ and $\widetilde{T}$ be the semiregular operator induced by T on $X/\mathrm{N}(T^p)$. All the assertions can be easily deduced from the preceding comments just by proving that

$$\mathrm{H_o}(\widetilde{T}) = \mathrm{H_o}(T)/\mathrm{N}(T^p), \tag{4.1}$$

$$\mathrm{K}(\widetilde{T}) = (\mathrm{K}(T) + \mathrm{N}(T^p))/\mathrm{N}(T^p). \tag{4.2}$$

Indeed, let $\pi : X \to X/\mathrm{N}(T^p)$ the canonical surjection. It is clear that $\pi(\mathrm{H_o}(T)) \subseteq \mathrm{H_o}(\widetilde{T})$. For the other inclusion, let $\pi(x)$ be such that $\lim \|\widetilde{T}^n(\pi(x))\|^{1/n} = 0$. There exists a sequence $\{u_n\} \subseteq \mathrm{N}(T^p)$ with $\lim \|T^n(x) + u_n\|^{1/n} = 0$. Thus

$$\|T^{p+n}(x)\|^{1/n} \leq \|T^p\|^{1/n} \|T^n(x) + u_n\|^{1/n},$$

and so $T^p(x) \in \mathrm{H_o}(T)$. Hence $x \in \mathrm{H_o}(T)$. This shows (4.1).

Now, take $\pi(x) \in \mathrm{K}(\widetilde{T})$. There exists a sequence $\{y_n\} \subseteq X$ such that $\pi(x) = \widetilde{T}^n(\pi(y_n))$ for every $n \in \mathbb{N}$. It follows that $T^p(x) = T^{n+p}(y_n)$ and $T^p(x) \in \mathrm{R}(T^n)$ for all $n \in \mathbb{N}$. Consequently, $T^p(x) \in \mathrm{K}(T)$, which implies that $x \in \mathrm{K}(T) + \mathrm{N}(T^p)$. We have proved that $\mathrm{K}(\widetilde{T}) \subseteq \pi(\mathrm{K}(T))$. For the other inclusion, note that

$$\begin{aligned}\pi(\mathrm{K}(T)) &= [\mathrm{K}(T) + \mathrm{N}(T^p)]/\mathrm{N}(T^p) \subseteq [\mathscr{R}^\infty(T) + \mathrm{N}(T^p)]/\mathrm{N}(T^p) \\ &\subseteq \bigcap_n [\mathrm{R}(T^n) + \mathrm{N}(T^p)]/\mathrm{N}(T^p) = \mathrm{K}(\widetilde{T}),\end{aligned}$$

proving (4.2). ∎

The following result illustrates the continuity of $\mathrm{H_o}(T)$ and $\mathrm{K}(T)$ for operators T such that $0 \notin \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$.

PROPOSITION 4.2. *Let $T \in \mathscr{L}(X)$ be such that $\mathrm{a_e}(T)$ is finite and $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed. There exists $\delta > 0$ such that for all λ with $0 < |\lambda| < \delta$, we have*:

(i) $\overline{\mathrm{H_o}(T-\lambda) + \mathrm{N}(T^{\mathrm{p}(T)})} = \overline{\mathrm{H_o}(T)}$,
(ii) $\mathrm{K}(T-\lambda) = \mathrm{K}(T) + \mathrm{N}(T^{\mathrm{p}(T)})$.

