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From restricted type to strong type estimates on quasi-Banach rearrangement invariant spaces

by

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Abstract. Let X be a quasi-Banach rearrangement invariant space and let T be an (ε,δ)-atomic operator for which a restricted type estimate of the form $\|T\chi_E\|_X \le D(|E|)$ for some positive function D and every measurable set E is known. We show that this estimate can be extended to the set of all positive functions $f \in L^1$ such that $\|f\|_\infty \le 1$, in the sense that $\|Tf\|_X \le D(\|f\|_1)$. This inequality allows us to obtain strong type estimates for T on several classes of spaces as soon as some information about the galb of the space X is known. In this paper we consider the case of weighted Lorentz spaces $X = \Lambda^q(w)$ and their weak version.

1. Introduction. It is well known (see [1], [3], [4] and [14]) that, for many interesting operators only a restricted estimate on characteristic functions is known, and it is of a general interest to show what kind of strong type estimate can be obtained from it. This is, for example, the principle of the weak type extrapolation theory where we have an operator satisfying

$$\|T\chi_E\|_{L^{p,\infty}} \le \frac{1}{p-1}\,|E|^{1/p}$$

for every $1 < p \le p_0$, and it is an open question to see if this implies that T is bounded from the Orlicz space $L\log L$ into $L^{1,\infty}$. A positive solution to this question will give us, when applied to the Carleson operator

$$Sf(x) = \sup_n |S_n f(x)|,$$

where $S_n f(x) = (D_n * f)(x)$, D_n is the Dirichlet kernel on $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ and $f \in L^1(\mathbb{T})$, the almost everywhere convergence of the Fourier series of a function in $L\log L(\mathbb{T})$.

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In a recent paper [4], it was proved that if the operator T is (ε, δ)-atomic approximable (see Definition 2.2), then an estimate of the form

$$(T\chi_E)^*(t) \leq h(t, |E|)$$

for every measurable set can be extended to every function f bounded by 1 and, from it, some strong type estimates on logarithmic type spaces were proved. In particular, if $h(t,s) \leq R(t)D(s)$, the above inequality is equivalent to $\|T\chi_E\|_X \leq D(|E|)$, where X is a weak weighted Lorentz space (see definition below).

The first purpose of this paper consists in proving, in Section 2, that a slight modification of the main theorem in [4] shows that if T is (ε, δ)-atomic approximable, and $\|T\chi_E\|_X \leq D(|E|)$ for some positive function D and every measurable set E where X is any quasi-Banach r.i. space, then $\|Tf\|_X \leq D(\|f\|_1)$ for every $f \in L^1$ such that $\|f\|_\infty \leq 1$.

Our second step will be to obtain, from this inequality, a strong type estimate, for which we need to have some information on $\mathrm{Galb}(X)$, which is defined (see [21]) by

$$\mathrm{Galb}(X) = \Big\{(c_n)_n;\ \sum_n c_n f_n \in X \text{ whenever } \|f_n\|_X \leq 1\Big\},$$

endowed with the norm $\|c\|_{\mathrm{Galb}(X)} = \sup_{\|f_n\|_X \leq 1} \|\sum_n c_n f_n\|_X$. In particular, we study, in Section 3, this galb for weighted Lorentz spaces $X = \Lambda^q(w)$, for $0 < q < \infty$, and also for the weak spaces $\Lambda^{q,\infty}(w)$. To this end, we use the following formula for the decreasing rearrangement of a sum of functions (see [9]): if $f = \sum_n c_n f_n$, then

$$f^*(3t) \leq \sum_n c_n \bigg(f_n^*(t) + \frac{1}{t} \int_{a_n t}^{t} f_n^*(s)\, ds\bigg),$$

where $\{a_n\}_n$ are any positive numbers such that $\sum_n a_n = 1$, and we need to solve the problem of computing, for $q \geq 1$,

$$\sup_{f\downarrow} \frac{\int_0^\infty (t^{-1} \int_{at}^t f(s)\, ds)^q w(t)\, dt}{\int_0^\infty f(t)^q w(t)\, dt},$$

where the supremum extends over the set of decreasing functions f. This problem will be solved in Section 5. Finally, in Section 4 we present some concrete examples and applications.

We shall denote by $L^0(\mathbb{R}^n)$ the class of Lebesgue measurable functions that are finite a.e., and $g^*(t) = \inf\{s : \lambda_g(s) \leq t\}$ is the *decreasing rearrangement* of g, where $\lambda_g(y) = |\{x \in \mathbb{R} : |g(x)| > y\}|$ is the *distribution function* of g with respect to Lebesgue measure. We refer the reader to [2] for further information about distribution functions, decreasing rearrangements and rearrangement invariant (r.i.) spaces.

If, in the definition of a norm, the triangle inequality is weakened to the requirement that for some constant c, $\|x+y\| \le c(\|x\|+\|y\|)$ holds for all x and y, then we have a *quasi-norm*. A complete quasi-normed space is called a quasi-Banach space. It is well known that the spaces ℓ^p for $0<p<1$ are *quasi-Banach spaces*. Observe that if X is a quasi-Banach r.i. space of measurable functions on $\mathbb{R}^n$ then there is a r.i. quasi-Banach space X^* of measurable functions on $\mathbb{R}$ such that $\|f\|_X = \|f^*\|_{X^*}$ for all $f \in X$. One simply defines $\|g\|_{X^*} = \|G\|_X$ where $G(x) = \omega g(|x|^n)$ with ω chosen so that g and G are equimeasurable. It is a simple matter to verify that X^* is a quasi-Banach space.

For a measurable set E, χ_E denotes the characteristic function of E, $|E|$ is the Lebesgue measure of E and, for simplicity in our arguments, we say that an operator T is *sublinear* if $T(\lambda f) = \lambda Tf$ and

$$\Big|T\Big(\sum_{n\in\mathbb{N}} f_n\Big)\Big| \le \sum_{n\in\mathbb{N}} |Tf_n|.$$

If we only know that $|T(f+g)| \le |Tf| + |Tg|$, then we need to assume some extra boundedness condition on our operator T, such as the boundedness of $T : L^1 + L^\infty \to L^0$, or to use some standard density argument to obtain our conclusions.

2. From restricted weak type to strong type. We shall work in $\mathbb{R}^n$, and Q will represent a cube with sides parallel to the axes. The results can be extended in the natural way to $\mathbb{T}^N$ (identifying $\mathbb{T}^N$ with $[0,1)^N$). In [4], the following definitions were introduced:

DEFINITION 2.1. Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a *δ-atom* if

(i) $\int_{\mathbb{R}^n} a(x)\,dx = 0$,
(ii) there exists a cube Q such that $|Q| \le \delta$ and $\operatorname{supp} a \subset Q$.

DEFINITION 2.2.

(a) A sublinear operator T, defined on $L^1 + L^\infty$ and taking values in L^0, is *(ε,δ)-atomic* if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$\|Ta\|_{L^1+L^\infty} \le \varepsilon \|a\|_1 \quad \text{for every } \delta\text{-atom } a. \tag{1}$$

(b) A sublinear operator T is *(ε,δ)-atomic approximable* if there exists a sequence $(T_n)_n$ of (ε,δ)-atomic operators such that $|T_n\chi_E| \le |T\chi_E|$ for every measurable set E and for every $f \in L^1$ such that $\|f\|_\infty \le 1$, and every $t > 0$,

$$(Tf)^*(t) \le \liminf_n (T_n f)^*(t).$$

In particular, any maximal operator of the form $\sup_j |K_j * f|$, where $K_j \in L^{p_j}$ for some $1 \leq p_j < \infty$, is (ε, δ)-atomic approximable (see [4] for more examples of this kind of operators). As we shall see in this paper, no operator bounded from L^p into L^p with $0 < p < 1$ is (ε, δ)-atomic approximable.

Definition 2.3. Given an operator T and a quasi-Banach r.i. space X, we define the *fundamental function* of T with respect to X by

$$\varphi_{X,T}(r) = \sup_{|E| \leq r} \|T\chi_E\|_X.$$

Observe that if T is the identity operator, then $\varphi_{X,T}$ is nothing but φ_X, the usual fundamental function of X.

Definition 2.4. Given $\delta > 0$, we say that $\mathcal{F}_\delta$ is a *δ-net* if it is a collection of open cubes of the following form:

$$\mathcal{F}_\delta = \{Q_j;\ |Q_j| = \delta,\ Q_j \text{ are pairwise disjoint}, \bigcup \overline{Q}_j = \mathbb{R}^n\}.$$

Theorem 2.1. *Let X be a quasi-Banach r.i. space and T a sublinear (ε, δ)-atomic approximable operator. Then, for every positive function $f \in L^1$ such that $\|f\|_\infty \leq 1$,*

$$\|Tf\|_X \leq \varphi_{X,T}(\|f\|_1).$$

Proof. In view of Definition 2.2, it is enough to prove the result for an (ε, δ)-atomic operator T.

Given X, let X^* be the space of measurable functions on $(0, \infty)$ such that $\|f\|_X = \|f^*\|_{X^*}$. Let $f \in L^1$ be a positive function such that $\|f\|_\infty \leq 1$ and, given $\varepsilon > 0$, consider a δ-net $\mathcal{F}_\delta$ where δ is associated to ε by the property that T is (ε, δ)-atomic.

