

Paraconvex functions and paraconvex sets

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Abstract. We study a class of functions which contains both convex functions and differentiable functions whose derivatives are locally Lipschitzian or Hölderian. This class is a subclass of the class of approximately convex functions. It enjoys refined properties. We also introduce a class of sets whose associated distance functions are of that type. We discuss the properties of the metric projections on such sets under some assumptions on the geometry of the Banach spaces in which they are embedded. We describe some relations between such sets and functions.

1. Introduction. Some classes of sets or functions are so favorable for smooth and nonsmooth analysis that they have been considered by a number of authors. This is the case for the class of (locally) p -paraconvex functions introduced in [56] and studied in [3], [5], [8], [9], [10], [12], [15], [30], [45], [51], [56]–[69], especially in the case $p = 2$ in which case the functions are also called semiconvex as in [32] and the book [11] which stresses their applications in optimal control and the study of Hamilton–Jacobi equations. This class contains the class of convex functions and those of class C^2 and enjoys interesting stability properties. In the present paper we study the notion of p -paraconvexity for sets introduced in [69] (in a global sense and for $p = 2$), studied in [15, Section 5] under the name of property (ω) and called sub-smoothness in [5]. We also introduce a notion of intrinsically p -paraconvex set. Both notions are given in terms of the distance function to the set. As in the case of the (more general) classes of approximately convex functions and sets in the sense of [5], [15], [18], [38], [39], [40] which differs from the one used in [24], [25], we provide characterizations. It turns out that the classes of p -paraconvex functions and sets form more structured classes in which one has a quantitative control of the variations of the subdifferentials of the functions or of the normal cone to the set at nearby points respectively. For instance, in the class of superreflexive Banach spaces, we show

2000 *Mathematics Subject Classification*: 49J52, 46N10, 46T20.

Key words and phrases: approximately convex function, approximately convex set, monotonicity, nonsmooth analysis, normal, projection, subdifferential.

that best approximations to p -paraconvex sets exist and we prove the Hölder continuity of projections, obtaining results somewhat similar to the case of convex subsets (see [1], [2], [50] and their references). We also present conditions ensuring that a marginal function is p -paraconvex (see Section 4), thus solving an unsettled problem. We relate the concepts of paraconvexity for sets and functions through epigraphs and sublevel sets in Section 7. Let us add that for the class of p -paraconvex functions (resp. sets) the usual notions of subdifferential (resp. normal) coincide; this fact makes these classes particularly attractive.

The concept of p -paraconvexity around a point we study is a localization of a notion introduced in [56]; let us note that the version we study is not global as in [10], [56]–[62], [69]. However, it amounts to p -paraconvexity in the sense of [56] on some neighborhood of the point. Since it has been proved in [56] that for $p > 2$ a p -paraconvex function whose restrictions to line segments are absolutely continuous is convex, in the following we restrict our attention to the case $p \in [1, 2]$.

DEFINITION 1. Given some $p \in [1, 2]$, a function $f : X \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a normed vector space X is said to be *p -paraconvex around $\bar{x} \in \text{dom } f := f^{-1}(\mathbb{R})$* if there exist $c, \delta > 0$ such that for any $x, y \in B(\bar{x}, \delta)$ and any $t \in [0, 1]$,

$$(1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + ct(1-t)\|x - y\|^p.$$

The function f is said to be (*locally*) *p -paraconvex on an open subset U* of X if it is finite and p -paraconvex around each point of U . The *index of p -paraconvexity* of f around \bar{x} is the infimum c_f of the set of real numbers c such that there exists some $\delta > 0$ for which relation (1) is satisfied for any $x, y \in B(\bar{x}, \delta)$ and $t \in [0, 1]$.

When c_f is negative, we have in fact a strong convexity property; the case $c_f = 0, p = 1$ corresponds to approximate convexity (see [38], [39]). The case $p = 2$ is of special interest, in particular in the setting of Euclidean spaces or Hilbert spaces. As mentioned above, it has been thoroughly studied, at least in finite dimensions, in Hilbert spaces, and in the locally Lipschitzian case. Here we get rid of these restrictions. We relate the notion of p -paraconvexity to the following notion of p -paramonotonicity. For $p = 2$ it corresponds to a localized version of a concept introduced by A. Pazy [42] (see also [17], [43] for related notions).

DEFINITION 2. A multimap (or set-valued map) $M : X \rightrightarrows X^*$ is said to be *p -paramonotone around \bar{x} on a subset E* of X if there exist some $m, \delta > 0$ such that for any $x_1, x_2 \in E \cap B(\bar{x}, \delta)$, $x_1^* \in M(x_1)$ and $x_2^* \in M(x_2)$,

$$(2) \quad \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -m\|x_1 - x_2\|^p.$$

For $E = X$ one simply says that M is p -paramonotone around \bar{x} . The index of p -paramonotonicity of M on E around \bar{x} is the infimum m_M of the set of real numbers $m > 0$ such that there exists some $\delta > 0$ for which relation (2) is satisfied for any $x_1, x_2 \in E \cap B(\bar{x}, \delta)$, $x_1^* \in M(x_1)$ and $x_2^* \in M(x_2)$.

Directional versions of the concepts we study can be introduced, but for brevity we do not consider them here. We also leave apart the generalization of p -paraconvexity consisting in replacing the term $ct(1-t)\|x-y\|^p$ in (1) by $ct(1-t)\alpha(\|x-y\|)$ where $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ is a hypermodulus, i.e. a nondecreasing function such that $\alpha(t)/t \rightarrow 0$ as $t \rightarrow 0_+$. The reason for that choice lies in the Hölder estimates we get for the negative of the distance function to an arbitrary closed subset of a uniformly smooth Banach space (Corollary 13) and for the metric projection onto a p -paraconvex subset of a superreflexive Banach space (Theorem 30); this last topic has been thoroughly studied for convex subsets (see [1], [2] for instance). The notion of p -paraconvex subset is studied in Sections 5 and 6. Although the choice $p = 2$ is the most important one, the case $p \neq 2$ may be of interest, for instance when working in some L_p space, as illustrated by Proposition 24 below.

2. Preliminaries. In the following, X is a Banach space with topological dual space X^* and $\mathcal{S}(X)$ denotes the set of lower semicontinuous (l.s.c.) functions $f : X \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. The open ball with center $\bar{x} \in X$ and radius $\rho > 0$ is denoted by $B(\bar{x}, \rho)$, while \bar{B}_X (resp. \bar{B}_{X^*}) stands for the closed unit ball of X (resp. X^*) and S_X stands for the unit sphere. Given a subset E of X , the distance function d_E associated with E is given by $d_E(x) := \inf_{e \in E} d(x, e)$. We write $(x_n) \xrightarrow{E} x$ to mean that $(x_n) \rightarrow x$ and $x_n \in E$ for each $n \in \mathbb{N}$.

Since the functions we consider are not necessarily smooth, we have to use some generalizations of derivatives. A first possibility is to take a primal approach by considering directional derivatives. A second approach is a dual one; it is a one-sided substitute to the derivative called a subdifferential. Several sets of axioms have been devised for such a concept. The notion of subdifferential we adopt here is as versatile as possible: a *subdifferential* on a Banach space X will just be a correspondence $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ which assigns a subset $\partial f(x)$ of the dual space X^* of X to any f in a subclass $\mathcal{F}(X)$ of the set $\mathcal{S}(X)$ of lower semicontinuous functions on X and any $x \in X$ at which f is finite; we assume it has the following natural property:

- (M) If $f \in \mathcal{F}(X)$ is a Lipschitzian function and if \bar{x} is a minimizer of f , then $0 \in \partial f(\bar{x})$.

