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# Sequences of 0's and 1's: sequence spaces with the separable Hahn property

by

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**Abstract.** In [3] it was discovered that one of the main results in [1] (Theorem 5.2), applied to three spaces, contains a nontrivial gap in the argument, but neither the gap was closed nor a counterexample was provided. In [4] the authors verified that all three above mentioned applications of the theorem are true and stated a problem concerning the topological structure of one of these three spaces. In this paper we answer the problem and give a counterexample to the theorem in doubt. Also we establish a new way of constructing separable Hahn spaces.

Let $\chi$ denote the set of all sequences of 0's and 1's and let $\chi(E)$ denote the linear hull of $\chi \cap E$. Given a sequence space $E$ we consider the natural order on it, i.e. for $x, y \in E$ with $x = (x_k)$, $y = (y_k)$ we set $x \le y$ whenever $x_k \le y_k$ for every $k \in \mathbb{N}$. This order defines the positive cone

$$E^+ := \{x \in E \mid x \ge 0\} = \{x \in E \mid x_k \ge 0 \ (k \in \mathbb{N})\}$$

on $E$.

For other notations and preliminary results we refer the reader to [1], [3] and [2].

**1. Introduction.** In [1] (see also [5] and [8]) the authors considered three types of Hahn properties. A sequence space $E$ is said to have the *Hahn property*, the *separable Hahn property* and the *matrix Hahn property* if the implication

$$\chi(E) \subset F \ \Rightarrow \ E \subset F$$

holds whenever $F$ is any FK-space, a separable FK-space and a matrix domain $c_A$ respectively. Evidently the Hahn property implies the separable Hahn property and the latter implies the matrix Hahn property. It was

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shown in [1, Theorem 5.3], and [8, Theorem 1.3] that the converse implications fail in general.

In [3] it was pointed out that the paper [1] by G. Bennett, J. Boos and T. Leiger contains a nontrivial gap in the proof of Theorem 5.2. This theorem is one of the main results of the paper and it was applied three times (cf. [1, (G) in Section 6, Theorem 6.4, Theorem 5.3]): for the space $|ac|_0$ of strongly almost-null sequences, $\ell^\infty \cap z^\alpha$ with $z \in \ell^\infty \setminus \ell^1$, and $\ell^\infty(|\lambda|)$, where $(\lambda_k)$ is an index sequence satisfying

$$\lambda_1 = 1 \quad \text{and} \quad \sup_k (\lambda_{k+1} - \lambda_k) = \infty. \tag{1.1}$$

In Theorem 5.2 of [1] the authors stated that for a monotone sequence space $E$ containing $\varphi$ the following conditions are equivalent:

(i) $E$ has the matrix Hahn property;
(ii) $E$ has the separable Hahn property;
(iii) $\chi(E)^\beta = E^\beta$.

However, in the proof of (iii)⇒(ii) a false argument was used (see [3] for details).

In [3, Theorem 2] J. Boos and T. Leiger showed that the equivalence (i)⇔(ii) holds for any monotone sequence space containing $\varphi$. Moreover, it is well known that (i)⇒(iii) is valid for any sequence space $E$. So only the implication (iii)⇒(ii) has not been settled.

In [7] it was shown that the theorem in doubt is true for $E = |ac|_0$ regardless of the validity of that theorem. In [4] a gliding hump argument was applied to show that two other applications (for $\ell^\infty(|\lambda|)$ and $\ell^\infty \cap z^\alpha$) of the theorem in [1] are valid.

We note that in the proof of the matrix Hahn property of $\ell^\infty(|\lambda|)$ the authors actually made use of the matrix Hahn property of $\ell^\infty$. Using the same idea of proof we will show in this paper that any sequence space

$$X(|\lambda|, 1) := \Big\{ x \in \omega \;\Big|\; \Big( \sum_{k=\lambda_n}^{\lambda_{n+1}-1} |x_k| \Big)_n \in X \Big\}$$

has the matrix Hahn property (even the separable Hahn property) whenever $X$ is *positively solid* and has the matrix Hahn property. This also gives us a way of constructing new separable Hahn spaces.

Now we will verify that the implication (iii)⇒(ii) does not hold in general.

THEOREM 1.1. *There exists a monotone sequence space $E$ satisfying $\chi(E)^\beta = E^\beta$, but failing to have the matrix Hahn property.*

