

(I)-envelopes of unit balls and James' characterization of reflexivity

by

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Abstract. We study the (I)-envelopes of the unit balls of Banach spaces. We show, in particular, that any nonreflexive space can be renormed in such a way that the (I)-envelope of the unit ball is not the whole bidual unit ball. Further, we give a simpler proof of James' characterization of reflexivity in the nonseparable case. We also study the spaces in which the (I)-envelope of the unit ball adds nothing.

1. Introduction. The present paper is a continuation of [13] where the (I)-envelopes of sets in Banach spaces were introduced and studied with the aim of using the result of [8] to obtain an easier proof of James' characterization of weak compactness in nonseparable spaces. It was shown there that an easy proof can be given in a large class of spaces. However, an example was exhibited showing that the general situation is not so easy ([13, Example 4.1]).

In this paper we study the (I)-envelopes of the unit balls and in particular two extreme classes of spaces: those where the (I)-envelope adds nothing (Section 4) and those where the (I)-envelope is as large as it can be (Sections 2 and 3). In Section 3 we give a simpler proof of James' characterization of reflexivity using recent results of [15].

Let us start by recalling the definition of the (I)-envelope and its basic properties.

Let X be a Banach space and $B \subset X^*$. The (I)-envelope of B is defined by the formula

$$(I)\text{-env}(B) = \bigcap \left\{ \overline{\text{conv} \bigcup_{n=1}^{\infty} \text{conv} C_n^{w*}}^{\|\cdot\|} : B = \bigcup_{n=1}^{\infty} C_n \right\}.$$

2000 *Mathematics Subject Classification*: Primary 46B26; Secondary 46A55, 46B10.

Key words and phrases: (I)-envelope, real-valued measurable cardinal, Grothendieck property, reflexive space, James theorem.

This work is part of the research project MSM 0021620839 financed by MSMT and partly supported by the research grant GA ĀR 201/06/0018 and by Universidad Politecnica de Valencia.

The motivation for this definition was the result [8, Theorem 2.3] stating (in the above notation) that $K = (\text{I})\text{-env}(B)$ whenever K is a convex weak* compact set in X^* and B a *boundary* of K (i.e., any element of X attains its maximum on K at some point of B). We will deal with the (I)-envelopes of closed convex sets in X considered canonically embedded into the bidual. For this case we have the following characterization ([13, Lemma 2.3]).

LEMMA 1.1. *Let X be a Banach space, $B \subset X$ a closed convex set and $G \in \overline{B}^{w^*}$. Then the following assertions are equivalent.*

- (1) $G \notin (\text{I})\text{-env}(B)$.
- (2) *There is a sequence of $\xi_n \in B_{X^*}$ such that*

$$\sup_{x \in B} \limsup_{n \rightarrow \infty} \xi_n(x) < \inf_{n \in \mathbb{N}} G(\xi_n).$$

- (3) *There is a sequence of $\xi_n \in B_{X^*}$ such that*

$$\sup_{x \in B} \limsup_{n \rightarrow \infty} \xi_n(x) < \liminf_{n \rightarrow \infty} G(\xi_n).$$

- (4) *There is a sequence of $\xi_n \in B_{X^*}$ such that*

$$\sup_{x \in B} \limsup_{n \rightarrow \infty} \xi_n(x) < \limsup_{n \rightarrow \infty} G(\xi_n).$$

Note that in all the conditions (2)–(4) we can replace the assumption that $\xi_n \in B_{X^*}$ by the assumption that the sequence ξ_n is bounded. As an easy consequence of this lemma we obtain the following one.

LEMMA 1.2. *Let X be a Banach space. Then the (I)-envelope of X in X^{**} consists exactly of the weak* sequentially continuous elements of X^{**} .*

Proof. Let ξ_n be a bounded sequence in X^* . Then

$$\sup_{x \in X} \limsup_{n \rightarrow \infty} \xi_n(x) = \begin{cases} 0 & \text{if } \xi_n \text{ weak* converges to } 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The statement now follows easily from Lemma 1.1. ■

2. Grothendieck spaces and reflexive spaces. A Banach space X is a *Grothendieck space* if weak and weak* convergences of sequences coincide in X^* . Clearly, any reflexive space is Grothendieck. There are also nonreflexive Grothendieck spaces: a classical example is the space ℓ_∞ (a result of Grothendieck [10], see also [4, Theorem VII.15]). Further examples can be found for example in [1, 20, 2, 17].

