Isomorphisms of some reflexive algebras

by

JIANKUI LI (Shanghai) and ZHIDONG PAN (University Center, MI)

Abstract. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices on complex separable Banach spaces $X$ and $Y$, respectively. We prove that under certain lattice-theoretic conditions every isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$ is quasi-spatial; in particular, if a subspace lattice $\mathcal{L}$ of a complex separable Banach space $X$ contains a sequence $E_i$ such that $(E_i)_- \neq X$, $E_i \subseteq E_{i+1}$, and $\bigvee_{i=1}^{\infty} E_i = X$ then every automorphism of $\text{alg} \mathcal{L}$ is quasi-spatial.

1. Introduction. Let $X$ and $Y$ be separable complex Banach spaces and let $B(X,Y)$ be the set of all bounded linear maps from $X$ into $Y$. When $X = Y$, we use $B(X)$ instead of $B(X,Y)$. When $X$ is a Hilbert space, we use $H$ instead of $X$. For vector spaces $\mathcal{U}$ and $\mathcal{V}$, we write $L(\mathcal{U},\mathcal{V})$ for the set of all linear maps from $\mathcal{U}$ to $\mathcal{V}$. By a subspace lattice on $X$, we mean a collection $\mathcal{L}$ of closed subspaces of $X$ with $0$ and $X$ in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\bigcap M_r$ and $\bigvee M_r$ belong to $\mathcal{L}$. If the operations of meet and join distribute over each other for any collections of subspaces in $\mathcal{L}$, then $\mathcal{L}$ is said to be completely distributive. If $L \in \mathcal{L}$, we denote by $L_-$ the subspace $\bigvee\{M \in \mathcal{L} : L \not\subseteq M\}$ and denote by $L_+$ the subspace $\bigcap\{M \in \mathcal{L} : M \not\subseteq L\}$. For a subspace lattice $\mathcal{L}$ of $X$, we use $\text{alg} \mathcal{L}$ to denote the algebra of all operators on $X$ that leave members of $\mathcal{L}$ invariant.

For Hilbert spaces, a common practice is to disregard the distinction between a subspace and the orthogonal projection onto it. A Hilbert space subspace lattice $\mathcal{L}$ is called a commutative subspace lattice if it consists of mutually commuting projections. If $\mathcal{L}$ is a commutative subspace lattice then $\text{alg} \mathcal{L}$ is called a CSL algebra.

If $\mathcal{L}$ is a subspace lattice on $X$, we define $\mathcal{J}_\mathcal{L} = \{L \in \mathcal{L} : L \neq 0$ and $L_- \neq X\}$. We say $\mathcal{J}_\mathcal{L}$ is sequentially dense in $X$ if there exists a sequence $E_i \in \mathcal{J}_\mathcal{L}$ such that $E_i \subseteq E_{i+1}$ and $\bigvee_{i=1}^{\infty} E_i = X$. Quasi-spatiality of isomorphisms has been studied in [1, 2, 4, 5]. The main task of [4] is to show that if $\mathcal{L}$ is a commutative subspace lattice on a Hilbert space $H$ such that $\mathcal{J}_\mathcal{L}$ is

2000 Mathematics Subject Classification: Primary 47B47, 47L35.

Key words and phrases: subspace lattice, isomorphism, reflexive.

This work was completed with the support of NSF of China.
sequentially dense in $H$ then every automorphism on $\text{alg} \mathcal{L}$ is quasi-spatial. In this paper, we generalize the above result, with a relatively simpler proof, to non-commutative subspace lattices on Banach spaces; more specifically, we show that if $\mathcal{L}$ is any subspace lattice on a Banach space $X$ such that $\mathcal{J}_{\mathcal{L}}$ is sequentially dense in $X$ then every automorphism on $\text{alg} \mathcal{L}$ is quasi-spatial. Our main result, Theorem 2.6, is stated in a slightly more general form; this also makes the presentation of the proof a little clearer.