Given $Q_i \in \mathcal{F}_\delta$, let $f_i = f\chi_{Q_i}$. Then

$$\int_{\mathbb{R}^n} f_i(x)\,dx \leq |Q_i|,$$

and hence we can take a cube $\widetilde{Q}_i \subset Q_i$ satisfying

$$|\widetilde{Q}_i| = \int_{\mathbb{R}^n} f_i(x)\,dx = \int_{Q_i} f(x)\,dx.$$

Then it is clear that the function $g_i = f_i - \chi_{\widetilde{Q}_i}$ is a δ-atom and

$$\|g_i\|_1 \leq \int_{Q_i} |f(x)|\,dx + |\widetilde{Q}_i| = 2\int_{Q_i} |f(x)|\,dx.$$

Now, $f = \sum_i f_i = \sum_i g_i + \chi_E$, where $E = \bigcup \widetilde{Q}_i$. Then, by sublinearity,

$$|Tf| \leq \sum_i |Tg_i| + |T\chi_E| \equiv G + |T\chi_E|.$$

For fixed $n > 1$, we have

$$\begin{aligned}
&(Tf)^*(x)\chi_{(1/n,n)}(x) \\
&\qquad \le G^*((1/n^2)x)\chi_{(1/n,n)}(x) + (T\chi_E)^*((1-1/n^2)x)\chi_{(1/n,n)}(x) \\
&\qquad \equiv Q_n(x) + R_n(x).
\end{aligned}$$

For $x \in (1/n, n)$, we have $0 \le x - 1/n \le (1-1/n^2)x$, and hence

$$R_n(x) \le (T\chi_E)^*(x-1/n)\chi_{(1/n,n)}(x);$$

it follows that $R_n^* \le (T\chi_E)^*$. On the other hand,

$$Q_n(x) \le G^*(1/n^3)\chi_{(1/n,n)}(x)$$

and

$$\begin{aligned}
G^*(1/n^3) = \Big(\sum_i |Tg_i|\Big)^*(1/n^3) &\le n^3 \int_0^{1/n^3} \Big(\sum_i |Tg_i|\Big)^* \\
&\le \sum_i n^3 \int_0^{1/n^3} (Tg_i)^* \le \sum_i n^3 \int_0^1 (Tg_i)^* \\
&\le n^3 \sum_i \|Tg_i\|_{L^1+L^\infty} \le n^3\varepsilon \sum_i \|g_i\|_1 \le 2n^3\varepsilon\|f\|_1.
\end{aligned}$$

Using these estimates for R_n and Q_n we have

$$\|(Tf)^*\chi_{(1/n,n)}\|_{X^*} \le 2n^2\varepsilon\|f\|_1\|\chi_{(1/n,n)}\|_{X^*} + \|T\chi_E\|_X.$$

We let first $\varepsilon \to 0$ and $n \to \infty$ to get

$$\|Tf\|_X \le \|T\chi_E\|_X.$$

Since $|E| = \sum_i |\widetilde{Q}_i| = \sum_i \int_{Q_i} f = \|f\|_1$, the result follows. ∎

Also, as a consequence of the previous remark we obtain the following:

PROPOSITION 2.1. *Let X be a quasi-Banach r.i. space and let T be a non-zero (ε,δ)-atomic approximable operator. Then $\varphi_{X,T}$ is quasi-concave.*

Proof. Clearly $\varphi_{X,T}(r)$ is non-decreasing. Suppose $s > r$. If $|E| \le s$ then $\|(r/s)\chi_E\|_\infty \le 1$ so

$$(1/s)\|T\chi_E\|_X \le (1/r)\varphi_{X,T}(\|(r/s)\chi_E\|_1) = (1/r)\varphi_{X,T}(r).$$

Since this holds for all such E, $(1/r)\varphi_{X,T}(r)$ is non-increasing. ∎

Every quasi-concave function is equivalent to a concave function so we shall assume from now on that D is a concave function with $\varphi_{X,T} \preceq D$.

REMARK 2.1. From the above proposition, we see that if X is any quasi-Banach r.i. space and $0 < p < 1$, then no operator T mapping L^p to X is (ε,δ)-atomic approximable. In particular, convolution operators on L^p with discrete measures with coefficients in ℓ^p are not (ε,δ)-atomic approximable.

DEFINITION 2.5. Given a sequence space $\mathcal{S} \subseteq \ell^1$ and a concave function D, we shall denote by $D(\mathcal{S})$ the set of all measurable functions f such that

$$\|f\|_{D(\mathcal{S})} = \inf\{\|(c_n D(\|f_n\|_1))_n\|_{\mathcal{S}}\}$$

is finite. Here the infimum extends over the set of all possible decompositions $f = \sum_n c_n f_n$ a.e. such that $\|f_n\|_\infty \leq 1$.

It is an exercise to prove the following.

THEOREM 2.2. *If D is concave then $L^1 \cap L^\infty \subseteq D(\mathcal{S})$. If, in addition, $s \preceq D(s)$, then $D(\mathcal{S}) \subseteq L^1$.*

Let us now give some concrete examples which will be useful in what follows:

EXAMPLES. (a) If $\mathcal{S} = \ell^p$ with $0 < p \leq 1$, then taking the decomposition

$$f = \sum_{n\in\mathbb{Z}} 2^n f_n,$$

where $f_n = 2^{-n} f \chi_{\{2^{n-1} \leq |f| < 2^n\}}$, we have

$$\|f\|_{D(\mathcal{S})} \preceq \Big(\sum_{n\in\mathbb{Z}} 2^{np} D^p(\lambda_f(2^n))\Big)^{1/p} \preceq \Big(\int_0^\infty y^{p-1} D^p(\lambda_f(y))\, dy\Big)^{1/p}$$

$$\sim \Big(\int_0^\infty f^*(t)^p \, dD^p(t)\Big)^{1/p} = \|f\|_{\Lambda^p(w)},$$

where λ_f is the distribution function of f and $\Lambda^p(w)$ is the weighted Lorentz space with weight $w(t) = dD^p(t)$, and hence we have proved that

$$\Lambda^p(dD^p) \subseteq D(\ell^p).$$

Therefore, using the previous theorem, we obtain

$$\Lambda^p(dD^p) + L^1 \cap L^\infty \subseteq D(\ell^p).$$

At this point, and since $0 < p \leq 1$, it will be good to know when this second space $\Lambda^p(dD^p) + L^1 \cap L^\infty$ is strictly larger than $\Lambda^p(dD^p)$. Obviously, these two spaces coincide if and only if $L^1 \cap L^\infty \subseteq \Lambda^p(dD^p)$. It follows from Theorem 3.3 in [17] that

PROPOSITION 2.2. $L^1 \cap L^\infty \subseteq \Lambda^p(dD^p)$ *if and only if*

$$\int_0^\infty \Big(\frac{\max(1,y)}{D^p(y)}\Big)^{p/(p-1)} dD^p(y) < \infty.$$

(b) If $\mathcal{S} = \ell \log \ell$, and $s \preceq D(s)$, then, taking the decomposition

$$f = \underline{f} + \sum_{n\geq 1} 2^n f_n,$$

where $\underline{f} = f\chi_{\{|f|\leq 1\}}$ and f_n are as before, we get

$$\|f\|_{D(\mathcal{S})} \preceq D(\|\underline{f}\|_1) + \Big(\int_1^\infty (\log^+ \log^+ y) D(\lambda_f(y))\, dy\Big).$$

From this, it follows using homogeneity that

$$L \log\log L(D) \subseteq D(\mathcal{S}),$$

where

$$\|f\|_{L\log\log L(D)} = \int_0^\infty f^*(t)(1 + \log^+ \log^+ f^*(t))\, dD(t).$$

In particular, if $D(s) = s(1 + \log^+(1/s))$, then

$$L \log\log L(D) = L \log L \log\log L.$$

Now, in this concrete case, it was proved in [8], applying the ideas of [1], that we can improve the above result by taking the decomposition

$$f = f_0 + \sum_{n\geq 1} 2^{2^n} f_n,$$

where $f_0 = f\chi_{\{|f|\leq 2\}}$ and $f_n = 2^{-2^n} f\chi_{\{2^{2^{n-1}} \leq |f| < 2^{2^n}\}}$. Using this decomposition, it can be proved that

$$L \log L \log\log\log L \subseteq D(\mathcal{S}),$$

and, in fact, it was proved in [8] that if $D(s) \geq s$ and $D(s^2) \leq sD(s)$, then

$$L \log\log\log L(D) \subseteq D(\mathcal{S}).$$

For our next purpose, we need the following concept which was introduced in [21].

DEFINITION 2.6. The *galb* of a quasi-Banach space X is defined by

$$\mathrm{Galb}(X) = \Big\{(c_n)_n;\ \sum_n c_n f_n \in X \text{ whenever } \|f_n\|_X \leq 1\Big\},$$

endowed with the "norm" $\|c\|_{\mathrm{Galb}(X)} = \sup_{\|f_n\|_X\leq 1} \|\sum_n c_n f_n\|_X$.

Now, since the motivation of our work is to obtain a certain type of estimates for an operator T for which a restricted estimate is known, on many occasions it will be enough to have a weak type estimate for the operator T or even to know that $Tf(x) < \infty$ for a.e. x, for every $f \in X$, in order to apply some Banach continuity principle. To this end, it will be enough to identify certain sets containing $\mathrm{Galb}(X)$.