It is easy to see that X is a Grothendieck space if and only if all elements of X^{**} are weak* sequentially continuous. Therefore the following theorem is an immediate consequence of Lemma 1.2.

THEOREM 2.1. *A Banach space X is Grothendieck if and only if $(\text{I})\text{-env}(X) = X^{**}$.*

While this theorem is quite easy, the situation with the (I)-envelopes of unit balls is more complicated. It is proved in [13, Example 4.1] that the (I)-envelope of the unit ball of ℓ_∞ is the whole bidual unit ball. It is asked there whether the same holds for any Grothendieck space. The next theorem shows that this is not the case. Namely, the only spaces for which $(\text{I-env}(B_X) = B_{X^{**}}$ for any equivalent norm are the reflexive ones.

THEOREM 2.2. *Let X be a nonreflexive Banach space. Then there is an equivalent norm $|\cdot|$ on X such that the (I)-envelope of the unit ball of $(X, |\cdot|)$ is a proper subset of the respective bidual unit ball.*

Proof. If X is separable, then $(\text{I-env}(B_X) = B_X$ (see [13, Remark 1.1(2)]), hence there is no need to renorm.

Suppose that X is nonseparable. Let $Y \subset X$ be a separable subspace which is not reflexive. (This is possible as reflexivity is separably determined, see e.g. [7, Exercise 4.24].)

Choose some $\xi \in S_{X^*}$ such that $\xi|_Y = 0$ and find $x_0 \in X$ with $\xi(x_0) = 1$. Notice that $\|x_0\| \geq 1$. Set

$$B = \{x \in X : \|x - \xi(x)x_0\| \leq 1 \text{ and } |\xi(x)| + \text{dist}(x - \xi(x)x_0, Y) \leq 1\}.$$

Then B is clearly a closed absolutely convex set and it is easy to check that

$$\frac{1}{2 + \|x_0\|} B_X \subset B \subset \|x_0\| B_X,$$

hence B is the closed unit ball of an equivalent norm on X . We will show that $\bar{B}^{w^*} \setminus (\text{I-env}(B) \neq \emptyset$.

As $Y \subset X$, the bidual Y^{**} can be canonically identified with a subspace of X^{**} , namely with the weak* closure of Y in X^{**} . Hence we can take some $F \in S_{Y^{**}} \setminus X$. Let Z be the span of Y and x_0 . Then Z is separable and $F \in Z^{**} \setminus Z$, hence $F|_{B_{Z^*}}$ is not weak* sequentially continuous. It follows that there are $\tilde{\xi}_n \in B_{Z^*}$ weak* converging to 0 and $c > 0$ such that $F(\tilde{\xi}_n) \geq c$ for each $n \in \mathbb{N}$. By the Hahn–Banach theorem extend $\tilde{\xi}_n$ to $\xi_n \in B_{X^*}$.

Set $G = x_0 + F$. Then $G \in \bar{B}^{w^*}$. Indeed, choose a net $y_\nu \in S_Y$ weak* converging to F . Then $x_0 + y_\nu$ weak* converges to G . Moreover, $x_0 + y_\nu \in B$ as

$$\begin{aligned} \xi(x_0 + y_\nu) &= 1, \\ \|x_0 + y_\nu - \xi(x_0 + y_\nu)x_0\| &= \|y_\nu\| \leq 1, \\ \text{dist}(x_0 + y_\nu - \xi(x_0 + y_\nu)x_0, Y) &= \text{dist}(y_\nu, Y) = 0. \end{aligned}$$

Further,

$$\begin{aligned} \liminf_{n \rightarrow \infty} G(\xi + \xi_n) &= \liminf_{n \rightarrow \infty} (\xi(x_0) + \xi_n(x_0) + F(\xi) + F(\xi_n)) \\ &= 1 + \lim_{n \rightarrow \infty} \xi_n(x_0) + \liminf_{n \rightarrow \infty} F(\xi_n) \geq 1 + c. \end{aligned}$$

We have used here the facts that $F(\xi) = 0$ (as $F \in \overline{Y}^{w*}$ and $\xi|_Y = 0$) and that $\xi_n(x_0) \rightarrow 0$ (as $x_0 \in Z$).