Proof. Let $\widetilde{T}$, p and π be as in the proof of the previous proposition. By [19, Lemme 1.3] there is $\delta > 0$ such that $\widetilde{T} - \lambda$ is semiregular, $\overline{\mathrm{H_o}(\widetilde{T}-\lambda)} = \overline{\mathrm{H_o}(\widetilde{T})}$ and $\mathrm{K}(\widetilde{T}-\lambda) = \mathrm{K}(\widetilde{T})$ for any λ with $|\lambda| < \delta$. Therefore we need only show that for $0 < |\lambda| < \delta$,

$$\overline{\mathrm{H_o}(\widetilde{T}-\lambda)} = \overline{(\mathrm{H_o}(T-\lambda) + \mathrm{N}(T^p))}/\mathrm{N}(T^p), \tag{4.3}$$

$$\mathrm{K}(\widetilde{T}-\lambda) = \mathrm{K}(T-\lambda)/\mathrm{N}(T^p). \tag{4.4}$$

Obviously, $\overline{\pi(\mathrm{H_o}(T-\lambda))} \subseteq \overline{\mathrm{H_o}(\widetilde{T}-\lambda)}$. For the reverse inclusion, we have

$$\begin{aligned}\overline{\mathrm{H_o}(\widetilde{T}-\lambda)} = \overline{\mathscr{N}^\infty(\widetilde{T}-\lambda)} &= \overline{\Big(\bigcup_n \mathrm{N}((T-\lambda)^n T^p)\Big)}/\mathrm{N}(T^p) \\ &= \overline{\bigcup_n (\mathrm{N}(T-\lambda)^n \oplus \mathrm{N}(T^p))}/\mathrm{N}(T^p) = \overline{\mathscr{N}^\infty(T-\lambda) \oplus \mathrm{N}(T^p)}/\mathrm{N}(T^p) \\ &\subseteq \overline{\mathrm{H_o}(T-\lambda) + \mathrm{N}(T^p)}/\mathrm{N}(T^p).\end{aligned}$$

Finally, for (4.4), observe that

$$\mathrm{K}(\widetilde{T}-\lambda) = \mathscr{R}^\infty(\widetilde{T}-\lambda) = \mathscr{R}^\infty(T-\lambda)/\mathrm{N}(T^p) = \mathrm{K}(T-\lambda)/\mathrm{N}(T^p). \ \blacksquare$$

Now let us recall an important property from local spectral theory. An operator $T \in \mathscr{L}(X)$ is said to have the *single-valued extension property*, SVEP for short, at $\lambda \in \mathbb{C}$, if for every open disc U centred at λ, the only analytic solution $f : U \to X$ of the equation $(T-\mu)f(\mu) = 0$ is the zero function. It is straightforward to see that if λ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ (respectively of $\sigma_{\mathrm{su}}(T)$), then T (respectively T^*) has the SVEP at λ. The converse does not hold in general. Note that $\partial\sigma(T)$ is contained in $\sigma_{\mathrm{ap}}(T)$ as well as in $\sigma_{\mathrm{su}}(T)$. So T and T^* have the SVEP at every point of $\partial\sigma(T)$. Also, the following implications hold (see [2]):

$$\mathrm{H_o}(T-\lambda) \text{ is closed} \Rightarrow \mathrm{H_o}(T-\lambda) \cap \mathrm{K}(T-\lambda) = \{0\} \Rightarrow T \text{ has the SVEP at } \lambda,$$
$$X = \mathrm{H_o}(T-\lambda) + \mathrm{K}(T-\lambda) \ \Rightarrow \ T^* \text{ has the SVEP at } \lambda.$$

The finiteness of the ascent and descent of an operator also has significant effects on the SVEP. More precisely,

$$\mathrm{a}(T-\lambda) < \infty \ \Rightarrow \ T \text{ has the SVEP at } \lambda,$$
$$\mathrm{d}(T-\lambda) < \infty \ \Rightarrow \ T^* \text{ has the SVEP at } \lambda.$$

Moreover, from [2], if T is either semiregular or semi-Fredholm, then these implications are actually equivalences. In the next propositions we generalize this fact to operators T with $0 \notin \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$.

PROPOSITION 4.3. *Let $T \in \mathscr{L}(X)$ be such that $\mathrm{a_e}(T)$ is finite and $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is closed. The following conditions are equivalent*:

(1) *T has the SVEP at* 0,

(2) 0 *is not an accumulation point of* $\sigma_{\mathrm{ap}}(T)$,
(3) T *has finite ascent*,
(4) $\mathrm{H_o}(T)$ *is closed*,
(5) $\mathrm{H_o}(T) \cap \mathrm{K}(T) = \{0\}$.