DEFINITION 2.7. The *weak galb* of a quasi-Banach space X is defined by

$$\mathrm{WGalb}(X) = \Big\{(c_n)_n;\ \sum_n c_n f_n \in M_X \text{ whenever } \|f_n\|_X \leq 1\Big\},$$

endowed with the norm $\|c\|_{\mathrm{WGalb}}(X) = \sup_{\|f_n\|_X \le 1} \|\sum_n c_n f_n\|_{M_X}$, where M_X is the maximal Marcinkiewicz space defined by

$$M_X = \{f;\ \|f\|_{M_X} = \sup_{t>0} f^*(t)\varphi_X(t) < \infty\}.$$

The *finite galb* of X is defined by

$$\mathrm{FGalb}(X) = \Big\{(c_n)_n;\ \sum_n c_n f_n \text{ is finite a.e. whenever } \|f_n\|_X \le 1\Big\}.$$

It is trivial that

$$\mathrm{Galb}(M_X) = \mathrm{WGalb}(M_X)$$

and

$$\mathrm{Galb}(X) \subseteq \mathrm{WGalb}(X) \subseteq \mathrm{FGalb}(X).$$

We shall see in Proposition 4.1 that the three concepts are different. Note that the advantage of the finite galb is that if two quasi-Banach spaces X and Y are such that $X \subseteq Y$ continuously, then

$$\mathrm{FGalb}(Y) \subseteq \mathrm{FGalb}(X).$$

A first general and important fact is the following:

Theorem 2.3. *Let X be a quasi-Banach r.i. space. Then*,

$$\mathrm{Galb}(X) \subseteq \mathrm{WGalb}(X) \subseteq \mathrm{FGalb}(X) \subseteq \ell^1 \cap L_{\varphi_X^{-1}}, \tag{2}$$

where

$$L_{\varphi_X^{-1}} = \Big\{(c_n)_n;\ \sum_n \varphi_X^{-1}(|c_n|) < \infty\Big\}.$$

Proof. The embedding in ℓ^1 is immediate. To show that $\mathrm{FGalb}(X) \subseteq L_{\varphi_X^{-1}}$ we suppose that $\sum_n \varphi_X^{-1}(|c_n|)$ diverges. It is a standard argument to select sets A_n of measure $\varphi_X^{-1}(|c_n|)$ such that $\sum_n \chi_{A_n} = \infty$ on a set of positive measure. Set $f_n = (1/c_n)\chi_{A_n}$; then $\|f_n\|_X = 1$ and so $(c_n)_n \notin \mathrm{FGalb}(X)$. ■

Remark 2.2. Obviously $\mathrm{Galb}(X) = \ell^1$ if and only if X is a Banach space. If this is not the case, we shall study conditions on our spaces to have the equality $\mathrm{Galb}(X) = L_{\varphi_X^{-1}} \cap \ell^1$.

Our second main result can now be formulated in the following way:

Theorem 2.4. *Let T be a sublinear (ε, δ)-atomic approximable operator and let X be a quasi-Banach r.i. space. Define $\varphi_{X,T}(\mathrm{Galb}(X))$ as in Definition* 2.5. *Then*:

(a) *$T : \varphi_{X,T}(\mathrm{Galb}(X)) \to X$ is bounded.*
(b) *$T : \varphi_{X,T}(\mathrm{WGalb}(X)) \to M_X$ is bounded.*
(c) *For every $f \in \varphi_{X,T}(\mathrm{FGalb}(X))$, $Tf(x) < \infty$ almost everywhere.*

Proof. We shall only prove (a), since the proofs of (b) and (c) are completely similar.

If $f = \sum_n c_n f_n$ then by sublinearity

$$\|Tf\|_X \le \|c_n T f_n\|_{\mathrm{Galb}(X)}.$$

If we suppose that $\|f_n\|_\infty \le 1$ for each n then by Theorem 2.1,

$$\|Tf_n\|_X \le \varphi_{X,T}(\|f_n\|_1),$$

and $\|Tf\|_X \le \|f\|_{\varphi_{X,T}(\mathrm{Galb}(X))}$ follows by taking the infimum over all such representations of f. ■

In particular, if T is a sublinear (ε,δ)-atomic approximable operator, the following corollaries follow from the examples given above.

COROLLARY 2.1. *If X is a Banach space, then $T : \Lambda^1(d\varphi_{X,T}) \to X$ is bounded.*

COROLLARY 2.2. *If* $\mathrm{Galb}(X) = \ell^p$ *with* $0 < p < 1$, *then*

$$T : \Lambda^p(d\varphi^p_{X,T}) + L^1 \cap L^\infty \to X$$

is bounded.

COROLLARY 2.3. *If* $\mathrm{Galb}(X) \subseteq \ell(\log \ell)^\alpha$ *and* $s \preceq \varphi_{X,T}(s)$, *then*

$$T : L(\log\log L)^\alpha(d\varphi_{X,T}) \to X$$

is bounded. If, in addition, $\varphi_{X,T}(s^2) \preceq s\varphi_{X,T}(s)$, *then*

$$T : L(\log\log\log L)^\alpha(d\varphi_{X,T}) \to X$$

is bounded.

Our next step will be to study the galb for the class of weighted Lorentz spaces.

3. The galb of weighted Lorentz spaces. The purpose of this section is to obtain information about the galb of the spaces $\Lambda^q(w)$ for $0 < q < \infty$ and of the weak type spaces $\Lambda^{q,\infty}(w)$. Hence, throughout this section,

$$f = \sum_{n=1}^{\infty} c_n f_n,$$

where $\|f_n\|_X \le 1$ and $X = \Lambda^q(w)$ or $X = \Lambda^{q,\infty}(w)$. We shall use the following formula for the decreasing rearrangement of a sum of functions (see [9]):

$$f^*(3t) \le \sum_n c_n \bigg(f_n^*(t) + \frac{1}{t} \int_{a_n t}^{t} f_n^*(s)\,ds \bigg), \tag{3}$$

where $\{a_n\}_n$ are any positive numbers such that $\sum_n a_n = 1$. It is easy to see that (3) remains valid when the condition $\sum_n a_n = 1$ is weakened to

$\sum_n a_n \le 1$. Recall that if $\Lambda^q(w)$ is quasi-Banach, then the primitive of the weight $W(t) = \int_0^t w$ satisfies the Δ_2 condition, and hence the number 3 on the left hand side of the above formula gives no problem at all.

We shall also need some estimates for the Steklov operator on decreasing functions. This operator is defined, for $0 < a < 1$, by

$$S_a f(t) = \frac{1}{t} \int_{at}^{t} f(s)\, ds.$$

Lemma 3.1.

$$\sup_{f\downarrow} \frac{(t^{-1}\int_{at}^t f(s)\, ds) W(t)}{\sup_{t>0} f(t)W(t)} = \sup_{t>0} \left(\frac{1}{t} \int_{at}^{t} \frac{1}{W(s)}\, ds \right) W(t).$$

Proof. The proof follows trivially since the largest function f with the property that $\sup_{t>0} f(t)W(t) = 1$ is $1/W$. ■

The meaning of the following two lemmas is that in estimating the norm of the Steklov operator on Lorentz spaces it is often sufficient to test it only on characteristic functions.

Lemma 3.2.

$$\sup_{f\downarrow} \frac{\int_0^\infty (t^{-1}\int_{at}^t f(s)\, ds) w(t)\, dt}{\int_0^\infty f(t)w(t)\, dt} = \sup_{r>0} \frac{1}{W(r)} \int_0^r \left(\int_s^{s/a} \frac{w(t)}{t}\, dt \right) ds.$$

Proof. This follows using Fubini and Theorem 2.12 of [10]. ■

Lemma 3.3. *If* $q > 1$, *then*

$$A := \sup_{f\downarrow} \frac{(\int_0^\infty (t^{-1}\int_{at}^t f(s)\, ds)^q w(t)\, dt)^{1/q}}{(\int_0^\infty f(t)^q w(t)\, dt)^{1/q}} < \infty$$

if and only if

$$B := \sup_{r} \left(\frac{1}{W(r)} \int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\, dt \right)^{1/q} < \infty. \tag{4}$$

Moreover,

(a) $B \le A \preceq 1 + B^q$,
(b) *if for some* $D > 1$, $W(s/a) \le DW(s)$ *for all* $s > 0$, *then*

$$(1-a) + B \preceq A \preceq (1-a) + B(\log D)^{1/q'}.$$

From this, *we can also conclude that*

(c) $(1-a) + B \preceq A \preceq (1-a) + B(\log(B/(\sqrt{a} - a)))^{1/q'}$.

The proof of this lemma will be postponed to the last section, since it is somewhat technical.