On the other hand, if $x \in B$ and $y \in Y$ is arbitrary, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} (\xi + \xi_n)(x) \\ &= \xi(x) + \limsup_{n \rightarrow \infty} (\xi_n(x - \xi(x)x_0 - y) + \xi_n(x_0)\xi(x) + \xi_n(y)) \\ &= \xi(x) + \limsup_{n \rightarrow \infty} \xi_n(x - \xi(x)x_0 - y) + \lim_{n \rightarrow \infty} (\xi_n(x_0)\xi(x) + \xi_n(y)) \\ &\leq \xi(x) + \|x - \xi(x)x_0 - y\|. \end{aligned}$$

(We have used the fact that $\xi_n(z) \rightarrow 0$ for all $z \in Z$.) As $y \in Y$ is arbitrary, we get

$$\limsup_{n \rightarrow \infty} (\xi + \xi_n)(x) \leq \xi(x) + \text{dist}(x - \xi(x)x_0, Y) \leq 1$$

for all $x \in B$. Hence, by Lemma 1.1 the sequence $\xi + \xi_n$ witnesses that G does not belong to the (I)-envelope of B . ■

As a corollary we get the following example.

EXAMPLE 2.3. *There is an equivalent norm on ℓ_∞ such that the (I)-envelope of the corresponding unit ball is not the whole bidual unit ball.*

We can also give an explicit description of an equivalent norm on ℓ_∞ with the property described in the above example. Its unit ball is

$$\{x \in \ell_\infty : \|x\|_\infty \leq 1 \text{ and } |x_1| + \limsup_{n \rightarrow \infty} |x_n| \leq 1\}.$$

This example, together with [13, Example 4.1], shows that “(I)-env(B_X) = $B_{X^{**}}$ ” is not an isomorphic property.

The following questions seems to be open.

QUESTION 2.4. *Let X be a Grothendieck space. Is there an equivalent norm on X such that the (I)-envelope of the corresponding unit ball is the whole bidual unit ball?*

QUESTION 2.5. *Is there a nonreflexive Grothendieck space X such that (I)-env(B_X) = B_X ?*

Of course, we cannot have positive answers to both questions. Using [13, Example 4.1] it is easy to see that if Question 2.5 has a positive answer, the corresponding space X cannot contain an isomorphic copy of ℓ_∞ . Hence a natural candidate for such an example is the space from [11].

3. Towards an easy proof of James’ characterization of reflexivity. In this section we discuss the possibility to give an easy proof of the famous James theorem [12, Theorem 5] saying that a Banach space X is reflexive provided any element of X^* attains its norm at some point

of B_X . For separable spaces there are easy proofs: see e.g. [3, Theorem 3.2] or [9, Theorem 5.7].

Up to our knowledge there is no easy proof of the general case. James' original proof can be divided into two steps. Suppose X is not reflexive. First, one can show that there is a sequence a_n in X such that

$$\inf_{n \in \mathbb{N}} \text{dist}(\text{span}\{a_i : i < n\}, \text{conv}\{a_i : i \geq n\}) > 0.$$

The second step is a construction of a functional not attaining its norm starting from such a sequence. Recently in [15] a new proof appeared for the second step in the general nonseparable case. Let us recall the basic steps of that proof (for details see [15]).

We first introduce some notions used in [15]. Let x_n be a sequence in a Banach space. A sequence b_n is a *block sequence* of x_n if there is a sequence F_n of pairwise disjoint finite subsets of \mathbb{N} and a sequence λ_i of real numbers such that $b_n = \sum_{i \in F_n} \lambda_i x_i$ for each $n \in \mathbb{N}$. If $\sum_{i \in F_n} |\lambda_i| = 1$ for each $n \in \mathbb{N}$, then b_n is called a *normalized block sequence*. If moreover $\lambda_i \geq 0$ for all i , then b_n is a *convex block sequence*.

(1) If B_{X^*} is *weak* convex block compact*, i.e. any sequence in B_{X^*} has a weak* convergent convex block sequence, then the second step of the proof of the James theorem follows from the Simons inequality [21]. This is contained in [15, Lemma 1 and Corollary 2].