*Proof.* Let $E := \langle (c_{0C_1} \cap \chi) \cup (\{x\} \cdot \chi) \rangle$, where $C_1$ is the Cesàro mean operator and $x$ is constructed as follows. Let $x_k = 1/k$ for $k = 1, \ldots, \lambda_1$, where $\lambda_1 \in \mathbb{N}$ is chosen such that

$$\frac{1}{\lambda_1}\sum_{k=1}^{\lambda_1} x_k \le 2^{-1}$$

(this can be done since $C_1$ is regular). Now we set $x_k := 1 - 1/k$ for $k = \lambda_1 + 1, \dots, \lambda_2$, where we choose $\lambda_2 > \lambda_1$ such that

$$\frac{1}{\lambda_2}\sum_{k=1}^{\lambda_2} x_k \ge 1 - 2^{-1}$$

(here we also make use of the regularity of $C_1$). Proceeding in this way, for $i > 1$ we set $x_k := 1/k$ for $k = \lambda_{2i-2} + 1, \dots, \lambda_{2i-1}$, where $\lambda_{2i-1} > \lambda_{2i-2}$ is chosen such that

$$\frac{1}{\lambda_{2i-1}}\sum_{k=1}^{\lambda_{2i-1}} x_k \le 2^{-i} \tag{1.2}$$

and then we set $x_k := 1 - 1/k$ for $k = \lambda_{2i-1} + 1, \dots, \lambda_{2i}$, where $\lambda_{2i} > \lambda_{2i-1}$ is taken such that

$$\frac{1}{\lambda_{2i}}\sum_{k=1}^{\lambda_{2i}} x_k \ge 1 - 2^{-i}. \tag{1.3}$$

Evidently, $E$ is monotone, $E \subset \ell^\infty$ and it can be verified that $E \cap \chi = c_{0C_1} \cap \chi$.

We will prove that $E^\beta = (E \cap \chi)^\beta$. First we note that since $\mathcal{T} \subset E \cap \chi \subset \ell^\infty$ and $\mathcal{T}^\beta = \ell^1$, we have $(E \cap \chi)^\beta = \ell^1$. In view of the inclusions $E \cap \chi \subset E \subset \ell^\infty$, also $E^\beta = \ell^1$. On the other hand, in view of (1.2) and (1.3), $x \notin c_{C_1}$, so $E \not\subset c_{C_1}$.

Hence the implication (iii)$\Rightarrow$(ii) of Theorem 5.2 in [1] does not hold.

Now coming back to the spaces $\ell^\infty \cap z^\alpha$ and $\ell^\infty(|\lambda|)$, we answer the problem stated in [4]. It was shown there that both $\ell^\infty \cap z^\alpha$ and $\ell^\infty(|\lambda|)$ as well as their $\beta$-dual spaces are solid BK-spaces. Moreover, the linear functional defined by any element $y$ of the $\beta$-dual was shown to be continuous. For $\ell^\infty(|\lambda|)$ the authors proved that the norm of this functional is equal to the norm of $y$ in the $\beta$-dual space while for $\ell^\infty \cap z^\alpha$ they only succeeded in verifying that the norm of the functional is less than or equal to the norm of $y$. So they asked whether equality holds. The following example demonstrates that the answer is negative.

Example. We use the notation of [4]:

$$E := \ell \cap z^\alpha \quad \text{and} \quad F := E^\beta = \ell^1 + \ell^\infty \cdot \{z\}.$$

We consider

$$z = \left(\frac{1}{k}\right) \quad \text{and} \quad y = \left(\frac{1}{2^k} + \frac{1}{k}\right).$$

Evidently $y \in \ell^1 + \ell^\infty \cdot \{z\}$. To evaluate $\|y\|_F$, consider a representation $y = v + wz$, where

$$v = \left(\frac{1}{2^k} - \frac{\alpha_k}{k}\right) \in \ell^1 \quad \text{and} \quad w = (1+\alpha_k) \in \ell^\infty.$$

For $\alpha_k = k/2^k$ ($k \in \mathbb{N}$) we get $\|w\|_\infty + \|v\|_1 = 3/2$, so $\|y\|_F \le 3/2$. We will show that $\|w\|_\infty + \|v\|_1 \ge 3/2$ for $(\alpha_k) \ne (k/2^k)$, hence $\|y\|_F = 3/2$ follows. Note that for all $(\alpha_k)$ we get

$$\|w\|_\infty + \|v\|_1 = \sup_k |1+\alpha_k| + \sum_k \left|\frac{1}{2^k} - \frac{\alpha_k}{k}\right| \ge |1+\alpha_1| + \left|\frac{1}{2} - \alpha_1\right|.$$

If $\alpha_1 \in [-1, 1/2]$, then $|1+\alpha_1| + |1/2 - \alpha_1| = 1 + \alpha_1 + 1/2 - \alpha_1 = 3/2$. For $\alpha_1 < -1$ we get $|1+\alpha_1| + |1/2-\alpha_1| = -2\alpha_1 - 1/2 > 3/2$ and for $\alpha_1 > 1/2$ we get $|1+\alpha_1| + |1/2-\alpha_1| = 2\alpha_1 + 1/2 > 3/2$. So $\|y\|_F = 3/2$.

In order to determine the operator norm $\|f_y\|$ of $f_y$ we fix $x \in E$ with $\|x\|_E = \|x\|_\infty + \sum_k |x_k/k| = 1$. We estimate

$$\begin{aligned}\Big|\sum_k x_k y_k\Big| &\le \sum_k \frac{|x_k|}{k} + \sum_k \frac{|x_k|}{2^k} \le \sum_k \frac{|x_k|}{k} + \|x\|_\infty \sum_k \frac{1}{2^k} \\ &= \sum_k \frac{|x_k|}{k} + \|x\|_\infty = 1.\end{aligned}$$

Therefore $\|f_y\| \le 1 < \|y\|_F$.