3.1. Galb($\Lambda^{q,\infty}(w)$). Let us start with the case of $\Lambda^{q,\infty}(w)$ defined by

$$\|f\|_{\Lambda^{q,\infty}(w)} = \sup_{t>0} f^*(t)W(t)^{1/q},$$

and observe that $\Lambda^{q,\infty}(w) = \Lambda^{1,\infty}(w_q)$, where $w_q(t) = W(t)^{1/q-1}w(t)$, and hence, the parameter q is somehow superfluous. However, it will be important for us that, for every $q > 1$,

$$\Lambda^{q,1}(w) \subseteq \Lambda^q(w) \subseteq \Lambda^{q,\infty}(w),$$

where $\Lambda^{q,1}(w) = \Lambda^1(w_q)$ with w_q as before. Moreover, by real interpolation,

$$\Lambda^q(w) = (\Lambda^{q,1}(w), \Lambda^{q,\infty}(w))_{1/q',q}.$$

As a first consequence of (2), we obtain the following result:

COROLLARY 3.1. *For every* $0 < q < \infty$,

$$\mathrm{Galb}(\Lambda^{q,\infty}(w)) \subseteq L_{(W^{1/q})^{-1}} \cap \ell^1.$$

THEOREM 3.1. *Let* $0 < q < \infty$ *and given* $0 < a < 1$, *let*

$$H(a) = \sup_{t>0} \bigg(\frac{1}{t}\int_{at}^{t} W(s)^{-1/q}\,ds\bigg) W(t)^{1/q}.$$

If $(c_n)_n \in \ell^1$ *and*

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n) < \infty,$$

then $(c_n)_n \in \mathrm{Galb}(\Lambda^{q,\infty}(w))$.

Proof. Using (3), we obtain

$$\|f\|_{\Lambda^{q,\infty}(w)} \preceq \sum_n c_n \|f_n\|_{\Lambda^{q,\infty}(w)} + \sum_n c_n \sup_{t>0} \bigg(\frac{1}{t}\int_{a_n t}^{t} f_n^*(s)\,ds\bigg) W(t)^{1/q}$$

for every positive sequence $(a_n)_n$ with $\sum a_n \le 1$, and so, by Lemma 3.1,

$$\|f\|_{\Lambda^{q,\infty}(w)} \preceq \sum_n c_n + \sum_n c_n H(a_n),$$

from which the result follows. ∎

EXAMPLE. If $w(t) = 1$, then $W(t) = t$ and $H(a) = q(1-a^{(q-1)/q})/(q-1)$. In particular, $H(a) \approx a^{(q-1)/q}$ if $q < 1$, $H(a) = \log(1/a)$ if $q = 1$, and $H(a) \approx 1$ if $q > 1$.

REMARK 3.1. If $H \in L^\infty$, we obtain $\mathrm{Galb}(\Lambda^{q,\infty}(w)) = \ell^1$; of course, this also follows from the fact that $H \in L^\infty$ if and only if $w \in B_q$, in which case $\Lambda^{q,\infty}(w)$ is a Banach space (see [19]).

Corollary 3.2. *If for every $t > 0$ and every $0 < a < 1$,*

$$\frac{1}{t}\int_{at}^{t} W(s)^{-1/q}\,ds \preceq \frac{a}{W(a)^{1/q}W(t)^{1/q}}, \tag{5}$$

then

$$\mathrm{Galb}(\Lambda^{q,\infty}(w)) = \mathrm{WGalb}(\Lambda^{q,\infty}(w)) = \mathrm{FGalb}(\Lambda^{q,\infty}(w)) = L_{(W^{1/q})^{-1}} \cap \ell^1.$$

Proof. The embedding $\mathrm{FGalb}(\Lambda^{q,\infty}(w)) \subseteq L_{(W^{1/q})^{-1}} \cap \ell^1$ follows from Theorem 2.3 and the opposite embedding follows from Theorem 3.1. Indeed, condition (5) reads $H(a) \preceq a/W(a)$, and if $(c_n)_n \in L_{(W^{1/q})^{-1}} \cap \ell^1$, we know that (c_n) tends to zero and hence we can assume that $(W^{1/q})^{-1}(c_n) \le 1$ for every n. Therefore,

$$\sum_n c_n H((W^{1/q})^{-1}(c_n)) \preceq \sum_n c_n \frac{(W^{1/q})^{-1}(c_n)}{c_n} = \sum_n (W^{1/q})^{-1}(c_n) < \infty,$$

and therefore $(c_n)_n \in \mathrm{Galb}(\Lambda^{q,\infty}(w))$ by Theorem 3.1. ∎

Corollary 3.3. *If $W(s)^{1/q}/s$ is equivalent to a decreasing function, then*

$$\ell \log \ell \subseteq \mathrm{Galb}(\Lambda^{q,\infty}(w)).$$

Proof. Applying the trivial fact that $H(a) \le \log(1/a)$ and taking $a_n = c_n$, we obtain the result. ∎

Observe that if $q = 1$ and $w = 1$, we obtain the well-known fact that $\ell \log \ell \subseteq \mathrm{Galb}(L^{1,\infty})$.

3.2. $\mathrm{Galb}(\Lambda^q(w))$ *for* $0 < q \le 1$

Theorem 3.2. *For every $0 < q \le 1$,*

$$\mathrm{Galb}(\Lambda^q(w)) \subseteq \ell^q.$$

Proof. Let $\alpha_1 > 0$ be small enough (if necessary) and choose α_k such that

$$W\Big(\sum_{j=1}^{k-1} \alpha_j\Big) \le \frac{1}{2} W(\alpha_k).$$

Let $\{A_k\}_{k=1}^N$ be a collection of disjoint sets such that $\alpha_k = |A_k|$ and define $\beta_k = W(\alpha_k)$. Obviously β_k is an increasing sequence.

Let $f_k = \beta_k^{-1/q}\chi_{A_k}$, so that $\|f_k\|_{\Lambda^q(w)} = 1$, and set

$$f = \sum_{k=1}^{N} c_k f_k.$$

Assume, without loss of generality, that $(c_k)_k$ is decreasing, and hence also $\beta_k^{-1/q} c_k$ is decreasing. Let $\gamma_0 = 0$ and $\gamma_k = \sum_{j=1}^k \alpha_j$. Then

$$f^*(t) = \beta_k^{-1/q} c_k$$

if $\gamma_{k-1} < t < \gamma_k$, and therefore

$$\begin{aligned}\int_0^\infty f^*(t)^q w(t)\,dt &= \sum_{k=1}^N c_k^q \beta_k^{-1} \int_{\gamma_{k-1}}^{\gamma_k} w(t)\,dt \\ &= \sum_{k=1}^N \frac{c_k^q}{W(\alpha_k)} \int_{\gamma_{k-1}}^{\gamma_k} w(t)\,dt > \frac{1}{2}\sum_{k=1}^N c_k^q,\end{aligned}$$

from which the result follows. ∎

THEOREM 3.3. *Let $0 < q \le 1$. Then* $\mathrm{Galb}(\Lambda^q(w)) = \ell^q$ *if and only if $W(t)/t$ is equivalent to a decreasing function.*

Proof. If $W(t)/t$ is equivalent to a decreasing function, then it is known (see [6]) that $\Lambda^1(w)$ is a Banach space, and since

$$|f|^q \le \sum_n c_n^q |f_n|^q \quad \text{and} \quad \|f\|^q_{\Lambda^q(w)} = \|f^q\|_{\Lambda^1(w)},$$

we obtain

$$\|f\|^q_{\Lambda^q(w)} \le \sum_n c_n^q \|f_n^q\|_{\Lambda^1(w)} \le \sum_n c_n^q;$$

therefore $\ell^q \subseteq \mathrm{Galb}(\Lambda^q(w))$ and hence they coincide. To prove the converse, we observe first that if $\mathrm{Galb}(\Lambda^q(w)) = \ell^q$, then

$$\|f\|^q_{\Lambda^q(w)} \le \inf \sum_n \|f_n\|^q_{\Lambda^q(w)},$$

where the infimum extends over all possible decompositions $f = \sum_n f_n$.

Now, we use the same argument as in [6]: let $k \in \mathbb{N}$ and $s > 0$ and set $f = \chi_{(0,2^k s)}$ and $f_j = \chi_{(js,(j+1)s)}$ with $j = 0, \dots, 2^k - 1$. Then, since $f = \sum_{j=0}^{2^k-1} f_j$, we obtain

$$W(2^k s) = \|f\|^q_{\Lambda^q(w)} \preceq \sum_{j=0}^{2^k-1} \|f_j\|^q_{\Lambda^q(w)} = 2^k W(s);$$

that is, $W(2^k s) \le 2^k W(s)$ and hence, if $s < r$ and k is such that $2^{k-1}s < r < 2^k s$, then

$$\frac{W(r)}{r} \le \frac{W(2^k s)}{2^{k-1}s} \preceq \frac{2^k W(s)}{2^{k-1}s} \preceq \frac{W(s)}{s},$$

as we wanted to prove. ∎

REMARK 3.2. In particular, if $X = L^{p,q}$ with $0 < q < \min(p,1)$, we recover the result proved in [13].

In general, if w does not satisfy the previous condition we have the following result:

THEOREM 3.4. *Given* $0 < a < 1$, *let*

$$H(a) = \sup_{r>0} \frac{1}{W(r)} \int_0^r \bigg(\int_s^{s/a} \frac{w(t)}{t}\, dt \bigg)\, ds.$$

If $(c_n)_n \in \ell^q$ *and*

$$\inf_{\sum_n a_n \le 1} \sum_n c_n^q H(a_n) < \infty,$$

then $(c_n)_n \in \mathrm{Galb}(\Lambda^q(w))$.