(2) If B_{X^*} is not weak* sequentially compact but is *weak* block compact* (i.e. any sequence in B_{X^*} has a weak* convergent normalized block sequence), then the second step of the proof can be replaced by an application of the Rosenthal ℓ_1 -theorem and the Simons inequality. This is proved in [15, Section 2.2].

(3) If B_{X^*} is not weak* block compact, then X contains an *asymptotically isometric copy of ℓ_1* , i.e. there is sequence a_n in B_X and a sequence of numbers $\delta_n \in (0, 1)$ converging to 1 such that

$$\left\| \sum_{i=1}^n \lambda_i a_i \right\| \geq \sum_{i=1}^n \delta_i |\lambda_i|$$

for each $n \in \mathbb{N}$ and each n -tuple $\lambda_1, \dots, \lambda_n$ of real numbers. This is [15, Theorem 2].

(4) If X contains an asymptotically isometric copy of ℓ_1 , then there is an element of X^* which does not attain its norm. This is the content of [15, Theorem 3].

We show here an argument replacing the first two steps. Moreover, our argument proves not only the second step but yields a complete proof. The key ingredient is the following proposition.

PROPOSITION 3.1. *Let X be a Grothendieck space such that the dual unit ball is weak* block compact. Then X is reflexive.*

Proof. Suppose that X is not reflexive. Then B_{X^*} is not weakly compact and hence it is not weakly sequentially compact by the Eberlein–Šmulian theorem [7, Theorem 3.59]. By the Grothendieck property it follows that B_{X^*} is not weak* sequentially compact. Let ξ_n be a sequence in B_{X^*} without a weak* converging subsequence. By the Rosenthal ℓ_1 -theorem [18] there is a subsequence of ξ_n equivalent to the standard basis of ℓ_1 . (Indeed, if ξ_n has no such subsequence, it contains a weakly Cauchy subsequence. Further, any weakly Cauchy sequence ζ_n in X^* converges in (X^{***}, w^*) to a point $\Xi \in X^{***}$. Then clearly ζ_n converges in (X^*, w^*) to $\Xi|_X$, hence is weak* convergent.) Hence we can suppose that the sequence ξ_n itself is equivalent to the basis of ℓ_1 , i.e. there is some constant $c > 0$ such that

$$\left\| \sum_{i=1}^n \lambda_i \xi_i \right\| \geq c \sum_{i=1}^n |\lambda_i|$$

for all $n \in \mathbb{N}$ and all n -tuples $\lambda_1, \dots, \lambda_n$ of real numbers. In particular the space

$$Y = \overline{\text{span}\{\xi_n : n \in \mathbb{N}\}}$$

is isomorphic to ℓ_1 . We claim that ξ_n has no weak* convergent normalized block sequence. Suppose that ζ_n is such a sequence. Then $\frac{1}{2}(\zeta_{2n} - \zeta_{2n+1})$ is a normalized block sequence of ξ_n weak* converging to 0, hence we can without loss of generality suppose that ζ_n weak* converges to 0. (This easy remark is made in [15, Section 1.2].)

Now, by the Grothendieck property ζ_n weakly converges to 0. As all ζ_n belong to Y and Y is isomorphic to ℓ_1 , it follows from the Schur theorem [7, Theorem 5.19] that ζ_n norm converges to 0. However, as ζ_n is a normalized block sequence of ξ_n , we get $\|\zeta_n\| \geq c$ for all $n \in \mathbb{N}$. This is a contradiction completing the proof. ■

This proposition enables us to give the following proof of the James theorem.

A proof of James' characterization of reflexivity. Let X be a Banach space such that every element of X^* attains its norm at some point of B_X . By [8, Theorem 2.3], the (I)-envelope of B_X is the whole $B_{X^{**}}$. By [16, Theorem 1.4] or [13, Theorem 3.4], X has the Grothendieck property. Suppose X is not reflexive. By Proposition 3.1, B_{X^*} is not weak* block compact. By [15, Theorem 2], X contains an asymptotically isometric copy of ℓ_1 and hence, by [15, Theorem 3], there is an element of X^* not attaining its norm. This completes the proof. ■

The proof just given is quite short but uses a large number of nontrivial results. Nonetheless, it seems to the author that this proof makes the James theorem more accessible. However, if we want an easy proof, we should give an easy proof of the fact that X contains an asymptotically isometric copy of ℓ_1 whenever X is a nonreflexive space with $(I)\text{-env}(B_X) = B_{X^{**}}$. Let us briefly discuss what would be needed for that.