2. The main result. For a subspace $E$ of a Banach space $X$, we define $E^\perp = \{ f^* \in X^* : f^*|_E = 0 \}$. For any $x \in X$ and $f^* \in X^*$, we use $x \otimes f^*$ to denote the rank-one operator satisfying $x \otimes f^*(u) = f^*(u)x$ for all $u \in X$. It follows from [3] that $x \otimes f^* \in \text{alg} \mathcal{L}$ if and only if there exists an $L \in \mathcal{J}_{\mathcal{L}}$ such that $x \in L$ and $f^* \in (L_\perp)^\perp$. In the following, we suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices on Banach spaces $X$ and $Y$, respectively; and $\text{alg} \mathcal{L}_1$ and $\text{alg} \mathcal{L}_2$ are the corresponding subalgebras of $\text{B}(X)$ and $\text{B}(Y)$, respectively.

We will break the proof of the main result into a few lemmas.

**Lemma 2.1.** Suppose $\mathcal{J}_{\mathcal{L}_2}$ is sequentially dense in $Y$, $\psi$ is an isomorphism from $\text{alg} \mathcal{L}_2$ to $\text{alg} \mathcal{L}_1$, and $E \in \mathcal{J}_{\mathcal{L}_1}$. Then for any $x \in E$, there exist $K \in \mathcal{J}_{\mathcal{L}_2}$, $y \in K$, $h^* \in (K_\perp)^\perp$, and $0 \neq g^* \in X^*$ such that $\psi(y \otimes h^*) = x \otimes g^*$.

**Proof.** Take any $x \in E$ and $0 \neq l^* \in (E_\perp)^\perp$. Then $x \otimes l^* \in \text{alg} \mathcal{L}_1$. Since $\psi$ is surjective, there exists a $B \in \text{alg} \mathcal{L}_2$ such that $\psi(B) = x \otimes l^*$. Since $\mathcal{J}_{\mathcal{L}_2}$ is sequentially dense in $Y$, there exists a $K \in \mathcal{J}_{\mathcal{L}_2}$ and $w \in K$ such that $y = Bw \neq 0$. Choose $0 \neq h^* \in (K_\perp)^\perp$ and set $A = \psi(w \otimes h^*)$ and $g^* = A^*l^*$. Then $\psi(y \otimes h^*) = \psi((Bw) \otimes h^*) = \psi(Bw \otimes h^*) = \psi(B)\psi(w \otimes l^*) = x \otimes l^*A = x \otimes g^*$. \hfill $\blacksquare$

**Remark 2.2.** Let $K$ be as in Lemma 2.1. From the proof of Lemma 2.1, one can see that, for any $L \in \mathcal{J}_{\mathcal{L}_2}$ with $K \subseteq L$, there exist $y_1 \in L$, $h_1^* \in (L_\perp)^\perp$, and $0 \neq g_1^* \in X^*$ such that $\psi(y_1 \otimes h_1^*) = x \otimes g_1^*$.

**Lemma 2.3.** Suppose $E_i \in \mathcal{J}_{\mathcal{L}_1}$ with $E_i \subseteq E_{i+1}$, $\bigvee_{i=1}^\infty E_i = X$, and $K_i \in \mathcal{J}_{\mathcal{L}_2}$ with $K_i \subseteq K_{i+1}$ and $\bigwedge_{i=1}^\infty K_i = Y$. If $\phi$ is an isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$, then there exist $K_{n_i} \in \mathcal{J}_{\mathcal{L}_2}$ with $K_{n_i} \subseteq K_{n_{i+1}}$, $\bigwedge_{i=1}^\infty K_{n_i} = Y$, and injective $T_i \in L(E_i, Y)$ with $\text{ran}(T_i) \subseteq K_{n_i}$ such that $\phi(A)T_ix = T_iAx$ for every $x \in E_i$ and $A \in \text{alg} \mathcal{L}_1$.