Proof. (1)⇒(2). The restriction of T to the closed subspace $\mathrm{K}(T)$ satisfies the SVEP at 0 and it is surjective, hence $T|_{\mathrm{K}(T)}$ is invertible. Consequently, by Theorem 2.3, for $\lambda \neq 0$ small enough, $\mathrm{N}(T-\lambda) = \mathrm{N}(T-\lambda) \cap \mathrm{K}(T) = \mathrm{N}(T_{|\mathrm{K}(T)} - \lambda) = \{0\}$, and $\mathrm{R}(T-\lambda)$ is closed. This shows that zero is an isolated point of $\sigma_{\mathrm{ap}}(T)$.

The implication (2)⇒(3) follows directly from Theorem 2.3.

(3)⇒(4). If T has finite ascent d, then $\mathscr{N}^\infty(T) = \mathrm{N}(T^d)$ and so by Proposition 4.1, $\mathrm{H_o}(T) = \mathrm{N}(T^d)$ is closed.

Finally, as mentioned above, the implications (4)⇒(5)⇒(1) hold without any restriction on T. ∎

Remark 4.4. Notice that Proposition 4.3 may fail to hold if $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ is not assumed to be closed. Indeed, consider the shift operator T on $\ell^2(\mathbb{N})$ given by

$$T(x_1, x_2, x_3, \ldots) = (x_2/2, x_3/3, x_4/4, \ldots).$$

It is straightforward to see that T is quasi-nilpotent and $\dim \mathrm{N}(T^k) = k$ for all $k \in \mathbb{N}$. Hence T has the SVEP at zero and $\mathrm{a_e}(T)$ is finite, while $\mathrm{a}(T)$ is infinite.

Dually we have the following result.

Proposition 4.5. *Let* $T \in \mathscr{L}(X)$ *be such that* $\mathrm{a_e}(T)$ *is finite and* $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ *is closed. The following are equivalent*:

(1) T^* *has the SVEP at* 0,
(2) 0 *is not an accumulation point of* $\sigma_{\mathrm{su}}(T)$,
(3) T *has finite descent*,
(4) $X = \mathrm{K}(T) + \mathrm{H_o}(T)$.

Proof. (1)⇒(2). Let $p = \mathrm{p}(T)$ and S be the restriction of T^* to the closed subspace $\mathrm{R}(T^{*p})$. Since

$$\dim \mathrm{R}(T^{*p})/\mathrm{R}(T^{*(p+1)}) = \operatorname{codim}(\mathrm{R}(T^*) + \mathrm{N}(T^{*p})) = \dim \mathrm{N}(T) \cap \mathrm{R}(T^p)$$

is finite, S is semi-Fredholm, and hence $\mathrm{K}(S) = \mathscr{R}^\infty(S) = \mathscr{R}^\infty(T^*) \subseteq \mathrm{K}(T^*)$. This shows that $\mathrm{K}(T^*) = \mathscr{R}^\infty(T^*)$ is closed. Now $T^*|_{\mathrm{K}(T^*)}$ is surjective and has the SVEP at 0, which implies that it is injective. Consequently, by Theorem 2.3, for small non-zero λ we have $\mathrm{N}(T^* - \lambda) = \mathrm{N}(T^* - \lambda) \cap \mathrm{K}(T^*) = \{0\}$ and $T - \lambda$ is semi-Fredholm. Therefore, $T - \lambda$ is surjective.

The implication (2)⇒(3) follows from Theorem 2.3, and (3) ⇒ (4) is a consequence of (1.1) and Proposition 4.1. Finally, (4)⇒(1) holds without any restriction on T. ∎

We mention that the equivalence between (2) and (4) has recently been established in [10] without any restriction on T.