In [15, Section 3] the following quantities are defined for any bounded sequence ξ_n in X^* :

$$\delta(\xi_n)_n = \sup_{x \in B_X} \limsup_{n \rightarrow \infty} \xi_n(x),$$

$$\varepsilon(\xi_n)_n = \inf\{\delta(\zeta_n)_n : (\zeta_n)_n \text{ is a normalized block sequence of } (\xi_n)_n\}.$$

The main part of the proof of [15, Theorem 2] is a construction of an asymptotically isometric copy of ℓ_1 in X starting from a bounded sequence ξ_n in X^* satisfying

$$\varepsilon(\xi_n)_n = \delta(\xi_n)_n > 0.$$

Such a sequence ξ_n can be constructed by the diagonal method ([15, Lemma 3]) from any sequence ξ_n^0 such that $\delta(\zeta_n)_n > 0$ for any normalized block sequence ζ_n of ξ_n^0 .

Now, if $(I)\text{-env}(B_X) = B_{X^{**}}$, then (by Lemma 1.1)

$$\delta(\xi_n)_n = \sup_{F \in B_{X^{**}}} \limsup_{n \rightarrow \infty} F(\xi_n)$$

for each bounded sequence ξ_n in X^* . Note that the right-hand side is zero if and only if ξ_n weakly converges to 0.

Hence we would need an easy proof of the following fact.

FACT. *Let X be a Banach space such that $(I)\text{-env}(B_X) = B_{X^{**}}$ and B_{X^*} is weakly block compact. Then X is reflexive.*

This fact follows from Rosenthal's and Schur's theorems. In fact, we have the following characterization of spaces not containing ℓ_1 :

PROPOSITION 3.2. *A Banach space Y contains no isomorphic copy of ℓ_1 if and only if B_Y is weakly block compact.*

Proof. Suppose that Y contains no isomorphic copy of ℓ_1 and y_n is a sequence in B_Y . By the Rosenthal theorem [18] there is a subsequence y_{n_k} which is weakly Cauchy. Now, $\frac{1}{2}(y_{n_{2k}} - y_{n_{2k+1}})$ is a normalized block sequence of y_n weakly converging to 0. Hence B_Y is weakly block compact.

Conversely, let y_n be a sequence in B_Y equivalent to the standard basis of ℓ_1 . Let z_n be a normalized block sequence of y_n weakly converging to 0. By the Schur theorem z_n norm converges to 0, which is impossible for any normalized block sequence of y_n . ■

Hence, the proof of the above Fact goes as follows: If B_{X^*} is weakly block compact, X^* contains no copy of ℓ_1 . Now, in this case B_{X^*} is weak* sequentially compact by the Rosenthal theorem. If, moreover, $(\text{I-env}(B_X)) = B_{X^{**}}$, we find that B_{X^*} is weakly sequentially compact and hence X is reflexive.

This argument is essentially the same as the one given above. The author does not know whether one can give an essentially simpler proof of the Fact. Neither is it clear whether one could give a comparably simple proof of the James characterization of weak compactness.

4. Spaces in which the (I)-envelope adds nothing. In this section we study the other extreme—the Banach spaces X for which $(\text{I-env}(X)) = X$ or at least $(\text{I-env}(B_X)) = B_X$. As an immediate consequence of Lemma 1.2 we get the following theorem.

THEOREM 4.1. *Let X be a Banach space. Then $(\text{I-env}(X)) = X$ if and only if the only weak* sequentially continuous elements of X^{**} are those from X .*

Note, that weak* continuous elements of X^{**} belong to X by [7, Theorem 3.16]. Hence the condition in the above theorem means that weak* sequential continuity of a linear function on X^* implies its weak* continuity. It is the case for example if (B_{X^*}, w^*) is angelic (cf. [13, Theorem 3.1]) or, more generally, if (B_{X^*}, w^*) is sequential (i.e., sequentially closed sets are closed). We will see below that these sufficient conditions are not necessary.

If $(\text{I-env}(X)) = X$, then $(\text{I-env}(B)) = B$ for all closed convex subsets $B \subset X$, in particular $(\text{I-env}(B_X)) = B_X$. It seems unclear whether the converse is true, i.e. we have the following problem.