**Proof.** For any $0 \neq f_i^* \in ((E_i)_\perp)^\perp$, there exist $E_{m_i}$ and $x_i \in E_{m_i}$ such that $f_i^*(x_i) = 1$. By Lemma 2.1, there exist $y_i \in K_{n_i} \in \mathcal{J}_{\mathcal{L}_2}$, $h_i^* \in ((K_{n_i})_\perp)^\perp$, and $0 \neq g_i^* \in X^*$ such that $\phi^{-1}(y_i \otimes h_i^*) = x_i \otimes g_i^*$. Since $E_i \subseteq E_{i+1}$ and $\bigwedge_{i=1}^\infty E_i = X$, there exist $E_{p_i}$ and $u_i \in E_{p_i}$ such that $g_i^*(u_i) = 1$. Define $T_i \in L(E_i, Y)$ by

$$T_ix = \phi(x \otimes f_i^*)y_i, \quad \forall x \in E_i$$

(2.1)
and define $S_i \in L(K_{n_i}, X)$ by
\[
S_i y = \phi^{-1}(y \otimes h_i^*) u_i, \quad \forall y \in K_{n_i}.
\]
It is clear from the definition of $T_i$ that $\text{ran}(T_i) \subseteq K_{n_i}$.

For any $x \in E_i$,
\[
S_i T_i x = \phi^{-1}(T_i x \otimes h_i^*) u_i = \phi^{-1}((\phi(x \otimes f_i^*) y_i) \otimes h_i^*) u_i
\]
\[
= (x \otimes f_i^*) \phi^{-1}(y_i \otimes h_i^*) u_i = (x \otimes f_i^*) (x_i \otimes g_i^*) u_i
\]
\[
= (x \otimes f_i^*) x = x.
\]
In particular, $T_i$ and $S_i |_{V_i}$ are injective, where $V_i = \text{ran}(T_i)$. Furthermore,
\[
\phi(A) T_i x = \phi(A) \phi(x \otimes f_i^*) y_i = \phi(A x \otimes f_i^*) y_i = T_i A x,
\]
\[
\forall x \in E_i, A \in \text{alg} \mathcal{L}_1.
\]

Similar to (2.1) and (2.2), we can construct $T_{i+1}$ and $S_{i+1}$; by Remark 2.2 we can assume $K_{n_i} \subseteq K_{n_{i+1}}$.

For any Banach space $X$, $f^* \in X^*$ and $E \subseteq X$, define
\[
[E \otimes f^*]_X = \{x \otimes f^* : x \in E\}.
\]

**Lemma 4.** Suppose $E_i \in \mathcal{J}_{\mathcal{L}_1}$ with $E_i \subseteq E_{i+1}$, $\bigvee_{i=1}^{\infty} E_i = X$, and $K_i \in \mathcal{J}_{\mathcal{L}_2}$ with $K_i \subseteq K_{i+1}$, $\bigvee_{i=1}^{\infty} K_i = Y$. If $\phi$ is an isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$, then for each $a_i^* \in ((E_i)_-)^\perp$, there is a $b_i^* \in Y^*$ such that $\phi([E_i \otimes a_i^*]_X) \subseteq [Y \otimes b_i^*]_Y$.

**Proof.** Let $T_i$ be as in Lemma 2.3. Then by (2.4) we have
\[
(2.5) \quad \phi(A) T_i x = T_i A x, \quad \forall x \in E_i, A \in \text{alg} \mathcal{L}_1.
\]
It follows that $B T_i x = T_i \phi^{-1}(B) x$ for $x \in E_i$ and $B \in \text{alg} \mathcal{L}_2$. This implies that whenever $B$ is a rank-one operator, $\phi^{-1}(B)$ is also a rank-one operator, since $\bigvee_{i=1}^{\infty} E_i = X$ and $T_i$ is injective. By the symmetry of $X$ and $Y$, $\phi$ also maps rank-one operators to rank-one operators.