**2. Generalization of $\ell^\infty(|\lambda|)$.** Throughout this section we assume that $1 \le p, q \le \infty$. For $p \in (1,\infty)$ we define $p'$ to be the number satisfying $1/p + 1/p' = 1$. We also use the usual convention that $p' = \infty$ for $p = 1$ and $p' = 1$ for $p = \infty$. Most of the proofs in this section are carried out for $p \in (1,\infty)$. The argument for the cases $p = 1$ and $p = \infty$ is analogous.

Given a subset $X$ of $\omega$, an index sequence $(\lambda_k)$ satisfying (1.1) and $p$ with $1 \le p \le \infty$ we define

$$X(|\lambda|, p) := \{x \in \omega \mid T^p(x) := ((\|x^{[\lambda_{n+1}-1]} - x^{[\lambda_n - 1]}\|_p)_n) \in X\}.$$

Note that setting $X = \ell^q$ we obtain the space $\ell(\lambda, p, q)$ introduced in [6].

If $X$ is a sequence space, then $X(|\lambda|, p)$ is closed under scalar multiplication. To guarantee that $X(|\lambda|, p)$ is also closed under vector addition, we need to demand that $X$ is *positively solid*, i.e., satisfies the condition

$$u \in X,\ 0 \le v \le u \ \Rightarrow\ v \in X.$$

Indeed, if $X$ is positively solid and $x, y \in X(|\lambda|, p)$, then by the Minkowski inequality,

$$0 \le T^p(x+y) \le T^p(x) + T^p(y),$$

hence $T^p(x+y)\in X$, implying $x+y\in X(|\lambda|,p)$. So if $X$ is a positively solid sequence space, then $X(|\lambda|,p)$ is a sequence space.

On the other hand, we will show that if $X$ is not positively solid, then $X(|\lambda|,p)$ is not a vector space . Indeed, by assumption we can find $u,v\in\omega$ with $0\le v\le u$ and $u\in X$, but $v\notin X$. We set $x_{\lambda_n}:=v_n$, $x_{\lambda_n+1}:=(u_n^p-v_n^p)^{1/p}$, $y_{\lambda_n}:=v_n$, $y_{\lambda_n+1}:=-(u_n^p-v_n^p)^{1/p}$ $(n\in\mathbb{N})$ and $x_k:=y_k:=0$ for $k\notin\{\lambda_n,\lambda_n+1\mid n\in\mathbb{N}\}$. Then

$$\Big(\sum_{k=\lambda_n}^{\lambda_{n+1}-1}|x_k|^p\Big)^{1/p}=\Big(\sum_{k=\lambda_n}^{\lambda_{n+1}-1}|y_k|^p\Big)^{1/p}=u_n\quad(n\in\mathbb{N}).$$

So $x$ and $y$ are in $X(|\lambda|,p)$. On the other hand,

$$\Big(\sum_{k=\lambda_n}^{\lambda_{n+1}-1}|x_k+y_k|^p\Big)^{1/p}=2v_n\quad(n\in\mathbb{N}),$$

so $x+y\notin X(|\lambda|,p)$. Hence $X(|\lambda|,p)$ is not a vector space.

Evidently any solid space is positively solid. On the other hand, $bs$ and $cs$ are positively solid spaces which are not solid. It is easy to verify that a sequence space is solid if and only if it is monotone and positively solid. Note also that if $X$ is positively solid, then $X(|\lambda|,p)$ is solid. Hereafter we suppose that $X$ is positively solid and contains $\varphi$. Hence the space $\langle X^+\rangle$ is solid.

Evidently, $X(|\lambda|,p)=X^+(|\lambda|,p)$. Hence in particular $\ell(|\lambda|,p)=cs(|\lambda|,p)$ $=bs(|\lambda|,p)$ and more generally, $X_1(|\lambda|,p)=X_2(|\lambda|,p)$ if $X_1^+=X_2^+$.

Proposition 2.1. *Let $\xi\in\{\alpha,\beta,\gamma\}$. Then $(X(|\lambda|,p))^\xi=(X^+)^\alpha(|\lambda|,p')$.*

*Proof.* Since $X(|\lambda|,p)$ is solid, it is sufficient to show that $(X(|\lambda|,p))^\alpha=(X^+)^\alpha(|\lambda|,p')$. Let $y\in(X^+)^\alpha(|\lambda|,p')$ and $x\in X(|\lambda|,p)$. Then by Hölder's inequality,

$$\begin{aligned}\sum_k|y_kx_k|&\le\sum_n\Big(\sum_{k=\lambda_n}^{\lambda_{n+1}-1}|x_k|^p\Big)^{1/p}\Big(\sum_{k=\lambda_n}^{\lambda_{n+1}-1}|y_k|^{p'}\Big)^{1/p'}\\&=\sum_n[T^p(x)]_n[T^{p'}(y)]_n<\infty.\end{aligned}$$

Hence $(X(|\lambda|,p))^\alpha\supset(X^+)^\alpha(|\lambda|,p')$.