Of course, usual subdifferentials also satisfy other conditions which turn them into calculus properties. These include a form of the mean value the-

orem we describe below. Another property is the nonemptiness of $\partial f(x)$ for x in a dense subset of X for any Lipschitzian function f on X ; then we say that X is a (Lipschitz) ∂ -subdifferentiability space. This property is satisfied if X is an Asplund space and ∂ is larger than the Fréchet subdifferential (sometimes called the firm subdifferential) defined by

$$x^* \in \partial^F f(x) \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0, \forall u \in \delta \bar{B}_X : f(x+u) \geq f(x) + \langle x^*, u \rangle - \varepsilon \|u\|.$$

If X is smooth enough, for instance if X is a Hilbert space, and if $p \in]1, 2]$, this property is satisfied by the p -proximal subdifferential ∂^p ; recall that for $p \in (1, +\infty)$, the p -proximal subdifferential of a function f on a n.v.s. X at $x \in X$ is given by

$$x^* \in \partial^p f(x) \Leftrightarrow \exists c > 0, \varrho > 0, \forall u \in \varrho \bar{B}_X : f(x+u) \geq f(x) + \langle x^*, u \rangle - c \|u\|^p.$$

Clearly $\partial^p f(x)$ is contained in the Fréchet subdifferential $\partial^F f(x)$. Albeit property (M) is satisfied for ∂^p for any $p > 1$, the practical use of ∂^p seems to be limited to the case $p \in]1, 2]$.

With any subdifferential ∂ is associated a notion of *normal cone* obtained by setting, for a subset E of X and $e \in E$,

$$N(E, e) := \mathbb{R}_+ \partial \iota_E(e),$$

where ι_E is the *indicator function* of E , given by $\iota_E(x) = 0$ if $x \in E$, $+\infty$ otherwise. In the cases $\partial = \partial^D$, $\partial = \partial^C$ and $\partial = \partial^F$ we get the *normal cones* $N^D(E, e)$, $N^C(E, e)$ and $N^F(E, e)$ to E at $e \in E$ in the senses of Bouligand, Clarke and Fréchet respectively, as defined below. Other notions have been devised by Ioffe [26], [27], Michel–Penot [36], Mordukhovich [37], Treiman [68] and several other authors.

Conversely, to a notion N of normal cone one can associate a notion of subdifferential ∂ by setting $E_f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$, $x_f := (x, f(x))$ and

$$\partial f(x) := \{x^* \in X^* : (x^*, -1) \in N(E_f, x_f)\}.$$

Now let us give the examples of those concepts we will use most. The (contingent or directional or Dini–Hadamard) *normal cone* $N^D(E, x)$ to a subset E of X at $x \in \text{cl}(E)$ is the polar cone of the directional (or Dini–Hadamard or Bouligand) *tangent cone* $T^D(E, x)$ to E at x , which is the set of vectors $v \in X$ such that there exist sequences $(t_n) \rightarrow 0_+$ and $(x_n) \xrightarrow{E} x$ for which $(t_n^{-1}(x_n - x)) \rightarrow v$. Both cones play a crucial role in nonlinear analysis and optimization.

The *Fréchet normal cone* (or *firm normal cone*) to E at x is given by $x^* \in N^F(E, x) \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0, \forall u \in E \cap B(x, \delta) : \langle x^*, u - x \rangle \leq \varepsilon \|u - x\|$. The *Clarke normal cone* $N^C(E, x)$ to a subset E of X at $x \in \text{cl}(E)$ is the polar cone of the *Clarke tangent cone* $T^C(E, x)$ to E at x , which is the set

of vectors $v \in X$ such that for any sequence $(x_n) \xrightarrow{E} x$ there exist sequences $(t_n) \rightarrow 0_+$ and (y_n) in E for which $(t_n^{-1}(y_n - x_n)) \rightarrow v$.

The (lower) *directional derivative* (or contingent derivative or lower Dini–Hadamard derivative) of f at x is given by

$$f^D(x, v) := \liminf_{(t,w) \rightarrow (0_+,v)} \frac{f(x + tw) - f(x)}{t}.$$

The *Clarke–Rockafellar derivative* or *circa-derivative* of a function $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ finite at x is given by the following formula:

$$f^C(x, v) := \inf_{r>0} \limsup_{\substack{f(y) \rightarrow f(x) \\ (t,y) \rightarrow (0_+,x)}} \inf_{w \in B(v,r)} \frac{f(y + tw) - f(y)}{t}.$$

Both derivatives are majorized by the *dag derivative* which is given by

$$f^\dagger(x, v) := \limsup_{(t,y) \rightarrow (0_+,x), f(y) \rightarrow f(x)} \frac{f(y + t(v + x - y)) - f(y)}{t}.$$

The interest of such a derivative seems to be limited to questions of generalized convexity and to the fact that the associated subdifferential is large. Here, the subdifferential *associated* with some generalized derivative f' is defined by

$$\partial f(x) := \{x^* \in X^* : x^*(\cdot) \leq f'(x, \cdot)\}.$$

For $f' = f^D$ (resp. $f' = f^C$) one recovers the Bouligand or Dini–Hadamard subdifferential ∂^D (resp. Clarke subdifferential ∂^C).

Given a subdifferential ∂ , following [31], [37], [47], one can associate to it a corresponding *limiting subdifferential* $\overline{\partial}$ by setting, for a l.s.c. function f and a point x of its domain,

$$\overline{\partial}f(x) := w^* \text{-} \limsup_{(u,f(u)) \rightarrow (x,f(x))} \partial f(u),$$

where the w^* -limsup is the set of weak* cluster points of bounded nets $(u_i^*)_{i \in I}$ with $u_i^* \in \partial f(u_i)$, $(u_i)_{i \in I} \rightarrow x$ and $(f(u_i))_{i \in I} \rightarrow f(x)$.

We will make use of mean value inequalities. Under a primal form, such estimates have been pointed out in [44]. In a dual form, as used here, they have been devised by Zagrodny for the Clarke subdifferential [71], [72], and by Loewen [34] for the Fréchet subdifferential. In [49], such a result is put in a general form including the Ioffe subdifferential [26], [27], the moderate subdifferential [36], and, in Asplund spaces, the Fréchet subdifferential, the Hadamard subdifferential and the limiting subdifferential. In [4] such a property is obtained under different assumptions.