For each fixed $m$, fix $0 \neq x_1 \in E_m$ and $0 \neq a_m^* \in ((E_m)_-)^\perp$ and suppose $\phi(x_1 \otimes a_m^*) = y_1 \otimes b_m^*$ for some $y_1 \in Y$ and $b_m^* \in Y^*$. We will show
\[
\phi([E_m \otimes a_m^*]_X) \subseteq [Y \otimes b_m^*]_Y.
\]
Take any $x_2 \in E_m$ such that $\{x_1, x_2\}$ is linearly independent. Suppose $\phi(x_2 \otimes a_m^*) = y_2 \otimes c_m^*$ for some $y_2 \in Y$ and $c_m^* \in Y^*$. We only need to show $\{b_m^*, c_m^*\}$ is linearly dependent.

Applying (2.5) with $A = x_1 \otimes a_m^*$ and $A = x_2 \otimes a_m^*$, respectively, we obtain
\[
(2.6) \quad b_m^* (T_i x) y_1 = a_m^* (x) T_i x_1, \quad \forall x \in E_i,
\]
and
\[
(2.7) \quad c_m^* (T_i x) y_2 = a_m^* (x) T_i x_2, \quad \forall x \in E_i,
\]
Since $E_i \subseteq E_{i+1}$ and $\bigvee_{i=1}^{\infty} E_i = X$, there exist $E_i$ and $x \in E_i$ such that $a_m^*(x) \neq 0$. Since $T_i$ is injective and $\{x_1, x_2\}$ is linearly independent, $\{T_ix_1, T_ix_2\}$ is linearly independent; so $\{y_1, y_2\}$ is linearly independent, by (2.6) and (2.7).

Since $\phi$ maps rank-one operators to rank-one operators, $\phi((x_1 + x_2) \otimes a_m^*)$ is a rank-one operator. Thus, $y_1 \otimes b_m^* + y_2 \otimes c_m^* = \phi((x_1 + x_2) \otimes a_m^*)$ is a rank-one operator. Since $\{y_1, y_2\}$ is linearly independent, $\{b_m^*, c_m^*\}$ is linearly dependent.

For a subspace $S$ of $L(\mathcal{U}, \mathcal{V})$, define $\text{ref}_a(S) = \{T \in L(\mathcal{U}, \mathcal{V}) : Tx \in Sx, \forall x \in \mathcal{U}\}$. We say $S$ is algebraically reflexive if $\text{ref}_a(S) = S$. It is well known and not hard to show that every one-dimensional subspace of $L(\mathcal{U}, \mathcal{V})$ is algebraically reflexive.

**Lemma 2.5.** Assuming the same hypotheses and notations as in Lemma 2.3, by rescaling $T_i$ we can have $T_{i+1}|E_i = T_i$ for $i = 1, 2, \ldots$.

**Proof.** Fix any $a_i^* \in ((E_i)_-)^\perp$ and $v \in Y$, and define $D \in L(E_i, Y)$ by $Dx = \phi(x \otimes a_i^*)v$ for $x \in E_i$. If $D$ is not the zero operator then $D$ is injective; indeed, by Lemma 2.4, there exists $b_i \in Y^*$ such that $\phi(x \otimes a_i^*) = \lambda x \otimes b_i^*$ for all $x \in E_i$. Since $\phi$ maps rank-one operators to rank-one operators, $\lambda x \neq 0$ for all $0 \neq x \in E_i$. If $D$ is not the zero operator then $b_i^*(v) \neq 0$, so $D$ is injective; in particular, the operators $T_i$ defined by (2.1) are injective (which we already knew). By the symmetry of $X$ and $Y$, the operators $S_i$ defined by (2.2) are also injective.