The next corollary follows immediately from Propositions 4.1, 4.3 and 4.5.

Corollary 4.6. *Let* $T \in \mathscr{L}(X)$ *be such that* $\mathrm{a_e}(T)$ *is finite and* $\mathrm{R}(T^{\mathrm{a_e}(T)+1})$ *is closed. The following assertions are equivalent*:

(i) T *and* T^* *has the SVEP at* 0,

(ii) $X = \mathrm{H_o}(T) \oplus \mathrm{K}(T)$,

(iii) 0 *is a pole of the resolvent of* T.

For a bounded operator T on X, we denote by $\mathscr{S}(T)$ the open set of complex numbers λ for which T fails to have the SVEP. It is easy to check that $\mathscr{S}(T)$ is a closed subset of $\sigma(T)$. The operator T is said to satisfy the SVEP if $\mathscr{S}(T)$ is empty. Clearly, if $\mathrm{a}(T-\lambda)$ is finite for every $\lambda \in \mathbb{C}$, then T has the SVEP. The following example reveals that this does not hold for the essential ascent.

Example 4.7. Let T be the standard shift operator on $\ell^2(\mathbb{N})$ given by $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. Since T is surjective and non-invertible, T does not have the SVEP. However, $\mathrm{a_e}(T-\lambda)$ is finite for all $\lambda \in \mathbb{C}$, because $\dim \mathrm{N}(T-\lambda) = 1$ for $|\lambda| < 1$, and $\mathrm{N}(T-\lambda) = \{0\}$ for $|\lambda| \geq 1$.

Corollary 4.8. *Every operator* $T \in \mathscr{L}(X)$ *satisfies*

$$\sigma_{\mathrm{asc}}(T) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \cup \mathscr{S}(T).$$

If T *has the SVEP*, *then* $\sigma_{\mathrm{asc}}(T) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \subseteq \sigma^{\mathrm{e}}_{\mathrm{asc}}(T^*)$.

Proof. It is clear that $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \cup \mathscr{S}(T) \subseteq \sigma_{\mathrm{asc}}(T)$, and the reciprocal inclusion follows from the last proposition. Thus, if T has the SVEP, $\sigma_{\mathrm{asc}}(T) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$. Let $\lambda \notin \sigma^{\mathrm{e}}_{\mathrm{asc}}(T^*)$ and $p = p(T^* - \lambda)$. It follows by Theorem 2.3 that for μ in a small punctured neighbourhood of λ, $T^* - \mu$ and hence $T - \mu$ is semi-Fredholm with

$$\begin{aligned}\dim \mathrm{N}(T-\mu) &= \operatorname{codim} \mathrm{R}(T^*-\mu) = \dim \mathrm{R}(T^*-\lambda)^p/\mathrm{R}(T^*-\lambda)^{p+1} \\ &= \dim \mathrm{N}(T-\lambda) \cap \mathrm{R}(T-\lambda)^p = \dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p.\end{aligned}$$

Moreover, as T has the SVEP, [21, Proposition 2.2] implies that $\mathrm{ind}(T-\mu) \leq 0$, and therefore $\dim \mathrm{N}(T-\mu) = \dim \mathrm{N}(T-\lambda)^{p+1}/\mathrm{N}(T-\lambda)^p$ is finite, as desired. ∎

Remark 4.9. Notice that dual results for the descent and essential descent are formulated in [6] and [7]. While this paper was in the refereeing process, Aiena in [1] has shown that Propositions 4.3, 4.5 and Corollary 4.6 hold for quasi-Fredholm operators by using different arguments. This is why we have kept the proofs here.

Acknowledgements. The authors would like to thank Professor Mostafa Mbekhta for several discussions concerning the improvement of this paper.

Also the authors would like to thank the referee for many interesting suggestions and corrections. The second author is partially supported by I+D MEC project no. MTM2005-02541 and Junta de Andalucía grant FQM1215.

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Received October 5, 2005
Revised version March 11, 2008 (5770)