Now suppose, contrary to our claim, that there exists $y\in(X(|\lambda|,p))^\alpha\setminus(X^+)^\alpha(|\lambda|,p')$. Then $u:=T^{p'}(y)\notin(X^+)^\alpha$, so we can find $z\in X^+$ such that $\sum_n|u_nz_n|=\sum_n u_nz_n=\infty$. We choose a sequence $(\varepsilon_n)$ with $0<\varepsilon_n<u_n$ $(n\in\mathbb{N})$ such that $\sum_n\varepsilon_nz_n<1$.

For every $n \in \mathbb{N}$ we consider the functional

$$f_n : l_p^{\lambda_{n+1}-\lambda_n} \to \mathbb{R}, \quad f_n(t) = \sum_{k=\lambda_n}^{\lambda_{n+1}-1} y_k t_{k-\lambda_n+1}.$$

These functionals are continuous and satisfy

$$\|f_n\| = \|(y_k)_{k=\lambda_n}^{\lambda_{n+1}-1}\|_{p'} = [T^{p'}(y)]_n = u_n \quad (n \in \mathbb{N}).$$

So we can find $x \in \omega$ such that

$$[T^p(x)]_n = z_n \quad \text{and} \quad |f_n((x_k)_{k=\lambda_n}^{\lambda_{n+1}-1})| > ([T^{p'}(y)]_n - \varepsilon_n) z_n \quad (n \in \mathbb{N}).$$

Hence $T^p(x) = z \in X^+ \subset X$, therefore $x \in X(|\lambda|, p)$. On the other hand,

$$\sum_k |x_k y_k| \ge \sum_n \Big| \sum_{k=\lambda_n}^{\lambda_{n+1}-1} x_k y_k \Big| \ge \sum_n ([T^{p'}(y)]_n - \varepsilon_n) z_n \ge \sum_n u_n z_n - 1 = \infty,$$

hence $y \notin (X(|\lambda|, p))^\alpha$. This contradiction proves that $y \in (X^+)^\alpha(|\lambda|, p')$. Hence $(X(|\lambda|, p))^\alpha = (X^+)^\alpha(|\lambda|, p')$.

LEMMA 2.2. *Let $(X, \tau_X)$ be a K-space with the topology generated by the system of seminorms $\{p \mid p \in \mathcal{P}\}$. Then $X(|\lambda|, q)$ is a K-space with the topology $\tau$ generated by the system of seminorms $\{\tilde{p} \mid p \in \mathcal{P}\}$ defined by*

$$\tilde{p}(x) = p(T^q(x)) \quad (x \in X(|\lambda|, q);\, p \in \mathcal{P}).$$

*Proof.* To show that $X(|\lambda|, q)$ is a K-space, we suppose that $(x^{(n)})$ converges to $x$ in $(X(|\lambda|, q), \tau)$. Then, since $X$ is a K-space, we have

$$[T^q(x^{(n)} - x)]_i = \Big( \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |x_k^{(n)} - x_k|^q \Big)^{1/q} \to 0 \quad \text{as } n \to \infty \ (i \in \mathbb{N}).$$

Since

$$|x_k^{(n)} - x_k| \le \Big( \sum_{j=\lambda_i}^{\lambda_{i+1}-1} |x_j^{(n)} - x_j|^q \Big)^{1/q} \quad (\lambda_i \le k < \lambda_{i+1};\, i, k \in \mathbb{N}),$$

the K-property of $(X(|\lambda|, q), \tau)$ follows.

In order to spread an FK-property from $X$ to $X(|\lambda|, q)$, we assume that the topology of $X$ is consistent with the natural order. More precisely, we asssume that seminorms $\{p_k\}$ generating the FK-topology of $X$ satisfy

$$u, v \in X,\ 0 \le u \le v \ \Rightarrow\ p_k(u) \le p_k(v) \quad (k \in \mathbb{N}). \tag{2.4}$$

Moreover, we require the condition

$$p_k(u) = \sup_n p_k(u^{[n]}) \quad (k \in \mathbb{N};\, u \in X^+). \tag{2.5}$$

Note that (2.5) is stronger than the AB-property and, on the assumption that (2.4) is satisfied, weaker than the AK-property. Obviously the norms

$\| \ \|_q$ $(1 \le q \le \infty)$ and $\| \ \|_{bs}$ satisfy both (2.4) and (2.5) while $\| \ \|_{bv}$ fails to have both (2.4) and (2.5).