QUESTION 4.2. *Let X be a Banach space such that $(\text{I-env}(B_X)) = B_X$. Is then $(\text{I-env}(X)) = X$?*

Note that the answer is positive if B_{X^*} is weak* sequentially compact. In this case any weak* sequentially continuous element of $B_{X^{**}}$ belongs to $(\text{I-env}(B_X))$ (see [13, Proposition 2.4]), hence the positive answer follows from Theorem 4.1. Below we show that the answer is also positive for $\ell_1(\Gamma)$ spaces. However, the general answer seems to be unknown (cf. also Question 2.5).

We also note that if $(\text{I-env}(B_X)) = B_X$, then $(\text{I-env}(B)) = B$ for each bounded closed convex subset of X . Hence the class of spaces in which $(\text{I-env}(B_X)) = B_X$ is stable with respect to isomorphisms and subspaces.

Now we will consider the space $X = \ell_1(\Gamma)$ for a set Γ with the standard norm. The dual X^* is canonically identified with the space $\ell_\infty(\Gamma)$ which is isometric to the space $C(\beta\Gamma)$ of continuous functions on the Čech–Stone

compactification of the discrete space Γ . The bidual X^{**} is then, by the Riesz theorem, identified with the space of all finite signed Radon measures on $\beta\Gamma$, which canonically correspond to finite signed finitely additive measures on the set Γ . We will use the latter description (see [6, Corollary IV.5.3]).

We have the following characterization of the (I)-envelope of the unit ball of $\ell_1(\Gamma)$.

PROPOSITION 4.3. *Let $X = \ell_1(\Gamma)$ and μ be a finite signed finitely additive measure on Γ with $\|\mu\| \leq 1$. The following are equivalent.*

- (i) μ is σ -additive.
- (ii) μ is weak* sequentially continuous.
- (iii) $\mu \in (\text{I-env}(B_X))$.

Proof. The implication (iii) \Rightarrow (ii) follows from Lemma 1.2, and (i) \Rightarrow (ii) follows from the Lebesgue dominated convergence theorem.

(ii) \Rightarrow (i). Suppose that μ is not σ -additive. Then there is a sequence A_n of pairwise disjoint sets such that

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \neq \sum_{n \in \mathbb{N}} \mu(A_n).$$

Let f_n be the characteristic function of $A_1 \cup \dots \cup A_n$ and f be the characteristic function of $\bigcup_{n \in \mathbb{N}} A_n$. Then $f_n \rightarrow f$ in the weak* topology of $\ell_\infty(\Gamma)$ but $\mu(f_n)$ does not converge to $\mu(f)$.

(i) \Rightarrow (iii). As the (I)-envelope of B_X is absolutely convex, it is enough to prove the implication for probability measures. Suppose that μ is a σ -additive probability on Γ . Let $B_n \nearrow B_X$. Set

$$A_n = \{\gamma \in \Gamma : e_\gamma \in B_n\},$$

where $\{e_\gamma : \gamma \in \Gamma\}$ is the canonical basis of $\ell_1(\Gamma)$. Then $A_n \nearrow \Gamma$ and hence $\mu(A_n) \nearrow 1$ (by σ -additivity of μ).

Let $\varepsilon > 0$ be arbitrary. There is some n such that $\mu(A_n) > 1 - \varepsilon/2$. As the set $\overline{\text{conv } B_n}^{w^*}$ clearly contains all probabilities supported by A_n , we get

$$\mu_n = \frac{\mu|_{A_n}}{\mu(A_n)} \in \overline{\text{conv } B_n}^{w^*}.$$

Further,

$$\|\mu - \mu_n\| = \mu(\Gamma \setminus A_n) + (1 - \mu(A_n)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get

$$\mu \in \bigcup_{n \in \mathbb{N}} \overline{\text{conv } B_n}^{w^* \|\cdot\|}.$$

As $B_n \nearrow B_X$ are arbitrary, we get $\mu \in (\text{I-env}(B_X))$. ■

From the above proposition we derive the following theorem.

THEOREM 4.4. *Let Γ be any set. The following are equivalent.*

- (a) *No cardinal number at most equal to $\text{card } \Gamma$ is real-valued measurable.*
- (b) $(\text{I-}\text{env}(B_{\ell_1(\Gamma)})) = B_{\ell_1(\Gamma)}$.
- (c) $(\text{I-}\text{env}(\ell_1(\Gamma))) = \ell_1(\Gamma)$.