Proof. Since $0 < q \le 1$, we have $|f|^q \le \sum_n c_n^q |f_n|^q$, and hence, using (3), we obtain

$$\|f\|^q_{\Lambda^q(w)} \preceq \sum_n c_n^q \|f_n\|^q_{\Lambda^q(w)} + \sum_n c_n^q \int_0^\infty \bigg(\frac{1}{t} \int_{a_n t}^t f_n^*(s)^q\, ds \bigg) w(t)\, dt$$

for every positive sequence $(a_n)_n$ with $\sum a_n \le 1$; so, by Lemma 3.2,

$$\|f\|_{\Lambda^q(w)} \preceq \sum_n c_n^q + \sum_n c_n^q H(a_n),$$

from which the result follows. ■

As a corollary of (2) we obtain:

COROLLARY 3.4.

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) \subseteq L_{(W^{1/q})^{-1}} \cap \ell^q.$$

COROLLARY 3.5. *If for every* $r > 0$ *and every* $0 < a < 1$,

$$\int_0^r \bigg(\int_s^{s/a} \frac{w(t)}{t}\, dt \bigg) ds \preceq \frac{aW(r)}{W(a)}, \tag{6}$$

then

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) = L_{(W^{1/q})^{-1}} \cap \ell^q.$$

Proof. Condition (6) reads $H(a) \preceq a/W(a)$, and hence the assertion follows as in Corollary 3.2. ■

Let us now assume that W is equivalent to a convex function.

LEMMA 3.4. *Let* $f_n \ge 0$, *and let* $g_n \ge 0$ *have disjoint supports and satisfy* $f_n^* = g_n^*$ *for every* n. *If* W *is equivalent to a convex function*, *then*

$$\Big\| \sum_n f_n \Big\|_{\Lambda^1(w)} \le \Big\| \sum_n g_n \Big\|_{\Lambda^1(w)}.$$

Proof. Let us start by proving that, under the above hypotheses,

$$\int_x^\infty \Big(\sum_n f_n\Big)^* \le \int_x^\infty \Big(\sum_n g_n\Big)^*$$

for every $x>0$. Since $\sum_n f_n$ and $\sum_n g_n$ have the same integral, it is enough to prove that

$$\int_0^x \Big(\sum_n g_n\Big)^* \le \int_0^x \Big(\sum_n f_n\Big)^*.$$

We have

$$\begin{aligned}
\int_0^x \Big(\sum_n g_n\Big)^* &= \sup\Big\{\int_E \sum_n g_n;\ |E|\le x\Big\} = \sup\Big\{\sum_n \int_{E_n} g_n;\ \sum_n |E_n|\le x\Big\}\\
&= \sup\Big\{\sum_n \int_0^{x_n} g_n^*;\ \sum_n x_n\le x\Big\} = \sup\Big\{\sum_n \int_0^{x_n} f_n^*;\ \sum_n x_n\le x\Big\}\\
&= \sup\Big\{\sum_n \int_{E_n} f_n;\ \sum_n |E_n|\le x\Big\}\\
&\le \sup\Big\{\int_E \sum_n f_n;\ |E|\le x\Big\} = \int_0^x \Big(\sum_n f_n\Big)^*.
\end{aligned}$$

Finally, since W is equivalent to a convex function, we can assume without loss of generality that w is an increasing function; hence, by the distribution formula for increasing weights, there exists a function $c_w(y)$ such that

$$\begin{aligned}
\Big\|\sum_n f_n\Big\|_{\Lambda^1(w)} &= \int_0^\infty \int_{c_w(y)}^\infty \Big(\sum_n f_n\Big)^*(t)\,dt\,dy\\
&\le \int_0^\infty \int_{c_w(y)}^\infty \Big(\sum_n g_n\Big)^*(t)\,dt\,dy = \Big\|\sum_n g_n\Big\|_{\Lambda^1(w)}. \ \blacksquare
\end{aligned}$$

Consequently, when computing $\mathrm{Galb}(\Lambda^1(w))$ for an increasing weight, we can assume that the functions f_n are disjointly supported. Also:

Theorem 3.5. *If W is a convex function, then, for every $0<q\le 1$,*

$$\mathrm{Galb}(\Lambda^q(w)) = \{(c_n)_n;\ (c_n^q)_n \in \mathrm{Galb}(\Lambda^1(w))\}.$$

Proof. Since $(\sum_n c_n f_n)^q \le \sum_n c_n^q f_n^q$, it is clear that

$$\{(c_n)_n;\ (c_n^q)_n \in \mathrm{Galb}(\Lambda^1(w))\} \subseteq \mathrm{Galb}(\Lambda^q(w)).$$

For the converse inclusion we observe that if $(c_n)_n \in \mathrm{Galb}(\Lambda^q(w))$, then $\sum_n c_n f_n \in \Lambda^q(w)$ for every $(f_n)_n$ disjointly supported with $\|f_n\|_{\Lambda^q(w)} \le 1$.

Since, in this case,

$$\Big(\sum_n c_n f_n\Big)^q = \sum_n c_n^q f_n^q,$$

we deduce that $\sum_n c_n^q g_n \in \Lambda^1(w)$ for every $(g_n)_n$ disjointly supported with $\|g_n\|_{\Lambda^1(w)} \le 1$. Since w is increasing, we conclude that $(c_n^q)_n \in \mathrm{Galb}(\Lambda^1(w))$. ∎

THEOREM 3.6. *Suppose that W is a convex function, and, for $0 < a < 1$, set*

$$H(a) = \sup_{at \le r \le t} \frac{W(t)r}{tW(r)}.$$

If $(c_n)_n \in \ell^q$ and

$$\inf_{\sum_n a_n \le 1} \sum_n c_n^q H(a_n) < \infty,$$

then $(c_n)_n \in \mathrm{Galb}(\Lambda^q(w))$.

Proof. By Theorem 3.5, it is enough to handle the case $q = 1$, and since W is convex, we can assume that the f_n are disjointly supported. Hence, for every positive sequence $(a_n)_n$ such that $\sum_n a_n \le 1$,

$$\begin{aligned}
\Big\|\sum_n c_n f_n\Big\|_{\Lambda^1(w)} &= \int_0^\infty W\Big(\sum_n \lambda_{f_n}(y/c_n)\Big)\,dy = \int_0^\infty W\bigg(\sum_n a_n \frac{\lambda_{f_n}(y/c_n)}{a_n}\bigg)\,dy \\
&\le \int_0^\infty \sum_n a_n W\bigg(\frac{\lambda_{f_n}(y/c_n)}{a_n}\bigg)\,dy \le \int_0^\infty \sum_n H(a_n) W(\lambda_{f_n}(y/c_n))\,dy \\
&\le \sum_n c_n H(a_n) \int_0^\infty W(\lambda_{f_n}(y))\,dy \le \sum_n c_n H(a_n),
\end{aligned}$$

and taking the infimum over all $(a_n)_n$ we obtain the result. ∎

3.3. $\mathrm{Galb}(\Lambda^q(w))$ *for* $q > 1$

THEOREM 3.7. *For every $q > 1$, $\mathrm{Galb}(\Lambda^q(w)) = \ell^1$ if and only if $w \in B_q$, that is, for every $r > 0$,*

$$r^q \int_r^\infty \frac{w(t)}{t^q}\,dt \preceq \int_0^r w(t)\,dt.$$

Proof. This is a consequence of the fact that for $q > 1$, $\Lambda^q(w)$ is Banach if and only if $w \in B_q$ (see [15]). ∎

THEOREM 3.8. *Given $0 < a < 1$, let*

$$H(a) = \sup_r \frac{1}{W(r)} \int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\,dt.$$

If $(c_n)_n \in \ell^1$ *and*

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n) < \infty,$$

then $(c_n)_n \in \mathrm{Galb}(\Lambda^q(w))$.

Proof. Using (3), we obtain

$$\begin{aligned} \|f\|_{\Lambda^q(w)} &\preceq \sum_n c_n \|f_n\|_{\Lambda^q(w)} \\ &\quad + \sum_n c_n \sup_{\|f\|_{\Lambda^q(w)}=1} \bigg(\int_0^\infty \bigg(\frac{1}{t}\int_{a_n t}^t f^*(s)\,ds\bigg)^q w(t)\,dt\bigg)^{1/q} \end{aligned}$$

for every positive sequence $(a_n)_n$ with $\sum a_n \le 1$, so, by Lemma 3.3,

$$\|f\|_{\Lambda^q(w)} \preceq \sum_n c_n + \sum_n c_n H(a_n),$$

from which the result follows. ∎

Remark 3.3. In particular, if we take $\Lambda^q(w) = L^{1,q}$, which means that $w(t) = t^{q-1}$, then

$$H(a) \sim \sup_r \frac{1}{r^q} \int_r^{r/a} (r - at)^q \frac{dt}{t} \sim \log \frac{1}{a},$$

and hence $\ell \log \ell \subseteq \mathrm{Galb}(L^{1,q})$. This estimate is not satisfactory since it is known that $\mathrm{Galb}(L^{1,q}) = \ell(\log \ell)^{1/q'}$ (see [18]).

However, if we use interpolation theory, we can improve the above result as follows.