Suppose $T_i$, $S_i$, $T_{i+1}$, and $S_{i+1}$ have been constructed as in Lemma 2.3. Then $S_{i+1}T_{i+1}x = x$ for all $x \in E_{i+1}$; in particular, $S_{i+1}T_{i+1}x = x$ for all $x \in E_i$. Let $V_i = \text{ran}(T_i)$ and note that $V_i \subseteq K_{n_i} \subseteq K_{n_{i+1}}$. Consider $S_i|V_i, S_{i+1}|V_i \in L(V_i, X)$. Since the one-dimensional subspace generated by the transformation $S_i|V_i$ is algebraically reflexive in $L(V_i, X)$ and

$$S_{i+1}T_i x = \phi^{-1}(T_i x \otimes h_{i+1}^*)u_{i+1} = \phi^{-1}(\phi(x \otimes f_i^*)y_i) \otimes h_{i+1}^*)u_{i+1} = (x \otimes f_i^*)\phi^{-1}(y_i \otimes h_{i+1}^*)u_{i+1} = (x \otimes f_i^*)t_{i+1},$$

where $t_{i+1} = \phi^{-1}(y_i \otimes h_{i+1}^*)u_{i+1}$, it follows that $S_{i+1}V_i = c_i S_i V_i$ for some scalar $c_i$. Since $S_{i+1}$ is injective, $c_i \neq 0$.

Replacing $S_{i+1}$ by $(1/c_i)S_{i+1}$ and $T_{i+1}$ by $c_iT_{i+1}$ and still calling them $S_{i+1}$ and $T_{i+1}$, respectively, we have $S_{i+1}V_i = S_i V_i$, and for any $x \in E_i$, $S_{i+1}T_i x = S_i T_i x = x = S_{i+1}T_{i+1}x$. It follows that $T_{i+1}x = T_i x$ for all $x \in E_i$.

We say $\phi$ is quasi-spatial if there exists an injective linear transformation $T \in L(D(T), Y)$, where $D(T)$ is the domain of $T$ such that $D(T)$ is dense.
in $X$ and invariant under $\text{alg} \mathcal{L}_1$, the range of $T$ is dense in $Y$, and
\begin{equation}
(2.8) \quad \phi(A)Tx = TAx, \quad \forall x \in D(T), \ A \in \text{alg} \mathcal{L}_1.
\end{equation}

**Theorem 2.6.** Suppose $\mathcal{J}_{\mathcal{L}_1}$ is sequentially dense in $X$ and $\mathcal{J}_{\mathcal{L}_2}$ is sequentially dense in $Y$. Then every isomorphism $\phi$ from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$ is quasi-spatial; in particular, $\phi$ preserves ranks of operators.

**Proof.** By the assumptions, there exist $E_i \in \mathcal{J}_{\mathcal{L}_1}$ with $E_i \subseteq E_{i+1}$, $\bigvee_{i=1}^{\infty} E_i = X$, and $K_i \in \mathcal{J}_{\mathcal{L}_2}$ with $K_i \subseteq K_{i+1}$, $\bigvee_{i=1}^{\infty} K_i = Y$. Now we can construct $T_i$ as in Lemma 2.3, with modifications as in Lemma 2.5. Let $E = \bigcup_{i=1}^{\infty} E_i$, the non-closed union of $E_i$, so $E$ is dense in $X$. Clearly, $E$ is invariant under $\text{alg} \mathcal{L}_1$, and if $x \in E$ then $x \in E_i$ for some $i$. Define $T x = T_i x$. By the agreement among $T_i$, it follows that $T$ is a well-defined, injective, linear transformation on $E$; moreover, $\phi(A)Tx = TAx$ for all $x \in E$ and $A \in \text{alg} \mathcal{L}_1$. Let $\text{ran}(T)$ be the range of $T$ and $K = \bigcup_{i=1}^{\infty} K_i$. Clearly $K$ is dense in $Y$ and $\text{ran}(T) \subseteq K$; we will show $\text{ran}(T) = K$. Take any $y \in K$. There exists $K_{n_i}$ such that $y \in K_{n_i}$. By (2.2) of Lemma 2.3, $S_i y = \phi^{-1}(y \otimes h_i^*)u_i \in E_{p_i} \subseteq E$. By (2.1) of Lemma 2.3,
\begin{align*}
T S_i y &= T_{p_i} S_i y = \phi(\phi^{-1}(y \otimes h_i^*)u_i \otimes f_{p_i}^*)y_{p_i} = (y \otimes h_i^*)\phi(u_i \otimes f_{p_i}^*)y_{p_i} \\
&= h_i^*(\phi(u_i \otimes f_{p_i}^*)y_{p_i})y = \mu_i y,
\end{align*}
where $\mu_i = h_i^*(\phi(u_i \otimes f_{p_i}^*)y_{p_i})$. Since $T_{p_i}$ and $S_i$ are injective, $\mu_i \neq 0$. Now $T(\mu_i^{-1}S_i y) = y$, so $\text{ran}(T) = K$.