DEFINITION 3 (Fuzzy mean value property). A subdifferential ∂ is said to be *valuable* on X if for any $\bar{x} \in X$ and $\bar{y} \in X \setminus \{\bar{x}\}$, any lower semicontinuous function $f : X \rightarrow \overline{\mathbb{R}} \cup \{+\infty\}$ finite at $\bar{x} \in X$, and any $r \in \mathbb{R}$ such

that $f(\bar{y}) \geq r$, there exist $u \in [\bar{x}, \bar{y})$ and sequences $(u_n) \rightarrow u$ and (u_n^*) such that $u_n^* \in \partial f(u_n)$, $(f(u_n)) \rightarrow f(u)$, and

$$(3) \quad \liminf_n \langle u_n^*, \bar{y} - \bar{x} \rangle \geq r - f(\bar{x}),$$

$$(4) \quad \liminf_n \langle u_n^*, x - u_n \rangle \geq \frac{\|x - u\|}{\|\bar{y} - \bar{x}\|} (r - f(\bar{x}))$$

$$\forall x \in (\bar{x} + \mathbb{R}_+(\bar{y} - \bar{x})) \setminus [\bar{x}, u),$$

$$(5) \quad \lim_n \|u_n^*\| d(u_n, [\bar{x}, \bar{y}]) = 0.$$

A simpler version due to Lebourg [33] is as follows. It is satisfied for $\partial := \partial^C$, the Clarke subdifferential, the moderate subdifferential ∂^M of [36], and if X is Asplund, by the Bouligand, Fréchet and limiting subdifferentials.

DEFINITION 4. A subdifferential ∂ is said to be *Lipschitz-valuable* on X if for any $\bar{x}, \bar{y} \in X$ and any Lipschitzian function $f : X \rightarrow \mathbb{R}$ one can find $w \in [\bar{x}, \bar{y}]$ and $w^* \in \partial f(w)$ such that $f(y) - f(x) = \langle w^*, y - x \rangle$.

We need some results about subdifferentials of distance functions.

LEMMA 5 ([40], [48]). *Let E be a nonempty subset of a Banach space X and let $x \in E$, $v \in X$. Then*

$$d_E^C(x, v) = \limsup_{t \rightarrow 0_+, e \xrightarrow{E} x} \frac{1}{t} d_E(e + tv).$$

The next result is close to [7, Lemma 6], [27, Lemma 5], [48, Lemma 1], [41, Lemma 3.6] and [5, Lemma 3.7], but it contains a crucial additional information obtained in [40]. Recall that the norm of X is said to satisfy the *Kadec-Klee property* if for every $x \in X$, a sequence (x_n) converges to x whenever it weakly converges to x and $(\|x_n\|) \rightarrow \|x\|$.

LEMMA 6. *Suppose that E is a closed nonempty subset of an Asplund space X , $w \in X \setminus E$ and $w^* \in \partial^F d_E(w)$. Then $\|w^*\| = 1$ and there exist sequences (x_n) , (x_n^*) in E and X^* respectively such that $x_n^* \in \partial^F d_E(x_n)$ for each $n \in \mathbb{N}$ and*

$$(\|x_n - w\|) \rightarrow d_E(w), \quad (\langle x_n^*, w - x_n \rangle) \rightarrow d_E(w), \quad (\|x_n^* - w^*\|) \rightarrow 0.$$

If moreover X is reflexive and if its norm has the Kadec-Klee property, then a subsequence of (x_n) converges to some best approximation x of w in E and $\langle w^, w - x \rangle = \|x - w\| = d_E(w)$.*

Now consider the *limiting normal cone*

$$\overline{N}(E, a) := \limsup_{x \xrightarrow{E} a} N(E, x)$$

and the limiting Fréchet normal cone

$$\overline{N^F}(E, a) := \limsup_{x \xrightarrow{E} a} N^F(E, x)$$

obtained from the normal cone (resp. the Fréchet normal cone) by the weak upper limit process described above.

COROLLARY 7 ([40]). *Let E be a closed nonempty subset of an Asplund space X and let $\bar{x}^* \in \overline{\partial^F} d_E(\bar{x})$ with $\bar{x} \in E$. Then $\|\bar{x}^*\| \leq 1$ and there exist sequences $(x_n), (x_n^*)$ in E and X^* respectively such that $(x_n) \rightarrow \bar{x}$, $(x_n^*) \rightarrow \bar{x}^*$ weak* and $x_n^* \in \partial^F d_E(x_n)$ for each $n \in \mathbb{N}$.*

3. Paraconvexity of functions. The following characterization of p -paraconvexity has been given in [14, Thm. 5.1] in the case where $p = 2$, X is finite-dimensional and the function is Lipschitzian; there, only the proximal subdifferential is used and equivalence with the lower- C^2 property in the sense of [54], [65] is proved. This characterization is extended to the case where X is a Hilbert space and the function is lower semicontinuous in [5, Thm. 4.1], using the Clarke–Rockafellar subdifferential. Here we consider the case of a general Banach space and of an arbitrary valuable subdifferential; the argument for the implication (d) \Rightarrow (a) is different.

THEOREM 8. *Let $p \in]1, 2]$, let $f \in \mathcal{F}(X)$ and let $\bar{x} \in \text{dom } f$. Suppose the subdifferential ∂f of f is contained in $\partial^\dagger f$. Then (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) below. If moreover ∂ is valuable, all these assertions are equivalent.*

- (a) f is p -paraconvex around \bar{x} ;
- (b) there exist $c \in \mathbb{R}$ and $\varrho > 0$ such that for any $x \in B(\bar{x}, \varrho) \cap \text{dom } f$ and $v \in B(0, \varrho)$,

$$f^\dagger(x, v) \leq f(x + v) - f(x) + c\|v\|^p;$$

- (c) there exist $c \in \mathbb{R}$ and $\varrho > 0$ such that for any $x \in B(\bar{x}, \varrho)$, $x^* \in \partial f(x)$ and $v \in B(0, \varrho)$,

$$(6) \quad \langle x^*, v \rangle \leq f(x + v) - f(x) + c\|v\|^p;$$

- (d) ∂f is p -paramonotone around \bar{x} .

Proof. (a) \Rightarrow (b). Given $c > c_f$, let $\delta > 0$ be such that for any $y, z \in B(\bar{x}, \delta)$ and $t \in]0, 1[$,

$$f((1-t)y + tz) \leq (1-t)f(y) + tf(z) + ct(1-t)\|y - z\|^p.$$

Set $\varrho := \delta/2$, fix $x \in B(\bar{x}, \varrho) \cap \text{dom } f$ and take $v \in B(0, \varrho)$ and $y \in B(x, \varrho) \cap \text{dom } f$. Then $z := x + v \in B(\bar{x}, \delta)$ and

$$\frac{f((1-t)y + t(x+v)) - f(y)}{t} \leq f(x+v) - f(y) + c(1-t)\|y - (x+v)\|^p,$$

so that, since $\limsup_{y \rightarrow x} -f(y) \leq -f(x)$,

$$f^\dagger(x, v) \leq f(x + v) - f(x) + c\|v\|^p.$$

(b) \Rightarrow (c) is an immediate consequence of the inclusion $\partial f \subset \partial^\dagger f$ and of the definition of $\partial^\dagger f$.

(c) \Rightarrow (d) is proved in [18, Thm. 2] and [5, Thm. 4.5] for the Clarke sub-differential. The general case is similar: given $c \in \mathbb{R}$ and ϱ as in (c), for any $x, y \in B(\bar{x}, \varrho/2)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$, taking $v := y - x$ and $v' := -v$ we have $x, y \in \text{dom } f$ and

$$\begin{aligned} \langle x^*, y - x \rangle &\leq f(y) - f(x) + c\|x - y\|^p, \\ \langle y^*, x - y \rangle &\leq f(x) - f(y) + c\|x - y\|^p, \end{aligned}$$

hence $\langle x^* - y^*, x - y \rangle \geq -2c\|x - y\|^p$.