PROPOSITION 2.3. *Let $X$ be an FK-space with the topology $\tau_X$ generated by a system of seminorms $\{p_k\}$ satisfying* (2.4) *and* (2.5)*. Then $X(|\lambda|, q)$ is an FK-space with the topology $\tau$ generated by the system of seminorms $\{\tilde{p}_k\}$ defined by*

$$\tilde{p}_k(x) = p_k(T^q(x)) \quad (x \in X(|\lambda|, q);\ k \in \mathbb{N}).$$

*Proof.* In view of Lemma 2.2 it is sufficient to prove that $(X(|\lambda|, q), \tau)$ is complete. Suppose that $(x^{(n)})$ is a Cauchy sequence in $(X(|\lambda|, q), \tau)$. By the K-property of $(X(|\lambda|, q), \tau)$ the sequence $(x_k^{(n)})$ is a Cauchy sequence for every $k \in \mathbb{N}$, hence $(x^{(n)})$ converges coordinatewise to some $x \in \omega$. Since

$$|[T^q(x^{(m)})]_i - [T^q(x^{(n)})]_i| \le [T^q(x^{(m)} - x^{(n)})]_i \quad (i \in \mathbb{N}),$$

(2.4) implies that $T^q(x^{(n)})$ is a Cauchy sequence in $(X, \tau_X)$, hence converges to some $z \in X$. By the K-property of $(X, \tau_X)$ we have

$$[T^q(x^{(n)})]_i = \Big( \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |x_k^{(n)}|^q \Big)^{1/q} \to z_i \quad \text{as } n \to \infty\ (i \in \mathbb{N}).$$

On the other hand, by the K-property of $(X(|\lambda|, q), \tau)$ it follows that

$$\Big( \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |x_k^{(n)}|^q \Big)^{1/q} \to \Big( \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |x_k|^q \Big)^{1/q} \quad \text{as } n \to \infty\ (i \in \mathbb{N}).$$

Hence $T^q(x) = z \in X$, implying $x \in X(|\lambda|, q)$. Now we prove that $x^{(n)} \to x$ in $(X(|\lambda|, q), \tau)$. We set $u^{(n)} := T^q(x^{(n)} - x)$ $(n \in \mathbb{N})$. Let $\varepsilon > 0$ and $k, s \in \mathbb{N}$. We choose $N \in \mathbb{N}$ such that $p_k(T^q(x^{(i)} - x^{(j)})) \le \varepsilon/2$ for $i, j \ge N$. In view of the K-property we can choose $i_0 \ge N$ such that

$$\sum_{r=1}^{\lambda_{s+1}-1} |x_r^{(i_0)} - x_r| p_k(e^r) \le \frac{\varepsilon}{2}.$$

Then for every $i \ge N$ by (2.4) we get

$$p_k((u^{(i)})^{[s]}) \le p_k(T^q(x^{(i)} - x^{(i_0)})) + \sum_{r=1}^{\lambda_{s+1}-1} |x_r^{(i_0)} - x_r| p_k(e^r) \le \varepsilon.$$

Then by (2.5) also $p_k(u^{(i)}) \le \varepsilon$ for $i \ge N$, hence $x^{(n)} \to x$ in $(X(|\lambda|, q), \tau)$, implying that $(X(|\lambda|, q), \tau)$ is complete.

LEMMA 2.4. *If $1 \le p < q \le \infty$, then $X(|\lambda|, p) \subset X(|\lambda|, q)$.*

*Proof.* For $p < q$, $n \in \mathbb{N}$ and $x \in \omega$ we get

$$0 \le [T^q(x)]_n \le [T^p(x)]_n \quad (n \in \mathbb{N}).$$

Since $X$ is positively solid, $T^p(x) \in X$ implies $T^q(x) \in X$.

REMARK 2.5. 1) Evidently $X(|\lambda|, p) = X(|\lambda|, q)$ for any $1 \le p, q \le \infty$ if $X^+ = |\omega|$. We also construct an example of $X$ which satisfies the first equality, but $X^+ \subsetneq |\omega|$. Let $(\lambda_n)$ be an index sequence satisfying (1.1). We set

$$X := \{z \in \omega \mid \sqrt[\lambda_{n+1}-\lambda_n]{|z_n|} \to 0\}$$

and note that $X$ is solid. Since

$$\sqrt[\lambda_{n+1}-\lambda_n]{[T^\infty(x)]_n} \le \sqrt[\lambda_{n+1}-\lambda_n]{[T^1(x)]_n} \le \sqrt[\lambda_{n+1}-\lambda_n]{(\lambda_{n+1}-\lambda_n)[T^\infty(x)]_n}$$

for every $n \in \mathbb{N}$, we get

$$\lim_n \sqrt[\lambda_{n+1}-\lambda_n]{[T^\infty(x)]_n} = \lim_n \sqrt[\lambda_{n+1}-\lambda_n]{[T^1(x)]_n}.$$

Hence Lemma 2.4 implies $X(|\lambda|, 1) = X(|\lambda|, p) = X(|\lambda|, \infty)$ for every $p > 1$.

2) For any $p, q$ with $1 < p < q < \infty$ we have

$$\ell^\infty(|\lambda|) = \ell^\infty(|\lambda|, 1) \subsetneq \ell^\infty(|\lambda|, p) \subsetneq \ell^\infty(|\lambda|, q) \subsetneq \ell^\infty(|\lambda|, \infty) = \ell^\infty.$$

To prove $\ell^\infty(|\lambda|, p) \subsetneq \ell^\infty(|\lambda|, q)$ for $1 \le p < q < \infty$ we define $x_k := 1/(\lambda_{n+1} - \lambda_n)^{1/q}$ for $\lambda_n \le k < \lambda_{n+1}$ and $n \in \mathbb{N}$. Then

$$\sup_n [T^q(x)]_n = 1, \quad \sup_n [T^p(x)]_n = \sup_n (\lambda_{n+1} - \lambda_n)^{(q-p)/qp} = \infty.$$

So $x \in \ell^\infty(|\lambda|, q) \setminus \ell^\infty(|\lambda|, p)$. To verify $\ell^\infty(|\lambda|, q) \subsetneq \ell^\infty$ we consider $x = e$.