Rank-preserving follows from (2.8) directly. \hfill \blacksquare

The following corollary is the main result of [4]. A special case of the corollary was proved earlier in [1] with an additional hypothesis of subspace lattices being completely distributive.

**Corollary 2.7 ([4, Theorem 17]).** Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative subspace lattices on a Hilbert space $H$ and $\mathcal{J}_{\mathcal{L}_1}$ is sequentially dense in $H$. Then every isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$ is quasi-spatial.

**Proof.** By [4, Theorem C], we can assume $\mathcal{L}_1 = \mathcal{L}_2$. Now the conclusion follows from Theorem 2.6. \hfill \blacksquare

Remark: The hypotheses in [4, Theorem 17] are stated differently from Corollary 2.7, but it is easy to check that they are equivalent.

**Theorem 2.8.** If $\mathcal{L}_1$ is a subspace lattice with $X_\sim \neq X$ and $\mathcal{L}_2$ is a subspace lattice with $Y_\sim \neq Y$, then every isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$ is spatially implemented and every bounded isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$ is spatially implemented by a bounded operator.

**Proof.** Suppose $\phi$ is an isomorphism from $\text{alg} \mathcal{L}_1$ to $\text{alg} \mathcal{L}_2$. Take $E_i = X$ and $K_i = Y$, then the hypotheses of Theorem 2.6 are satisfied. Let $T_i$ be defined by (2.1) and $S_i$ be defined by (2.2) in Lemma 2.3. By (2.3),
$S_i \in L(Y, X)$ is surjective. By the first paragraph of the proof of Lemma 2.5, $S_i$ is injective, so $S_i$ has an inverse. Now the equality $S_iT_ix = x$ for all $x \in E_i (= X)$ implies $T_i$ is invertible with $T_i^{-1} = S_i$. Finally, (2.5) of Lemma 2.4 implies $\phi$ is spatially implemented. If $\phi$ is bounded, then so are $T_i$ and $S_i$. ■

**Corollary 2.9.** If $L$ is a subspace lattice on a Hilbert space $H$ with $0_+ \neq 0$, then every automorphism of $\text{alg} \ L$ is spatial.

**Proof.** Suppose $L$ satisfies $0_+ \neq 0$ and $\phi$ is an automorphism of $\text{alg} \ L$. Let $L^\perp = \{I - L : L \in L\}$, where $I$ is the identity operator on $H$. Then $L^\perp$ satisfies $H^+ \neq H$.

Define $\phi^*(A^*) = (\phi(A))^*$ for $A^* \in \text{alg} \ L^\perp$. Then $\phi^*$ is an automorphism of $\text{alg} \ L^\perp$. By Theorem 2.8, we have $\phi^*(A^*) = (\phi(A))^* = TA^*T^{-1}$ for some $T \in B(H)$. So $\phi$ is spatial. ■

**References**