(d) \Rightarrow (a) when ∂ is valuable. Assume ∂f is p -paramonotone around \bar{x} , that is, there exist $\varrho, m > 0$ such that for any $x, y \in B(\bar{x}, \varrho)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$,

$$(7) \quad \langle x^* - y^*, x - y \rangle \geq -m\|x - y\|^p.$$

Let $\delta := \varrho/2$, $x, y \in B(\bar{x}, \delta)$, $t \in]0, 1[$, $z := tx + (1 - t)y$ and $r < f(z)$. Without loss of generality, to prove (1) we may assume that $x \neq y$ and $f(x), f(y) < +\infty$. Since ∂ is valuable we can find $u \in [y, z]$ and sequences $(u_n) \rightarrow u$ and (u_n^*) in X^* such that $u_n^* \in \partial f(u_n)$ for each n and

$$(8) \quad \liminf_n \left\langle u_n^*, \frac{x - u_n}{\|x - u_n\|} \right\rangle > \frac{r - f(y)}{\|z - y\|}.$$

Let $s \in]0, 1[$ be such that $z = x + s(u - x)$ and set $z_n = x + s(u_n - x)$. Since $(u_n) \rightarrow u$, one has $(z_n) \rightarrow z$, and since f is l.s.c. at z , for n larger than a certain $n_0 \in \mathbb{N}$ one has $u_n, z_n \in B(\bar{x}, \varrho)$ and $r < f(z_n)$. Moreover $\|z_n - x\| = (1 - t_n)\|y - x\|$ for some sequence $(t_n) \rightarrow t$. Using again the fact that ∂ is valuable, we obtain $v_n \in [x, z_n]$ and sequences $(v_{n,k}) \rightarrow v_n$ and $(v_{n,k}^*)$ such that $v_{n,k} \in B(\bar{x}, \varrho)$ and $v_{n,k}^* \in \partial f(v_{n,k})$ for any $n \geq n_0$ and $k \in \mathbb{N}$, and

$$(9) \quad \liminf_k \left\langle v_{n,k}^*, \frac{u_n - v_{n,k}}{\|u_n - v_{n,k}\|} \right\rangle > \frac{r - f(x)}{\|z_n - x\|} = \frac{r - f(x)}{(1 - t_n)\|y - x\|}.$$

From (8) there exists some $n_1 \geq n_0$ such that for $n \geq n_1$, since x, z_n, u_n are aligned,

$$(10) \quad \left\langle u_n^*, \frac{v_n - u_n}{\|v_n - u_n\|} \right\rangle = \left\langle u_n^*, \frac{x - u_n}{\|x - u_n\|} \right\rangle > \frac{r - f(y)}{\|y - z\|} = \frac{r - f(y)}{t\|y - x\|}.$$

On the other hand, since $(v_{n,k}) \rightarrow v_n$ for each n , by (10) and (9), for each

$n \geq n_1$ one can find $q(n)$ such that for $k \geq q(n)$,

$$\begin{aligned} \left\langle u_n^*, \frac{v_{n,k} - u_n}{\|v_{n,k} - u_n\|} \right\rangle &> \frac{r - f(y)}{t\|y - x\|}, \\ \left\langle v_{n,k}^*, \frac{u_n - v_{n,k}}{\|u_n - v_{n,k}\|} \right\rangle &> \frac{r - f(x)}{(1 - t_n)\|y - x\|}. \end{aligned}$$

Adding these inequalities and using (7), we get

$$m\|u_n - v_{n,k}\|^{p-1} \geq \frac{r - f(y)}{t\|y - x\|} + \frac{r - f(x)}{(1 - t_n)\|y - x\|}.$$

Passing to the limit as $k, n \rightarrow \infty$, we get

$$mt(1 - t)\|y - x\|^p \geq (1 - t)(r - f(y)) + t(r - f(x)).$$

Since r is arbitrarily close to $f(z)$, we obtain (1). ■

COROLLARY 9. *Suppose $f \in \mathcal{F}(X)$ is finite at $\bar{x} \in X$ and p -paraconvex around \bar{x} with $p \in]1, 2]$. Then, for any subdifferential ∂ such that $\partial^p f \subset \partial f \subset \partial^\dagger f$ and for x near \bar{x} , one has $\partial f(x) = \partial^p f(x) = \partial^\dagger f(x)$; in particular, $\partial^F f(x) = \partial^C f(x)$.*

REMARK. The proof we have given yields a quantitative approach: the infimum of the constants c appearing in assertions (b) and (c) is majorized by c_f and the index of p -paramonotonicity of ∂f around \bar{x} is majorized by $2c_f$. On the other hand, the implication (d) \Rightarrow (a) shows that c_f is majorized by the index of p -paramonotonicity of ∂f around \bar{x} . We refer to [73] for such estimates in the case of uniformly convex functions which may serve as more precise models. ■

COROLLARY 10. *Suppose $f : U \rightarrow \mathbb{R}$ is a differentiable function on some open subset U of X with a locally $(p-1)$ -Hölderian derivative, with $p \in]1, 2]$. Then f is locally p -paraconvex on U .*

Proof. Since f is of class C^1 , its Clarke subdifferential is just the derivative Df of f . Moreover, Df is locally p -paramonotone: for every $\bar{x} \in U$, there exists some $m > 0$ and some neighborhood V of \bar{x} contained in U such that for $x, y \in V$,

$$\langle Df(x) - Df(y), x - y \rangle \geq -m\|x - y\|^{p-1}\|x - y\|.$$

Thus the result follows from the implication (d) \Rightarrow (a) of Theorem 8. ■

4. Paraconvexity of marginal functions. The statements of the following criteria for p -paraconvexity are simpler than the ones in [39] but they rely on differentiability assumptions. We use the following notation. Given an arbitrary set S , an open subset X_0 of a n.v.s. X , $\bar{x} \in X_0$ and

$g : S \times X_0 \rightarrow \mathbb{R}$, let $f : X_0 \rightarrow \overline{\mathbb{R}}$ be the marginal function

$$f(x) := \sup_{s \in S} g(s, x), \quad x \in X_0.$$

We assume f is finite at \bar{x} , and for $x \in X_0$ and $\varepsilon > 0$ we set

$$S(x, \varepsilon) := \begin{cases} \{s \in S : g(s, x) \geq f(x) - \varepsilon\} & \text{if } f(x) < +\infty, \\ \{s \in S : g(s, x) \geq \varepsilon^{-1}\} & \text{if } f(x) = +\infty, \end{cases}$$

$$T(x, \varepsilon) := \{s \in S : g(s, x) \geq f(\bar{x}) - \varepsilon\}.$$

THEOREM 11. *If the following conditions are satisfied for some $m \geq 0$, $p \in]1, 2]$, $\varrho > 0$, then the marginal function f is Lipschitzian and p -paraconvex on $B(\bar{x}, \delta)$ for some $\delta \in]0, \varrho[$:*

- (C₁) for each $s \in S$ the function $g_s := g(s, \cdot)$ is Fréchet differentiable on the ball $B(\bar{x}, \varrho)$;
- (C₂) for every $x \in B(\bar{x}, \varrho)$ one can find $\gamma := \gamma_x > 0$ with $\|g'_s(x)\| \leq m$ for each $s \in S(x, \gamma)$;
- (C₃) there exist $\sigma, \eta > 0$ such that $\|g'_s(x) - g'_s(x')\| \leq \sigma \|x - x'\|^{p-1}$ for any $x, x' \in B(\bar{x}, \varrho)$ and $s \in T(x, \eta) \cap T(x', \eta)$.