In [4, Proposition 2.1] it was shown that

$$(\chi \cap \ell^\infty(|\lambda|, 1))^\alpha = (\ell^\infty(|\lambda|, 1))^\alpha = \ell^1(|\lambda|, \infty) = |\ell^\infty|^\alpha(|\lambda|, \infty).$$

We will prove that the same statement holds if we take $X$ instead of $\ell^\infty$ on assumption $\chi(X)^\alpha = (X^+)^\alpha$ (which is satisfied for $X = \ell^\infty$).

LEMMA 2.6. *Let $X$ satisfy $\chi(X)^\alpha = (X^+)^\alpha$. Then*

1) $(\chi \cap X(|\lambda|, 1))^\alpha = (\chi \cap X(|\lambda|, 1))^\beta = (X^+)^\alpha(|\lambda|, \infty)$;
2) $(\chi \cap X(|\lambda|, \infty))^\alpha = (\chi \cap X(|\lambda|, \infty))^\beta = (X^+)^\alpha(|\lambda|, 1)$.

*Proof.* 1) Since $\chi \cap X(|\lambda|, 1)$ is monotone and $(\chi \cap X(|\lambda|, 1))^\alpha \supset (X(|\lambda|, 1))^\alpha$, it is sufficient to prove that $(\chi \cap X(|\lambda|, 1))^\alpha \subset (X(|\lambda|, 1))^\alpha = (X^+)^\alpha(|\lambda|, \infty)$. Let $y \in (\chi \cap X(|\lambda|, 1))^\alpha \setminus (X^+)^\alpha(|\lambda|, \infty)$. Then $u := T^\infty(y) \notin (X^+)^\alpha = \chi(X)^\alpha$, so we can find $z \in \chi \cap X$ with $\sum_n |u_n z_n| = \sum_n u_n z_n = \infty$. We put $x_{\xi_n} = z_n$ and $x_i := 0$ for $i \notin \{\xi_k \mid k \in \mathbb{N}\}$, where $\xi_n$ is the minimal index $i_0 \in [\lambda_n, \lambda_{n+1})$ with $|y_{i_0}| = \max\{|y_i| \mid \lambda_n \le i < \lambda_{n+1}\}$ $(n \in \mathbb{N})$. Evidently $x \in \chi \cap X(|\lambda|, 1)$. On the other hand,

$$\sum_k |y_k x_k| = \sum_n u_n z_n = \infty,$$

which contradicts $y \in (\chi \cap X(|\lambda|, 1))^\alpha$.

2) The proof of 2) is analogous to 1) except that the definition of $x$ is now $x_k := z_n$ for $\lambda_n \leq k < \lambda_{n+1}$ $(n \in \mathbb{N})$.

REMARK 2.7. The equality $(\chi \cap X(|\lambda|, p))^\alpha = (X^+)^\alpha(|\lambda|, p')$ for $p > 1$ may fail even for $X$ satisfying $\chi(X)^\alpha = (X^+)^\alpha$. Note that for $q \in [1, \infty)$ we get

$$\chi \cap \ell^\infty(|\lambda|, q) = \{x \in \chi \mid \sup_n |\{k \in \mathbb{N} \mid x_k = 1\} \cap [\lambda_n, \lambda_{n+1})| < \infty\}.$$

So by Lemma 2.6, $(\chi \cap \ell^\infty(|\lambda|, p))^\alpha = \ell^1(|\lambda|, \infty)$.

We will now verify that the converse statement for Lemma 2.6 holds even if we replace 1 with $p$ and $\infty$ with $p'$.

LEMMA 2.8. *If* $X(|\lambda|, p)^\alpha = (\chi \cap X(|\lambda|, p))^\alpha$, *then* $(X^+)^\alpha = (\chi \cap X^+)^\alpha$.