Proof. The implication (c) \Rightarrow (b) is trivial.

(b) \Rightarrow (a). Suppose that there is some real-valued measurable cardinal number at most equal to $\text{card } \Gamma$. This means that there is a σ -additive probability measure μ on Γ such that $\mu(\{\gamma\}) = 0$ for all $\gamma \in \Gamma$. By the previous proposition $\mu \in (\text{I-}\text{env}(B_{\ell_1(\Gamma)}))$. Further, clearly $\mu \notin \ell_1(\Gamma)$.

(a) \Rightarrow (c). Suppose that $\mu \in (\text{I-}\text{env}(\ell_1(\Gamma))) \setminus \ell_1(\Gamma)$. By Lemma 1.2 we see that μ is weak* sequentially continuous, and so by Proposition 4.3 the measure $\mu/\|\mu\|$ (and hence also μ) is σ -additive. Set

$$\nu = \sum_{\gamma \in \Gamma} \mu(\{\gamma\})e_\gamma.$$

Then $\nu \in \ell_1(\Gamma)$ and hence $\mu - \nu \neq 0$. So $\mu - \nu$ is a nonzero σ -additive finite measure on Γ which is zero on singletons, hence there is some real-valued measurable cardinal at most equal to $\text{card } \Gamma$. ■

As a consequence we get the following example.

EXAMPLE 4.5.

- (a) *Let $X = \ell_1([0, \omega_1])$. Then $(\text{I-}\text{env}(X)) = X$. In this case (B_{X^*}, w^*) is not angelic (even not sequential).*
 - (i) *Under the continuum hypothesis B_{X^*} is not weak* sequentially compact.*
 - (ii) *Under Martin's axiom and negation of the continuum hypothesis B_{X^*} is weak* sequentially compact.*
- (b) *Let $X = \ell_1([0, 1])$. Then (B_{X^*}, w^*) is not weak* sequentially compact.*
 - (i) *If there is no real-valued measurable cardinal at most equal to the continuum, then $(\text{I-}\text{env}(X)) = X$.*
 - (ii) *If there is a real-valued measurable cardinal at most equal to the continuum, then $(\text{I-}\text{env}(B_X)) \supsetneq B_X$.*
- (c) *If there are no real-valued measurable cardinals (for example under $V = L$), then $(\text{I-}\text{env}(\ell_1(\Gamma))) = \ell_1(\Gamma)$ for any set Γ .*

Proof. (a) The cardinal \aleph_1 is not real-valued measurable by [22, §1, Satz (A)]. Therefore $(\text{I-}\text{env}(X)) = X$. The fact that B_{X^*} is not sequential is easy: (B_{X^*}, w^*) is homeomorphic to $[-1, 1]^{[0, \omega_1]}$ and the set

$$\{x \in [-1, 1]^{[0, \omega_1]} : \{\gamma : x(\gamma) \neq 0\} \text{ is countable}\}$$

is sequentially closed but not closed. Assertion (i) follows from the well-known fact that $\{0, 1\}^{[0,1]}$ is not sequentially compact. Assertion (ii) follows from the result of Malykhin and Shapirovskii [14] that under Martin's axiom all compact spaces of cardinality at most equal to the continuum are sequentially compact.

(b) We use again the fact that $\{0, 1\}^{[0,1]}$ is not sequentially compact and Theorem 4.4.

Assertion (c) follows from Theorem 4.4. The fact that under $V = L$ there are no real-valued measurable cardinals follows from [22, §2, Satz 3] and [19]. ■

5. Final remarks. All the results of this paper are formulated for real spaces. However, they can be easily transferred to the complex case as well using the method described in [13, Section 5] and the following observations:

- The unit balls constructed in Theorem 2.2 and Example 2.3 are also absolutely convex in the complex case.
- James' characterization of reflexivity of complex spaces follows easily from the real version.
- Propositions 3.1 and 3.2 are true for complex spaces, as also are the Rosenthal and Schur theorems. (This is easy for the Schur theorem, and for the Rosenthal theorem it is proved in [5].)

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Received May 18, 2006
Revised version June 26, 2007

(5953)