Proof. Let us apply the characterization of Theorem 8: we want to find $\beta, \sigma > 0$ such that

$$(11) \quad \forall x \in B(\bar{x}, \beta), \forall v \in \beta \overline{B}_X : \quad f^C(x, v) \leq f(x + v) - f(x) + \sigma \|v\|^p.$$

Note that f is l.s.c. on $B(\bar{x}, \varrho)$; we will first show that f is Lipschitzian with constant $\ell := m + \varrho^{p-1}\sigma$ on some ball $B(\bar{x}, 2\delta)$, where σ, η are as in (C₃). Let $\varepsilon \in]0, 1[$ be such that $(\ell + 2)\varepsilon < \eta$ and let $\delta \in]0, \varrho/4[\cap]0, \varepsilon/4[$ be such that $f(w) > f(\bar{x}) - \varepsilon$ for each $w \in B(\bar{x}, 2\delta)$. Given $x, y \in B(\bar{x}, 2\delta)$, we pick $\gamma := \gamma_y \in]0, \varepsilon[$ such that $\|g'_s(y)\| \leq m$ for each $s \in S(y, \gamma)$, as in (C₂). Given $s \in S(y, \gamma)$ let $h : [0, 1] \rightarrow \mathbb{R}$ and $\bar{t} \in [0, 1]$ be defined by $h(t) := g(s, y + t(x - y))$ and

$$\bar{t} := \sup\{t \in [0, 1] : h(r) \geq g(s, y) - \ell\varepsilon \forall r \in [0, t]\}.$$

By continuity of g_s , for any $t \in [0, \bar{t}]$ and $z_t := y + t(x - y)$ we have

$$h(t) = g(s, z_t) \geq g(s, y) - \ell\varepsilon \geq f(y) - \gamma - \ell\varepsilon \geq f(\bar{x}) - \eta,$$

hence $s \in T(z_t, \eta)$. In particular, taking $t = 0$, we see that $s \in T(y, \eta)$, hence, by (C₃), $\|g'_s(z_t) - g'_s(y)\| \leq \sigma \|z - y\|^{p-1}$. It follows that $\|g'_s(z_t)\| \leq \ell$. The mean value theorem yields some $z \in [y, y + \bar{t}(x - y)]$ such that

$$(12) \quad h(0) - h(\bar{t}) = \bar{t}g'_s(z)(x - y).$$

Thus $h(0) - h(\bar{t}) \leq 4\bar{t}\ell\delta < \ell\varepsilon$. If $\bar{t} < 1$, then by continuity, we have $h(\bar{t}) = g(s, y) - \ell\varepsilon = h(0) - \ell\varepsilon$, a contradiction. Therefore $\bar{t} = 1$ and (12) yields

$$(13) \quad g(s, y) - g(s, x) = g'_s(z)(x - y) \leq \ell \|x - y\|.$$

Since s is arbitrary in $S(y, \gamma)$, we get

$$f(y) - f(x) \leq \sup_{s \in S(y, \gamma)} g(s, y) - g(s, x) \leq \ell \|x - y\|$$

and f is Lipschitzian with constant ℓ on $B(\bar{x}, 2\delta)$.

Now let us estimate $f^C(x, u)$ for $(x, u) \in B(\bar{x}, \delta) \times S_X$. Since f is Lipschitzian, we can pick a sequence $((x_n, t_n)) \rightarrow (x, 0_+)$ such that $f^C(x, u) = \lim_n t_n^{-1}(f(x_n + t_n u) - f(x_n))$. Set $y_n := x_n + t_n u$, $\gamma_n := \min(t_n^2, \gamma_{y_n}, \varepsilon)$ and choose $s_n \in S(x_n + t_n u, \gamma_n)$. Taking $n \geq k$, with k large enough, we may suppose that $x_n \in B(\bar{x}, \delta)$ and $x_n + t_n u \in B(\bar{x}, \delta)$. Using relation (13), we obtain

$$g_{s_n}(x) \geq g_{s_n}(y_n) - \ell \|x - y_n\| \geq f(y_n) - \gamma_n - \ell t_n,$$

hence, as f is l.s.c. at x , $\liminf_n g_{s_n}(x) \geq f(x)$ or $(g_{s_n}(x)) \rightarrow f(x)$.

Moreover, using again (13), for every $w \in B(\bar{x}, \delta)$ and $n \geq k$ we have

$$(14) \quad \begin{aligned} g_{s_n}(w) &\geq g_{s_n}(y_n) - \ell \|w - y_n\| \geq f(y_n) - \gamma_n - 2\ell\delta \\ &\geq f(\bar{x}) - 2\varepsilon - \ell\varepsilon \geq f(\bar{x}) - \eta, \end{aligned}$$

so that $s_n \in T(w, \eta)$.

Taking $t \in]0, \delta]$, from the mean value theorem we get some $r_n, r'_n \in [0, 1]$ such that

$$\begin{aligned} \frac{1}{t} (g(s_n, x + tu) - g(s_n, x)) &= g'_{s_n}(x + r_n t u)(u), \\ \frac{1}{t_n} (g(s_n, x_n + t_n u) - g(s_n, x_n)) &= g'_{s_n}(x_n + r'_n t_n u)(u). \end{aligned}$$

Choosing $w = x + r_n t u$ and then $w = x_n + r'_n t_n u$ in (14), we find that $s_n \in T(x + r_n t u, \eta) \cap T(x_n + r'_n t_n u, \eta)$, so that we may apply (C_3) . Thus

$$\begin{aligned} \frac{1}{t_n} (g(s_n, x_n + t_n u) - g(s_n, x_n)) - \frac{1}{t} (g(s_n, x + tu) - g(s_n, x)) \\ \leq \|g'_{s_n}(x_n + r'_n t_n u) - g'_{s_n}(x + r_n t u)\| \leq \sigma \|x_n - x + r'_n t_n u - r_n t u\|^{p-1}. \end{aligned}$$

From these inequalities, the choice of s_n and the fact that $(g(s_n, x)) \rightarrow f(x)$, we deduce that

$$\begin{aligned} \lim_n \frac{1}{t_n} (f(x_n + t_n u) - f(x_n)) &\leq \liminf_n \frac{1}{t_n} (g(s_n, x_n + t_n u) - g(s_n, x_n) + t_n^2) \\ &\leq \liminf_n \left[\frac{1}{t} (g(s_n, x + tu) - g(s_n, x)) + \sigma \|x_n - x + r'_n t_n u - r_n t u\|^{p-1} \right] \\ &\leq \frac{1}{t} (f(x + tu) - f(x)) + \sigma t^{p-1} \|u\|^{p-1}. \end{aligned}$$

Therefore, for any $v \in \delta \bar{B}_X$, setting $v = tu$ with $t \in]0, \delta]$ and $u \in S_X$, we get (11). ■

The following immediate consequence partially generalizes [14, Thm. 5.1] which is set in finite-dimensional spaces and for suprema of functions of class C^2 .