Theorem 3.9. *If* $W(t)^{1/q}/t$ *is equivalent to a decreasing function*, *then*

$$\ell(\log \ell)^{1/q'} \subseteq \mathrm{Galb}(\Lambda^q(w)).$$

Proof. If $W(t)^{1/q}/t$ is equivalent to a decreasing function, then $\Lambda^1(w_q)$ is a Banach space with $w_q(t) = W(t)^{1/q-1} w(t)$, and consequently $\mathrm{Galb}(\Lambda^1(w_q)) = \ell^1$. On the other hand, by Corollary 3.3, $\ell \log \ell \subseteq \mathrm{Galb}(\Lambda^{1,\infty}(w_q))$, and hence, using interpolation (see [5] and [11]), we obtain the result. ∎

As a corollary of Theorem 3.8, we also obtain the following result.

Corollary 3.6. *If*, *for every* $0 < a < 1$ *and every* $r > 0$,

$$\int_r^{r/a} (r - at)^q \, \frac{w(t)}{t^q}\, dt \le \frac{W(r)a}{W(a)^{1/q}},$$

then

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) = L_{(W^{1/q})^{-1}} \cap \ell^1.$$

3.4. *Weak galb and finite galb.* The purpose of this subsection is to obtain information about the weak and finite galbs of the spaces $\Lambda^q(w)$.

THEOREM 3.10. *Given* $0 < a < 1$, *let*

$$H(a) = \begin{cases} \sup_t \frac{W(t)^{1/q}}{t} \left(\int_{at}^{t} \left(\frac{u - at}{W(u)} \right)^{q'-1} du \right)^{1/q'} & \text{if } q > 1, \\ \sup_{t,r} \frac{W(t)^{1/q}}{t} \frac{(\min(r,t) - at)_+}{W(r)^{1/q}} & \text{if } q \le 1. \end{cases}$$

If $(c_n)_n \in \ell^1$ *and*

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n) < \infty,$$

then $(c_n)_n \in \mathrm{WGalb}(\Lambda^q(w))$.

Proof. Using (3), we obtain

$$\|f\|_{\Lambda^{q,\infty}(w)} \preceq \sum_n c_n \|f_n\|_{\Lambda^q(w)} + \sum_n c_n \sup_t \frac{W(t)^{1/q}}{t} \sup_{\|f\|_{\Lambda^q(w)}=1} \frac{1}{t} \int_{a_n t}^{t} f^*(s)\, ds$$

for every positive sequence $(a_n)_n$ with $\sum a_n \le 1$, and the result follows, in the case $q > 1$, from Sawyer's duality formula (see [15]) and, if $q \le 1$, from Theorem 2.12 of [10]. ∎

Using a completely similar argument to that of the previous theorem, we can prove the following result.

THEOREM 3.11. *Given* $0 < a < 1$, *let*

$$H(a;t) = \begin{cases} \left(\int_{at}^{t} \left(\frac{u - at}{W(u)} \right)^{q'-1} du \right)^{1/q'} & \text{if } q > 1, \\ \sup_r \frac{(\min(r,t) - at)_+}{W(r)^{1/q}} & \text{if } q \le 1. \end{cases}$$

If $(c_n)_n \in \ell^1$ *and, for every* $t > 0$,

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n; t) < \infty,$$

then $(c_n)_n \in \mathrm{FGalb}(\Lambda^q(w))$.

Sometimes, we can use the embedding properties of the weak and finite galbs in order to obtain some information about the galb, as in the following corollary:

COROLLARY 3.7. *Let $q \geq 1$. If $W(s)/s^q$ is equivalent to a bounded, decreasing function, then*

(a) $\quad \ell \log \ell = \mathrm{Galb}(\Lambda^{q,\infty}(w)) = \mathrm{WGalb}(\Lambda^{q,\infty}(w)) = \mathrm{FGalb}(\Lambda^{q,\infty}(w)),$

(b) $\quad \ell(\log \ell)^{1/q'} = \mathrm{Galb}(\Lambda^{q}(w)) = \mathrm{WGalb}(\Lambda^{q}(w)) = \mathrm{FGalb}(\Lambda^{q}(w)).$

Proof. (a) By Corollary 3.3,

$$\ell \log \ell \subseteq \mathrm{Galb}(\Lambda^{q,\infty}(w)) \subseteq \mathrm{WGalb}(\Lambda^{q,\infty}(w)) \subseteq \mathrm{FGalb}(\Lambda^{q,\infty}(w)).$$

Now, since $W(s) \preceq s^q$, we have $L^{1,\infty} \subseteq \Lambda^{q,\infty}(w)$, and therefore

$$\mathrm{FGalb}(\Lambda^{q,\infty}(w)) \subseteq \mathrm{FGalb}(L^{1,\infty}).$$

Since it is known that $\mathrm{FGalb}(L^{1,\infty}) = \ell \log \ell$ (see [13]), we obtain the result.

(b) The proof of this part is completely similar since, by Theorem 3.9,

$$\ell(\log \ell)^{1/q'} \subseteq \mathrm{Galb}(\Lambda^{q}(w));$$

next since $W(s) \preceq s^q$, we have $L^{1,q} \subseteq \Lambda^q(w)$; finally, we use the fact that $\mathrm{FGalb}(L^{1,q}) = \ell(\log \ell)^{1/q'}$ (see [18]). ∎

4. Some examples and applications. If we apply our result to the classical case of $L^{p,q}$ we obtain the following result:

COROLLARY 4.1.

1) *If $q > 1$ and $p > 1$, then* $\mathrm{Galb}(L^{p,q}) = \ell^1$.
2) *If $p = 1$, then* $\ell \log \ell = \mathrm{Galb}(L^{1,\infty})$.
3) *If $p < 1$, then* $\mathrm{Galb}(L^{p,\infty}) = \ell^p$.
4) *If $0 < q \leq 1$ and $q \leq p$, then* $\mathrm{Galb}(L^{p,q}) = \ell^q$.
5) *If $0 < p \leq 1$ and $p \leq q \leq \infty$, then* $\mathrm{Galb}(L^{p,q}) = \ell^p$.
6) *If $q > 1$, then* $\ell(\log \ell)^{1/q'} = \mathrm{Galb}(L^{1,q})$.

Proof. 1) is clear because the spaces are Banach spaces, and 2) has already been mentioned several times. 3) is consequence of (5); 4) of Theorem 3.3; and 5) of Theorems 3.5 and 3.6 for the case $q \leq 1$, while for $q > 1$, we have to proceed by interpolation since we already know that $\mathrm{Galb}(L^{p,\infty}) = \mathrm{Galb}(L^{p,1}) = \ell^p$. Finally, the embedding $\ell(\log \ell)^{1/q'} \subset \mathrm{Galb}(L^{1,q})$ in 6) is a consequence of Theorem 3.9, and for the converse we have to refer to [18]. ∎

Another example:

COROLLARY 4.2. *If $W(t) = t(1 + \log^+(1/t))^{-\alpha}$ with $\alpha > 0$, then, for every $0 < q \leq 1$,*

$$\ell^q(\log \ell)^\alpha = \mathrm{Galb}(\Lambda^q(w)).$$

Proof. This is a consequence of Theorem 3.6 and (2), since one can easily check that, in this case, $H(a) \preceq (1 + \log^+(1/a))^\alpha$, and $L_{\varphi_X^{-1}} = \ell^q(\log \ell)^\alpha$. ∎

If T is of restricted weak type (p,p) with constant $1/(p-1)$, as happens with the Carleson operator given in the introduction, then

$$(T\chi_E)^*(t) \le \frac{|E|}{t}\left(1+\log^+\frac{t}{|E|}\right) \le |E|\left(1+\log^+\frac{1}{|E|}\right)\frac{1}{t}(1+\log^+ t),$$

that is,

$$\|T\chi_E\|_X \le D(|E|),$$

where $X = \Lambda^{1,\infty}(w)$ with $W(t) = t/(1+\log^+ t)$ and $D(s) = s(1+\log^+(1/s))$. Also, when dealing with the bilinear Hilbert transform, the space that appears naturally is $X = \Lambda^{p,\infty}(w)$ with $p = 2/3$ and $W(t) = t/(1+\log^+ t)^{4/3}$ (see [7]). These examples motivate the study of the galb of the above spaces.

COROLLARY 4.3. *If $W(t) = t(1+\log^+ t)^{-\alpha}$ with $\alpha > 0$, then*

$$\mathrm{Galb}(\Lambda^{q,\infty}(w)) = \begin{cases} \ell^q & \text{for } 0<q<1, \\ \ell\log\ell & \text{for } q=1. \end{cases}$$

Proof. This is a consequence of (5). ∎

Finally, it is important to mention that, in general, $\mathrm{Galb}(X)$, $\mathrm{WGalb}(X)$ and $\mathrm{FGalb}(X)$ do not coincide, as is shown in the following proposition:

PROPOSITION 4.1. *If $0<q<p<1$, then*

1) $\mathrm{Galb}(L^{p,q}(\mathbb{R}) \cap L^1(\mathbb{R})) = \ell^q$,
2) $\mathrm{WGalb}(L^{p,q}(\mathbb{R}) \cap L^1(\mathbb{R}))) = \ell^p$,
3) $\mathrm{FGalb}(L^{p,q}(\mathbb{R}) \cap L^1(\mathbb{R}))) = \ell^1$.