*Proof.* It suffices to prove that $(\chi \cap X^+)^\alpha \subset (X^+)^\alpha$. So let $w \in (\chi \cap X^+)^\alpha$. Then $\sum_k |w_k u_k| < \infty$ for every $u \in \chi \cap X^+$. We set $y_{\lambda_i} := w_i$ $(i \in \mathbb{N})$ and $y_k := 0$ for $k \notin \{\lambda_i \mid i \in \mathbb{N}\}$. We verify that $y = (y_i) \in (\chi \cap X(|\lambda|, p))^\alpha$. Indeed, let $x \in \chi \cap X(|\lambda|, p)$ and set $u_i := \tilde{x}_{\lambda_i} := x_{\lambda_i}$ $(i \in \mathbb{N})$, $\tilde{x}_k := 0$ for $k \notin \{\lambda_i \mid i \in \mathbb{N}\}$. Since $X(|\lambda|, p)$ is solid, $\tilde{x} \in \chi \cap X(|\lambda|, p)$. Then $u = (u_i) = T^p(\tilde{x}) \in \chi \cap X^+$. Therefore

$$\sum_k |y_k x_k| = \sum_i |y_{\lambda_i} x_{\lambda_i}| = \sum_i w_i u_i < \infty.$$

Therefore $y \in (\chi \cap X(|\lambda|, p))^\alpha = X(|\lambda|, p)^\alpha$, hence $\sum_k |y_k x_k| < \infty$ for every $x \in X(|\lambda|, p)$. So if we take $u \in X^+$ and consider $x$ with $x_{\lambda_i} = u_i$ $(i \in \mathbb{N})$ and $x_k = 0$ for $k \notin \{\lambda_i \mid i \in \mathbb{N}\}$ we obtain $u = T^p(x)$ and

$$\sum_i |w_i u_i| = \sum_k |x_k y_k| < \infty.$$

Hence $w \in (X^+)^\alpha$.

THEOREM 2.9. *If* $\varphi \subset X$ *and* $\langle X^+ \rangle$ *has the matrix Hahn property*, *then* $X(|\lambda|, 1)$ *and* $X(|\lambda|, \infty)$ *have the separable Hahn property.*

*Proof.* First we verify that $X(|\lambda|, 1)$ has the separable Hahn property. In view of [3, Proposition 1 and Theorem 2] it is sufficient to prove that $\chi \cap X(|\lambda|, 1) \subset c_{0A}$ implies $X(|\lambda|, 1) \subset c_{0A}$. We define

$$b_{ni} := \max_{\lambda_i \leq k < \lambda_{i+1}} |a_{nk}| \quad (n, i \in \mathbb{N})$$

and verify that $\chi \cap X \subset c_{0B}$. Since $(a_{nk})_k \in (\chi \cap X(|\lambda|, 1))^\beta$, by Lemma 2.6

we have $(b_{nk})_k \in (X^+)^\alpha$ $(n \in \mathbb{N})$. If we suppose on the contrary that

$$\sum_k b_{n_i k} u_k \geq 4\varepsilon \quad (i \in \mathbb{N})$$

for some $\varepsilon > 0$, $u \in \chi \cap X$ and an index sequence $(n_i)$, then by the usual gliding hump argument we may choose an index sequence $(k_i)$ and a subsequence $(m_i)$ of $(n_i)$ such that

$$\begin{gathered}\sum_{k=k_{p-1}+1}^{k_p} b_{m_p k} u_k \geq 3\varepsilon, \qquad \sum_{k=k_p+1}^{\infty} b_{m_p k} u_k \leq \varepsilon, \\ \sum_{k=1}^{k_{p-1}} b_{m_p k} u_k \leq \varepsilon \quad (p \in \mathbb{N}).\end{gathered} \tag{2.6}$$

For $p \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k_{p-1} < k \leq k_p$ let $\xi_k$ denote the minimal index $j$ with $\lambda_k \leq j < \lambda_{k+1}$ such that $|a_{m_p j}| = \max\{|a_{m_p i}| \mid \lambda_k \leq i < \lambda_{k+1}\}$. We set $x_{\xi_k} := \operatorname{sgn}(a_{m_p \xi_k}) u_k$ for $k \in \mathbb{N}$ with $k_{p-1} < k \leq k_p$ and $x_i := 0$ for $i \notin \{\xi_k \mid k \in \mathbb{N}\}$. Then $x \in \chi(X(|\lambda|, 1))$. Applying (2.6) for every $p \geq 2$ we have

$$\begin{aligned}\Big|\sum_{i=1}^{\lambda_{k_{p-1}+1}-1} a_{m_p i} x_i\Big| &\leq \sum_{i=1}^{k_{p-1}} \max_{\lambda_i \leq k < \lambda_{i+1}} |a_{m_p k}| \Big|\sum_{k=\lambda_i}^{\lambda_{i+1}-1} x_k\Big| = \sum_{i=1}^{k_{p-1}} b_{m_p i} u_i < \varepsilon, \\ \Big|\sum_{i=\lambda_{k_p+1}}^{\infty} a_{m_p i} x_i\Big| &\leq \sum_{i=k_p}^{\infty} \max_{\lambda_i \leq k < \lambda_{i+1}} |a_{m_p k}| \Big|\sum_{k=\lambda_i}^{\lambda_{i+1}-1} x_k\Big| = \sum_{i=k_p}^{\infty} b_{m_p i} u_i < \varepsilon.\end{aligned}$$

Now for every $p \in \mathbb{N}$ we get

$$\sum_i a_{m_p i} x_i = \sum_{i=1}^{\lambda_{k_{p-1}+1}-1} a_{m_p i} x_i + \sum_{k=k_{p-1}+1}^{k_p} b_{m_p k} u_k + \sum_{i=\lambda_{k_p+1}}^{\infty} a_{m_p i} x_i \geq \varepsilon,$$

contrary to $x \in c_{0A}$. Hence $\chi \cap X \subset c_{0B}$, implying $X^+ \subset c_{0B}$.