COROLLARY 12. *Suppose S is a compact topological space, and $g : S \times X \rightarrow \mathbb{R}$ is continuous with a partial derivative with respect to $x \in B(\bar{x}, \rho)$ which is jointly continuous and Lipschitz with respect to x , uniformly with respect to $s \in S$. Then the marginal function f is Lipschitzian and 2-paraconvex on $B(\bar{x}, \delta)$ for some $\delta > 0$.*

Another consequence will be used later on.

COROLLARY 13. *Suppose X is a uniformly smooth Banach space with modulus of smoothness ϱ_X satisfying for some $k > 0$ and $p > 1$ the estimate $\varrho_X(t) \leq kt^p$ for $t \geq 0$. Then, for any nonempty closed subset E of X , the function $-d_E$ is p -paraconvex on $X \setminus E$.*

Proof. Let j be the derivative of the norm on $X \setminus \{0\}$. Let us first prove that, given $s > r > 0$, there exists some $c > 0$ such that

$$(15) \quad \|j(x) - j(y)\| \leq c\|x - y\|^{p-1} \quad \forall x, y \in s\bar{B}_X \setminus r\bar{B}_X.$$

By [6, Prop. A.5] we have, for any $x, y \in s\bar{B}_X \setminus r\bar{B}_X$,

$$\begin{aligned} \|j(x) - j(y)\| &\leq k2^p \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^{p-1} \leq k2^p r^{2-2p} \| \|y\|x - \|x\|y \|^{p-1} \\ &\leq k2^p r^{2-2p} (\|y\| \|x - y\| + \|y\| \| \|x\| - \|y\| \|)^{p-1} \\ &\leq k2^{2p-1} r^{2-2p} s^{p-1} \|x - y\|^{p-1}. \end{aligned}$$

Given $\bar{x} \in X \setminus E$, let $\eta = \rho = r = d_E(\bar{x})/2$ and set $S := E$, $g(e, x) = -\|x - e\|$ for $x \in B(\bar{x}, \rho)$ and $e \in E$. Then $f(x) = -d_E(x)$ for $x \in B(\bar{x}, \rho)$, and for $e \in T(x, \eta)$, i.e. for $e \in E$ satisfying $\|x - e\| \leq d_E(\bar{x}) + \eta$, we have $r \leq \|x - e\| \leq 3r$. Thus (15) shows that condition (C_3) is satisfied. Since the norm is differentiable on $B(\bar{x}, \rho)$ with derivative j bounded by 1, conditions (C_1) and (C_2) are also satisfied. ■

5. Paraconvexity of sets. We observe that using the notion of p -paraconvexity for the indicator function ι_E of a subset E of X would lead to convexity of E and not to a relaxed form of convexity. Therefore, we rather use the distance function d_E . In the following, \bar{x} is a point of E and $p \in]1, 2]$.

DEFINITION 14. A subset E of X is said to be p -paraconvex around \bar{x} if its associated distance function d_E is p -paraconvex around \bar{x} .

This notion is different from the one studied in [35]. Clearly, a p -paraconvex set is approximately convex in the sense of [38].

EXAMPLE. The set $E := \{(r, s) \in \mathbb{R}^2 : s \geq |r| - r^2\}$ is p -paraconvex but nonconvex. This example is a special instance of Proposition 25 below.

It is not obvious to decide whether the preceding definition depends on the choice of the norm in an equivalence class; by contrast, the variant presented in the next section will not depend on the choice of the norm inducing the topology.

The following result is an easy consequence of [18] when $\partial = \partial^C$. However, since we use here an arbitrary subdifferential contained in the Clarke subdifferential, we have to use Theorem 8 with the distance function.

THEOREM 15. *Let ∂ be a subdifferential on the family of Lipschitz functions such that $\partial f \subset \partial^C f$ for any Lipschitz function f on X . Then (a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (c') \Rightarrow (d) below. If moreover ∂ is Lipschitz-valuable on X , all these assertions are equivalent.*

- (a) E is p -paraconvex around \bar{x} in the sense that d_E is p -paraconvex around \bar{x} ;
 - (b) there exist $c, \rho > 0$ such that for any $x \in B(\bar{x}, \rho)$ and $v \in B(0, \rho)$,
- $$(16) \quad d_E^C(x, v) \leq d_E(x + v) - d_E(x) + c\|v\|^p;$$
- (c) there exist $c, \rho > 0$ such that for any $x \in B(\bar{x}, \rho)$, $x^* \in \partial d_E(x)$ and $(u, t) \in S_X \times]0, \rho)$,
- $$(17) \quad \langle x^*, u \rangle \leq \frac{d_E(x + tu) - d_E(x)}{t} + ct^{p-1};$$
- (c') there exist $c, \rho > 0$ such that for any $x \in B(\bar{x}, \rho)$, $x^* \in \partial d_E(x)$ and $v \in \rho \bar{B}_X$,
- $$(18) \quad \langle x^*, v \rangle \leq d_E(x + v) - d_E(x) + c\|v\|^p;$$
- (d) ∂d_E is p -paramonotone around \bar{x} .

The following corollary is a consequence of Corollary 9.

COROLLARY 16. *If E is p -paraconvex around \bar{x} for some $p > 1$, then, for any subdifferential ∂ such that $\partial^p f \subset \partial f \subset \partial^C f$ for any Lipschitz function f , one has $\partial^p d_E(x) = \partial^F d_E(x) = \partial d_E(x) = \partial^C d_E(x)$ for x close to \bar{x} .*

6. Intrinsic p -paraconvexity. As in the case of approximate convexity ([40]), the terminology of the definition we now adopt is justified by the fact that the notion we introduce is obtained by restricting the requirement on the distance function to the subset E . Thus, this new notion is more general than the preceding one. We will show later that the two notions do not coincide.

DEFINITION 17. Given $p \in [1, +\infty)$, a subset E of X is said to be *intrinsically p -paraconvex around $\bar{x} \in E$* if there exist $c, \rho > 0$ such that for

any $x_1, x_2 \in E \cap B(\bar{x}, \varrho)$ and $t \in [0, 1]$,

$$(19) \quad d_E((1-t)x_1 + tx_2) \leq ct(1-t)\|x_1 - x_2\|^p.$$

It is *intrinsically p -paraconvex* if it is intrinsically p -paraconvex around each of its points.

Note that there is no loss of generality in assuming that E is closed. Also note that this definition does not depend on the choice of the norm among the ones inducing the same topology.

Characterizations can be given as follows. When one of the assertions (c)–(d) below holds, we say that E is *∂ -intrinsically p -paraconvex around \bar{x}* .