Proof. 1) Recall that $\mathrm{Galb}(L^{p,q}) = \ell^q$, while $\mathrm{Galb}(L^1) = \ell^1$. In particular, if $(c_n)_n$ is in ℓ^q and $\|f_n\|_{L^{p,q}\cap L^1} \le 1$, then

$$\Big\|\sum_n c_n f_n\Big\|_{L^{p,q}} \le c\Big(\sum_n |c_n|^q\Big)^{1/q},$$

$$\Big\|\sum_n c_n f_n\Big\|_{L^1} \le c\sum_n |c_n| \le c\Big(\sum_n |c_n|^q\Big)^{1/q}.$$

Hence $\ell^q \subseteq \mathrm{Galb}(L^{p,q}\cap L^1)$. To prove the converse inclusion, given a positive sequence $(c_n)_n$, choose a strictly increasing sequence of integers $k_n \ge 1$ so that $2^{-k_n/p}|c_n|$ is decreasing. Then choose disjoint sets $\{A_n\}_n$ with $|A_n| = 2^{k_n}$. Finally, let $f_n = |A_n|^{-1/p}\chi_{A_n}$, so that

$$\begin{aligned}\|f_n\|_{L^{p,q}\cap L^1} &= \max\{|A_n|^{-1/p}\|\chi_{A_n}\|_{L^{p,q}}, |A_n|^{-1/p}\|\chi_{A_n}\|_{L^1}\} \\ &= \max\{1, |A_n|^{1-1/p}\} = 1.\end{aligned}$$

It is routine to calculate the rearrangement of the simple function $\sum_n c_n f_n$

and, using the fact that $|A_n|$ is rapidly increasing, to get

$$\begin{aligned}\Big\|\sum_n c_n f_n\Big\|_{L^{p,q}} \\ &= \Big(\sum_n (p/q)|c_n|^q |A_n|^{-q/p}\Big(\Big(\sum_{j=1}^{n}|A_j|\Big)^{q/p} - \Big(\sum_{j=1}^{n-1}|A_j|\Big)^{q/p}\Big)\Big)^{1/q} \\ &\succeq \Big(\sum |c_n|^q\Big)^{1/q}.\end{aligned}$$

We conclude that if $(c_n)_n \in \mathrm{Galb}(L^{p,q}\cap L^1)$, then $(c_n)_n \in \ell^q$.

2) Let $(c_n)_n \in \ell^p$ and let $\|f_n\|_{L^{p,q}\cap L^1} = 1$. Since $\|f_n\|_{L^{p,\infty}} \preceq \|f_n\|_{L^{p,q}}$ and $\mathrm{Galb}(L^{p,\infty}) = \ell^p$, we have

$$\Big\|\sum_n c_n f_n\Big\|_{L^{p,\infty}} \preceq \Big(\sum_n |c_n|^p\Big)^{1/p}.$$

Similarly, $\|f_n\|_{L^{1,\infty}} \preceq \|f_n\|_{L^1}$ and since $\mathrm{Galb}(L^{1,\infty}) = \ell\log\ell$,

$$\Big\|\sum_n c_n f_n\Big\|_{L^{1,\infty}} \preceq \|(c_n)_n\|_{\ell\log\ell} \preceq \Big(\sum_n |c_n|^p\Big)^{1/p}.$$

Hence $\ell^p \subseteq \mathrm{WGalb}(L^{p,q}\cap L^1)$. To prove the converse inclusion, given a finite sequence $(c_n)_n$, choose k and a sequence $\{A_n\}_n$ of disjoint sets with $|A_n| = k|c_n|^p \geq 1$. Let $f_n = |A_n|^{-1/p}\chi_{A_n}$, so that $\|f_n\|_{L^{p,q}\cap L^1} = 1$. Then $|\sum_n c_n f_n| = k^{-1/p}$ on a set of measure $\sum_n |A_n| = k\sum_n |c_n|^p$. In particular, the norm of $\sum_n c_n f_n$ in $L^{p,\infty}$ is $(\sum_n |c_n|^p)^{1/p}$, and hence $(c_n)_n \in \ell^p$.

3) First observe that if $(c_n)_n \in \mathrm{FGalb}(L^{p,q}\cap L^1)$, then in particular $\sum_n |c_n| < \infty$. To prove the converse, observe that if $\sum_n |c_n| < \infty$ and if $\|f_n\|_{L^{p,q}\cap L^1} = 1$, then $\sum_n c_n f_n(x)$ converges in L^1 and therefore it is finite almost everywhere. ∎

5. Proof of Lemma 3.3. (a) The inequality $B \leq A$ is trivial since it is just taking the supremum over all functions of the form $\chi_{(0,r)}$. To prove the converse, fix a decreasing function f and an $r > 0$. Let $r_j = r/a^j$ and write

$$\int_0^\infty \Big(\frac{1}{t}\int_{at}^{t} f\Big)^q w(t)\,dt \leq 2^{q-1}\sum_{j=-\infty}^{\infty}(U_j + V_j)$$

where

$$U_j = \int_{r_j}^{r_{j+1}}\Big(\int_{r_j}^{t} f\Big)^q \frac{w(t)}{t^q}\,dt \quad\text{and}\quad V_j = \int_{r_j}^{r_{j+1}}\Big(\int_{at}^{r_j} f\Big)^q \frac{w(t)}{t^q}\,dt.$$

Since f is decreasing, for each j we have

$$\begin{aligned}
U_j &= q \int_{r_j}^{r_{j+1}} \int_{r_j}^{t} \Big(\int_{r_j}^{s} f \Big)^{q-1} f(s)\, ds \, \frac{w(t)}{t^q}\, dt \\
&\leq q \int_{r_j}^{r_{j+1}} \int_{r_j}^{t} \left(\frac{s-r_j}{s-as} \int_{as}^{s} f \right)^{q-1} f(s)\, ds \, \frac{w(t)}{t^q}\, dt \\
&= q(1-a)^{1-q} \int_{r_j}^{r_{j+1}} \int_{r_j}^{t} (s-r_j)^{q-1} \int_{g(t)}^{g(s)} dy\, ds \, \frac{w(t)}{t^q}\, dt \\
&\quad + q(1-a)^{1-q} \int_{r_j}^{r_{j+1}} \int_{r_j}^{t} (s-r_j)^{q-1}\, ds\, g(t) \, \frac{w(t)}{t^q}\, dt \\
&\equiv (1-a)^{1-q} U_j^{(1)} + (1-a)^{1-q} U_j^{(2)}
\end{aligned}$$

where $g(s) = (s^{-1} \int_{as}^{s} f)^{q-1} f(s)$. Note that g is also decreasing.

Let λ_g denote the distribution function of g. To estimate $U_j^{(1)}$, we expand the region of integration by observing that

$$\left\{ \begin{array}{c} r_j < t < r_{j+1} \\ r_j < s < t \\ g(t) < y < g(s) \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} g(r_{j+1}) \leq y \leq g(r_j) \\ \lambda_g(y) \leq t \leq \lambda_g(y)/a \\ at \leq s \leq \lambda_g(y) \end{array} \right\}.$$

Performing the inner (ds) integration and using the hypothesis (4) we see that

$$\begin{aligned}
U_j^{(1)} &\leq \int_{g(r_{j+1})}^{g(r_j)} \int_{\lambda_g(y)}^{\lambda_g(y)/a} (\lambda_g(y) - at)^q \, \frac{w(t)}{t^q}\, dt\, dy \\
&\leq B^q \int_{g(r_{j+1})}^{g(r_j)} \Big(\int_{0}^{\lambda_g(y)} w \Big)\, dy.
\end{aligned}$$

The estimate for $U_j^{(2)}$ is simpler,

$$U_j^{(2)} = \int_{r_j}^{r_{j+1}} (t-r_j)^q \, ds\, g(t) \, \frac{w(t)}{t^q}\, dt \leq \int_{r_j}^{r_{j+1}} gw.$$

Now

$$\sum_{j=-\infty}^{\infty} (U_j^{(1)} + U_j^{(2)}) \leq B^q \int_0^{\infty} \Big(\int_0^{\lambda_g(y)} w \Big)\, dy + \int_0^{\infty} gw = (B^q + 1) \int_0^{\infty} gw.$$

The estimate for V_j begins similarly. With g as above,

$$\begin{aligned}
V_j &= q\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}\Big(\int_{at}^{s} f\Big)^{q-1} f(s)\,ds\,\frac{w(t)}{t^q}\,dt \\
&\le q\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}\left(\frac{s-at}{s-as}\int_{as}^{s} f\right)^{q-1} f(s)\,ds\,\frac{w(t)}{t^q}\,dt \\
&= q(1-a)^{1-q}\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}(s-at)^{q-1}\int_{g(r_j)}^{g(s)} dy\,ds\,\frac{w(t)}{t^q}\,dt \\
&\quad + q(1-a)^{1-q}g(r_j)\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}(s-at)^{q-1}\,ds\,\frac{w(t)}{t^q}\,dt \\
&\equiv (1-a)^{1-q}V_j^{(1)} + (1-a)^{1-q}V_j^{(2)}.
\end{aligned}$$

Interchange and expand the region of integration for $V_j^{(1)}$ by observing that

$$\left\{\begin{array}{c} r_j < t < r_{j+1} \\ at < s < r_j \\ g(r_j) < y < g(s) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} g(r_j) \le y \le g(r_{j-1}) \\ \lambda_g(y) \le t \le \lambda_g(y)/a \\ at \le s \le \lambda_g(y) \end{array}\right\}.$$