Now for every $x \in X(|\lambda|, 1)$ we get

$$\Big|\sum_k a_{nk} x_k\Big| \leq \sum_i \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |a_{nk}|\, |x_k| \leq \sum_i b_{ni} [T^1(x)]_i \quad (n \in \mathbb{N}),$$

so $x \in c_{0A}$.

For $X(|\lambda|, \infty)$ we use the same idea of proof except that we define $B = (b_{ni})$ and $x$ by setting $b_{ni} := \sum_{k=\lambda_i}^{\lambda_{i+1}-1} |a_{nk}|$ $(n, i \in \mathbb{N})$ and $x_j := u_k \operatorname{sgn}(a_{m_p j})$ for $p, k, j \in \mathbb{N}$ with $k_{p-1} < k \leq k_p$ and $\lambda_k \leq j < \lambda_{k+1}$.

REMARK 2.10. 1) Note that $X(|\lambda|, 1)$ may fail to have the Hahn property even if $\langle X^+\rangle$ has the Hahn property. As an example consider the space $\ell^\infty(|\lambda|, 1)$ (cf. [4, Corollary 2.5]).

2) In view of Remark 2.7 the space $X(|\lambda|, p)$ for $p \in (1, \infty)$ fails in general to have the matrix Hahn property.

The following result demonstrates that $X(|\lambda|, 1)$ has the separable Hahn property if and only if $\langle X^+\rangle$ does.

PROPOSITION 2.11. *If $X(|\lambda|, p)$ has the matrix Hahn property, then $\langle X^+\rangle$ has the separable Hahn property.*

*Proof.* Since $\langle X^+\rangle$ is solid, by [3, Proposition 1 and Theorem 2] it is sufficient to prove that $\chi \cap \langle X^+\rangle \subset c_{0B}$ implies $\langle X^+\rangle \subset c_{0B}$.

Suppose on the contrary that we can find a matrix $B = (b_{nk})$ and $u \in X^+$ such that $\chi \cap X^+ \subset c_{0B}$, but $u \notin c_{0B}$.

We define the matrix $A = (a_{nk})$ and the sequence $x = (x_k)$ by $a_{n\lambda_i} := b_{ni}$, $x_{\lambda_i} := u_i$ $(n, i \in \mathbb{N})$ and $a_{nk} := x_k := 0$ for $k \notin \{\lambda_i \mid i \in \mathbb{N}\}$ and $n \in \mathbb{N}$. Evidently, $x \in X(|\lambda|, p)$.

We will verify that $\chi \cap X(|\lambda|, p) \subset c_{0A}$, but $x \notin c_{0A}$, which would imply that $X(|\lambda|, p)$ does not have the matrix Hahn property.

To prove the first statement let $y \in \chi \cap X(|\lambda|, p)$ and set $v_k := y_{\lambda_k}$ $(k \in \mathbb{N})$. Since $0 \le v_k \le [T^p(y)]_k$ $(k \in \mathbb{N})$, we have $v = (v_k) \in \chi \cap X^+ \subset c_{0B}$. So

$$\lim_n \sum_i a_{ni} y_i = \lim_n \sum_k a_{n\lambda_k} y_{\lambda_k} = \lim_n \sum_k b_{nk} v_k = 0.$$

Hence $y \in c_{0A}$, implying $\chi \cap X(|\lambda|, p) \subset c_{0A}$.

On the other hand, $(\sum_k a_{nk} x_k)_n = (\sum_k b_{nk} u_k)_n \notin c_0$, that is, $x \notin c_{0A}$.

## References


[1] G. Bennett, J. Boos, and T. Leiger, *Sequences of 0's and 1's,* Studia Math. 149 (2002), 75–99.

[2] J. Boos, *Classical and Modern Methods in Summability,* Oxford Univ. Press, New York, Oxford, 2000.

[3] J. Boos and T. Leiger, *Addendum: Sequences of 0's and 1's (Studia Math. 149 (2002), 75–99)*, Studia Math. 171 (2005), 305–309.

[4] —, —, *Sequences of 0's and 1's: special sequence spaces with the separable Hahn property*, Acta Math. Hungar., to appear (2006).

[5] J. Boos and M. Zeltser, *Sequences of 0's and 1's. Classes of concrete 'big' Hahn spaces*, Z. Anal. Anwend. 22 (2003), 819–842.

[6] K.-G. Grosse-Erdmann, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality Series*, Lecture Notes in Math. 1679, Springer, Berlin, 1998.

[7] C. E. Stuart and P. Abraham, *Generalizations of the Nikodym boundedness and Vitali–Hahn–Saks theorems*, J. Math. Anal. Appl. 300 (2004), 351–361.

[8] M. Zeltser, J. Boos and T. Leiger, *Sequences of 0's and 1's: New results via double sequence spaces*, J. Math. Anal. Appl. 275 (2002), 883–899.

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