THEOREM 18. *Let E be a nonempty closed subset of X and let ∂ be a subdifferential such that $\partial f \subset \partial^C f$ for any Lipschitz function f on X . Then (a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftarrow (e), and when X is a ∂ -subdifferentiability space, also (e) \Rightarrow (a), where*

$$(20) \quad \begin{aligned} & \text{(a) } E \text{ is intrinsically } p\text{-paraconvex around } \bar{x}; \\ & \text{(b) there exist } c, \delta > 0 \text{ such that for any } x, x' \in E \cap B(\bar{x}, \delta), \\ & \quad d_E^C(x, x' - x) \leq c\|x - x'\|^p; \\ & \text{(c) there exist } c, \delta > 0 \text{ such that for any } x, x' \in E \cap B(\bar{x}, \delta) \text{ and } x^* \in \\ & \quad \partial d_E(x), \end{aligned}$$

$$(21) \quad \begin{aligned} & \langle x^*, x' - x \rangle \leq c\|x - x'\|^p; \\ & \text{(d) } \partial d_E(\cdot) \text{ is } p\text{-paramonotone around } \bar{x} \text{ on } E: \text{ there exist } c, \delta > 0 \text{ such} \\ & \text{that for any } x_1, x_2 \in E \cap B(\bar{x}, \delta), x_1^* \in \partial d_E(x_1) \text{ and } x_2^* \in \partial d_E(x_2), \end{aligned}$$

$$(22) \quad \begin{aligned} & \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -c\|x_1 - x_2\|^p; \\ & \text{(e) there exist } c, \varrho > 0 \text{ such that for any } w \in B(\bar{x}, \varrho), x \in E \cap B(\bar{x}, \varrho) \\ & \text{and } w^* \in \partial d_E(w), \end{aligned}$$

$$(23) \quad d_E(w) + \langle w^*, x - w \rangle \leq c\|x - w\|^p.$$

Proof. (a) \Rightarrow (b). Let $\varrho > 0$ be as in Definition 17 and let $x, x' \in E \cap B(\bar{x}, \varrho)$. By Lemma 5, we have

$$d_E^C(x, x' - x) \leq \limsup_{t \rightarrow 0+, e \xrightarrow{E} x} \frac{1}{t} d_E(e + t(x' - x)).$$

Now, since for $t \in]0, 1[$, $d_E(e + t(x' - x)) \leq d_E(e + t(x' - e)) + t\|e - x\| \leq ct(1-t)\|x' - e\|^p + t\|e - x\|$, we get

$$d_E^C(x, x' - x) \leq c\|x' - x\|^p.$$

(b) \Rightarrow (c) is a consequence of the inclusion $\partial d_E(x) \subset \partial^C d_E(x)$ and of the definition of $\partial^C d_E(x)$. When $\partial = \partial^C$, the reverse implication follows from the relation $d_E^C(x, u) = \sup\{\langle x^*, u \rangle : x^* \in \partial^C d_E(x)\}$.

(c) \Rightarrow (d). Let $c, \delta > 0$ be as in (c) and let $x_1, x_2 \in E \cap B(\bar{x}, \delta)$, $x_1^* \in \partial d_E(x_1)$ and $x_2^* \in \partial d_E(x_2)$. Taking $x = x_1$, $x^* = x_1^*$, $x' = x_2$ and adding inequality (21) to the corresponding one obtained by choosing $x = x_2$, $x^* = x_2^*$, $x' = x_1$ we get (22) with $2c$ in place of c .

(d) \Rightarrow (c) is obtained by taking $x_1 = x$, $x_1^* = x^*$, $x_2 = x'$, $x_2^* = 0$ in (d), using the fact that x_2 is a minimizer of d_E , so that $0 \in \partial d_E(x_2)$.

(e) \Rightarrow (c) is obvious (take $w = x' \in E$).

(e) \Rightarrow (a) when X is a ∂ -subdifferentiability space. Given $c, \varrho > 0$ as in (e), let $x_1, x_2 \in E \cap B(\bar{x}, \varrho)$, $t \in [0, 1]$, and $w := (1-t)x_1 + tx_2$. Since X is a ∂ -subdifferentiability space, there exist sequences $(w_n) \rightarrow w$ and (w_n^*) such that $w_n^* \in \partial d_E(w_n)$ for each $n \in \mathbb{N}$. Then, as $\partial d_E(w_n) \subset \partial^C d_E(w_n) \subset \overline{B}_{X^*}$, we have $(\langle w_n^*, w - w_n \rangle) \rightarrow 0$. Since by convexity $w \in B(\bar{x}, \varrho)$, we have $w_n \in B(\bar{x}, \varrho)$ for n large enough, it follows that

$$\begin{aligned} (1-t)d_E(w_n) + (1-t)\langle w_n^*, x_1 - w_n \rangle &\leq (1-t)c\|x_1 - w_n\|^p, \\ td_E(w_n) + t\langle w_n^*, x_2 - w_n \rangle &\leq tc\|x_2 - w_n\|^p. \end{aligned}$$

Adding these relations, we get

$$d_E(w_n) + \langle w_n^*, w - w_n \rangle \leq (1-t)c\|x_1 - w_n\|^p + tc\|x_2 - w_n\|^p,$$

and, passing to the limit,

$$d_E(w) \leq c(1-t)t^p\|x_1 - x_2\|^p + ct(1-t)^p\|x_2 - x_1\|^p \leq 2ct(1-t)\|x_1 - x_2\|^p. \blacksquare$$

Now let us give specializations to some specific subdifferentials and normal cones. We start with the Fréchet normal cone.

COROLLARY 19. *For the Fréchet subdifferential ∂^F , assertion (c) of the preceding theorem is equivalent to the following:*

(f) *there exist $c, \delta > 0$ such that for any $x, x' \in E \cap B(\bar{x}, \delta)$ and $x^* \in N^F(E, x)$,*

$$(24) \quad \langle x^*, x' - x \rangle \leq c\|x^*\| \|x - x'\|^p.$$

If (c) holds with $\partial = \partial^C$, then $N^C(E, x) = N^F(E, x)$ for all $x \in E \cap B(\bar{x}, \delta)$.

If X is an Asplund space then all the assertions of Theorem 18 are equivalent to (f).

Proof. The relation $\partial^F d_E(x) = N^F(E, x) \cap \overline{B}_{X^*}$ for each $x \in E$ yields equivalence of (c) and (f) by a homogeneity argument which does not require any assumption on X . Moreover, (c) shows that $\partial^C d_E(x) = \partial^F d_E(x)$ for each $x \in E \cap B(\bar{x}, \delta)$ and that $\partial^F d_E(x)$ is weak* closed. Therefore, by [13, Prop. 2.4.2], denoting by $\overline{\text{co}}^*(S)$ the weak* closed convex hull of a subset S of X^* , we get

$$N^C(E, x) = \overline{\text{co}}^*(\mathbb{R}_+ \partial^C d_E(x)) = \overline{\text{co}}^*(\mathbb{R}_+ \partial^F d_E(x)) = N^F(E, x)$$

since $N^F(E, x)$ is convex and weak* closed by (f).