Performing the ds integration and using the hypothesis (4) yields

$$V_j^{(1)} \le \int_{g(r_j)}^{g(r_{j-1})}\int_{\lambda_g(y)}^{\lambda_g(y)/a}(\lambda_g(y)-at)^q\,\frac{w(t)}{t^q}\,dt\,dy \le B^q\int_{g(r_j)}^{g(r_{j-1})}\Big(\int_0^{\lambda_g(y)} w\Big)\,dy.$$

Thus,

$$\sum_{j=-\infty}^{\infty} V_j^{(1)} \le B^q\int_0^{\infty}\Big(\int_0^{\lambda_g(y)} w\Big)\,dy = B^q\int_0^{\infty} gw.$$

To estimate $V_j^{(2)}$ we use the fact that $g(r_j)$ is a decreasing sequence. For each $k>1$,

$$\sum_{j=-\infty}^{k-1}\int_{r_j}^{r_{j+1}}(r_j-at)^q\,\frac{w(t)}{t^q}\,dt \le (1-a)^q\int_0^{r_k} w$$

and, by (4),

$$\int_{r_k}^{r_{k+1}}(r_k-at)^q\,\frac{w(t)}{t^q}\,dt \le B^q\int_0^{r_k} w.$$

It follows that

$$\sum_{j=-\infty}^{k}\int_{r_j}^{r_{j+1}}(r_j-at)^q\,\frac{w(t)}{t^q}\,dt \le \sum_{j=-\infty}^{k}\big((1-a)^q+B^q\big)\int_{r_{j-1}}^{r_j} w$$

for all k and, because $g(r_j)$ is a decreasing sequence,

$$\sum_{j=-\infty}^{\infty} V_j^{(2)} \le \sum_{j=-\infty}^{\infty} g(r_j)((1-a)^q + B^q) \int_{r_{j-1}}^{r_j} w$$
$$\le ((1-a)^q + B^q) \int_0^\infty gw.$$

Combining the inequalities above, we get

$$\int_0^\infty \left(\frac{1}{t}\int_{at}^t f\right)^q w(t)\,dt \preceq (1+B^q) \int_0^\infty gw$$
$$\le (1+B^q)\left(\int_0^\infty \left(\frac{1}{t}\int_{at}^t f\right)^q w(t)\,dt\right)^{1/q'} \left(\int_0^\infty f^q w\right)^{1/q},$$

and we conclude (by approximating f by integrable functions if necessary) that $A \preceq 1 + B^q$.

To prove (b), we shall use some ideas of Stepanov and Ushakova [20, Theorem 3]. For the converse, we apply Theorem 3.1 of [16], although Theorem 1 of [15] will also do. We have

$$A = \sup_{\substack{\|f\|_{L^q(w)}\le 1\\ f\downarrow}} \left(\int_0^\infty \left(\frac{1}{t}\int_{at}^t f(s)\,ds\right)^q w(t)\,dt\right)^{1/q}$$
$$= \sup_{\substack{\|f\|_{L^q(w)}\le 1\\ f\downarrow}} \sup_{\|g\|_{L^{q'}(w)}\le 1} \int_0^\infty \frac{1}{t}\int_{at}^t f(s)\,ds\, g(t)w(t)\,dt$$
$$= \sup_{\|g\|_{L^{q'}(w)}\le 1} \sup_{\substack{\|f\|_{L^q(w)}\le 1\\ f\downarrow}} \int_0^\infty f(s)\left(\frac{1}{w(s)}\int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\right)w(s)\,ds$$
$$\approx \sup_{\|g\|_{L^{q'}(w)}\le 1} \left(\int_0^\infty \left(\frac{\int_0^x G(s)w(s)\,ds}{\int_0^x w(s)\,ds} + \frac{\int_0^\infty G(s)w(s)\,ds}{\int_0^\infty w(s)\,ds}\right)^{q'} w(x)\,dx\right)^{1/q'}$$

where $G(s) = \frac{1}{w(s)}\int_s^{s/a} g(t)w(t)\,\frac{dt}{t}$. Now

$$\int_0^x Gw = \int_0^x \int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\,ds = \int_0^{x/a} \frac{1}{t}\int_{at}^{\min(x,t)} ds\, g(t)w(t)\,dt$$
$$= (1-a)\int_0^x g(t)w(t)\,dt + \int_x^{x/a} (x/t - a)g(t)w(t)\,dt$$

and

$$\int_0^\infty Gw = \int_0^\infty \frac{1}{w(s)} \int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\,w(s)\,ds$$
$$= \int_0^\infty \int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\,ds = (1-a)\int_0^\infty gw.$$

Therefore $A \approx A_1 + A_2 + A_3$ where

$$A_1 = (1-a) \sup_{\|g\|_{L^{q'}(w)}\le 1} \left(\int_0^\infty \left(\frac{\int_0^x g(t)w(t)\,dt}{\int_0^x w(t)\,dt} \right)^{q'} w(x)\,dx \right)^{1/q'},$$

$$A_2 = (1-a) \sup_{\|g\|_{L^{q'}(w)}\le 1} \left(\int_0^\infty \left(\frac{\int_0^\infty g(t)w(t)\,dt}{\int_0^\infty w(t)\,dt} \right)^{q'} w(x)\,dx \right)^{1/q'},$$

$$A_3 = \sup_{\|g\|_{L^{q'}(w)}\le 1} \left(\int_0^\infty \left(\frac{\int_x^{x/a} (x/t - a)g(t)w(t)\,dt}{\int_0^x w(t)\,dt} \right)^{q'} w(x)\,dx \right)^{1/q'}.$$

The first two are easy. Hardy's inequality says that $A_1 = (1-a)q$ and Hölder's inequality yields $A_2 = 1-a$. For A_3 we use Theorem 4.4 of [12]. By replacing x by s and $g(t)w(t)$ by $f(t)t$ we recognize A_3 as the best constant in the inequality

$$\left(\int_0^\infty \left(\int_s^{s/a} (s-ay)f(y)\,dy \right)^{q'} \left(\int_0^s w \right)^{-q'} w(s)\,ds \right)^{1/q'}$$
$$\le A_3 \left(\int_0^\infty f(t)^{q'} t^{q'} w(t)^{1-q'}\,dt \right)^{1/q'}.$$

It is trivial to check that the so called GHO condition in [12] holds for the kernel $k(s,y) = s - ay$ and so $A_3 \approx \max(A_{3,1}, A_{3,2})$ where

$$A_{3,1} = \sup_{s\le x\le s/a} \left(\int_s^x (t-s)^{q'} \left(\int_0^t w \right)^{-q'} w(t)\,dt \right)^{1/q'} \left(\int_x^{s/a} t^{-q} w(t)\,dt \right)^{1/q},$$

$$A_{3,2} = \sup_{s\le x\le s/a} \left(\int_s^x \left(\int_0^t w \right)^{-q'} w(t)\,dt \right)^{1/q'} \left(\int_x^{s/a} (s-at)^q t^{-q} w(t)\,dt \right)^{1/q}.$$

Since

$$B = \sup_r \frac{\left(\int_r^{r/a} (r-at)^q \frac{w(t)}{t^q}\,dt \right)^{1/q}}{\left(\int_0^r w(t)\,dt \right)^{1/q}},$$

we have

$$A_{3,2} \le \sup_s \left(\int_s^\infty \left(\int_0^t w \right)^{-q'} w(t)\,dt \right)^{1/q'} \left(\int_s^{s/a} (s-at)^q w(t)\,dt \right)^{1/q} \preceq B.$$

Also

$$
\begin{aligned}
A_{3,1} &= \sup_{s\le x\le s/a}\Big(\int_s^x\Big(\int_x^{s/a}(t-s)^q y^{-q}w(y)\,dy\Big)^{q'-1}\Big(\int_0^t w\Big)^{-q'}w(t)\,dt\Big)^{1/q'}\\
&\le \sup_{s\le x\le s/a}\Big(\int_s^x\Big(\int_t^{t/a}(t-ay)^q y^{-q}w(y)\,dy\Big)^{q'-1}\Big(\int_0^t w\Big)^{-q'}w(t)\,dt\Big)^{1/q'}\\
&\le B\sup_{s\le x\le s/a}\Big(\int_s^x\Big(\int_0^t w\Big)^{-1}w(t)\,dt\Big)^{1/q'}\\
&\le B\sup_{s\le x\le s/a}\Big(\log\Big(\int_0^x w\Big)-\log\Big(\int_0^s w\Big)\Big)^{1/q'}\\
&\le B(\log(D))^{1/q'}.
\end{aligned}
$$

This completes the proof of (b).

To prove (c), it is enough to show that

$$
\int_0^{s/a} w\le D\int_0^s w,\quad s>0,
$$

with $D=(B/(\sqrt{a}-a))^{2q}$. Now, for any $r>0$, we have $r/a\ge r/\sqrt{a}$ so

$$
B^q\int_0^r w\ge\int_r^{r/\sqrt{a}}(r-at)^q t^{-q}w(t)\,dt\ge(\sqrt{a}-a)^q\int_0^{r/\sqrt{a}} w.
$$

Hence

$$
\int_0^{s/a} w\le(B/(\sqrt{a}-a))^q\int_0^{s/\sqrt{a}} w\le(B/(\sqrt{a}-a))^{2q}\int_0^s w.
$$

This completes the proof. ∎

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