(c) \Rightarrow (e) when X is an Asplund space. Let $c, \delta > 0$ be as in (c). Let $\varrho := \delta/2$, $w \in B(\bar{x}, \varrho) \setminus E$ (note that (23) coincides with (21) if $w \in E$), $x \in E \cap B(\bar{x}, \varrho)$ and $w^* \in \partial^F d_E(w)$. By Lemma 6 we can find sequences (x_n) in E and (x_n^*) in X^* such that $x_n^* \in \partial^F d_E(x_n)$ for each $n \in \mathbb{N}$ and

$$(25) \quad (\|x_n - w\|) \rightarrow d_E(w), \quad (\langle x_n^*, w - x_n \rangle) \rightarrow d_E(w), \quad (\|x_n^* - w^*\|) \rightarrow 0.$$

Since $d_E(w) \leq \|\bar{x} - w\| < \varrho$ and $\|x - w\| > 0$, we may suppose that $\|x_n - w\| \leq \varrho$, $\|x_n - w\| \leq 2\|x - w\|$ for each $n \in \mathbb{N}$; then $x_n \in B(\bar{x}, 2\varrho) \subset B(\bar{x}, \delta)$ and $\|x - x_n\| \leq \|x - w\| + \|w - x_n\| \leq 3\|x - w\|$. Now, for n large enough, (25) implies the first inequality below, while (c) ensures the third:

$$\begin{aligned} d_E(w) + \langle w^*, x - w \rangle & \leq \limsup_n (\langle x_n^*, w - x_n \rangle + (\langle x_n^*, x - w \rangle + \|x_n^* - w^*\| \|x - w\|)) \\ & \leq \limsup_n (\langle x_n^*, x - x_n \rangle + \|x_n^* - w^*\| \|x - w\|) \\ & \leq \limsup_n (c\|x - x_n\|^p + \|x_n^* - w^*\| \|x - w\|). \end{aligned}$$

Passing to the limit, and using the estimate $\|x - x_n\| \leq 3\|x - w\|$ obtained above, we get

$$d_E(w) + \langle w^*, x - w \rangle \leq c(3^p + 1)\|x - w\|^p.$$

Finally, since X is Asplund, it is a ∂^F -subdifferentiability space, so that (e) ensures that E is intrinsically p -paraconvex by Theorem 18. ■

The case of the limiting subdifferential can easily be derived from the preceding corollary.

COROLLARY 20. *Suppose that E is a closed subset of an Asplund space X and let ∂ be the limiting Fréchet subdifferential $\overline{\partial^F}$. Then all the assertions of Theorem 18 are equivalent.*

Proof. In view of Corollary 7, the passages from the assertions using ∂^F to those involving $\overline{\partial^F}$ result from a passage to the weak* limit for a bounded net; the reverse implications are obvious. ■

Now let us turn to the Clarke subdifferential. It would be interesting to know whether one can get rid of the assumption of the last assertion of the next corollary that X is an Asplund space.

COROLLARY 21. *For a closed subset E of a Banach space X and $\partial = \partial^C$, we have (e) \Rightarrow (a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) in Theorem 18. If moreover X is an Asplund space, all these assertions are equivalent.*

Proof. The implications follow from the choice $\partial = \partial^C$ in Theorem 18 since any Banach space is a ∂^C -subdifferentiability space. When X is an Asplund space the equivalences can be deduced from the preceding corollary:

since $\partial^C d_E(x) = \overline{co}^*(\overline{\partial^F} d_E(x))$ one has the equivalence $(e^F) \Leftrightarrow (\overline{e^F}) \Leftrightarrow (e^C)$ (where (e^C) , $(\overline{e^F})$, (e^F) are (e) for $\partial = \partial^C, \overline{\partial^F}, \partial^F$ respectively), hence $(c^C) \Rightarrow (c^F) \Rightarrow (e^F) \Rightarrow (e^C) \Rightarrow (a)$. ■

The preceding local property can be related to a global one as defined and studied in [10], [15], [19], [20], [22] in the case $p = 2$, X being a Hilbert space.

DEFINITION 22. Given a subset E of X , $p \in]1, 2]$ and a continuous function $\varphi : E \times E \rightarrow \mathbb{R}_+$, the subset E of X is said to be φ - p -convex if for any $x, y \in E$ and $x^* \in N^F(E, x)$,

$$(26) \quad \langle x^*, y - x \rangle \leq \varphi(x, y) \|x^*\| \|x - y\|^p.$$

In this definition the choice of the Fréchet normal cone is natural since if the above condition holds for some other cone $N(E, x)$ contained in $N^C(E, x)$ then $N(E, x) \subset N^F(E, x)$. The following result clarifies the links between φ - p -convexity and p -paraconvexity.

PROPOSITION 23. Let $p \in]1, 2]$, E a subset of a Banach space X , and $\varphi : E \times E \rightarrow \mathbb{R}_+$ a continuous function such that E is φ - p -convex. Then E is ∂^F -intrinsically p -paraconvex around each point of E in the sense that assertion (c) of Theorem 18 is satisfied with $\partial = \partial^F$.

Conversely, if E is ∂^F -intrinsically p -paraconvex around each point of E , then E is φ - p -convex for some continuous function $\varphi : E \times E \rightarrow \mathbb{R}_+$.

Proof. The first assertion is immediate: given $c > \varphi(\bar{x}, \bar{x})$ for some $\bar{x} \in E$, relation (21) is satisfied whenever $\delta > 0$ is small enough. Let us prove the converse by first showing that there exist an open neighborhood V of the diagonal of $E \times E$ and a continuous function $\psi : V \rightarrow \mathbb{R}_+$ such that for any $(x, y) \in V$ and $x^* \in N^F(E, x)$,

$$(27) \quad \langle x^*, y - x \rangle \leq \psi(x, y) \|x^*\| \|x - y\|^p.$$

Given $(x, y) \in E \times E$, let

$$\omega_p(x, y) = \sup \left\{ \left\langle x^*, \frac{y - x}{\|x - y\|^p} \right\rangle : x^* \in N^F(E, x) \cap \overline{B}_{X^*} \right\}.$$

Assertion (c) of Theorem 18 shows that for any $e \in E$ there exist $c, \rho > 0$ such that $\omega_p(x, y) \leq c$ for any $x, y \in B(e, \rho)$. Clearly, the infimum $c_p(e)$ of such constants c defines an upper semicontinuous function on E . Since E is a metric space, there exists a continuous function $\psi_p : E \rightarrow \mathbb{R}$ such that $c_p(e) + 1 \leq \psi_p(e)$ for every $e \in E$. Then, by definition of c_p , for each $e \in E$ one can find $\rho(e) > 0$ such that $\omega_p(x, y) \leq c_p(e) + 1 \leq \psi_p(e)$ for all $x, y \in B(e, \rho(e))$. Since E is paracompact, we can find a locally finite open

covering $(V_e)_{e \in E}$ of E such that $V_e \subset B(e, \varrho(e))$ for every $e \in E$. Set

$$V := \bigcup_{e \in E} V_e \times V_e, \quad \psi(x, y) := \sum_{e \in E} \lambda_e(x, y) \psi_p(e),$$

where $(\lambda_e)_{e \in E}$ is a continuous partition of unity subordinated to the covering $(V_e \times V_e)_{e \in E}$ of V . Then ψ is continuous and for every $(x, y) \in V$ one has

$$\omega_p(x, y) = \sum_{e \in E} \lambda_e(x, y) \omega_p(x, y) \leq \sum_{e \in E} \lambda_e(x, y) \psi_p(e) = \psi(x, y),$$

so that inequality (27) holds.

Let U be a closed neighborhood of the diagonal of $E \times E$ contained in V and let φ be given by

$$\varphi(x, y) = (1 - t(x, y))\psi(x, y) + t(x, y) \frac{1}{\|x - y\|^{p-1}}$$

where

$$t(x, y) := \frac{d((x, y), U)}{d((x, y), (E \times E) \setminus V) + d((x, y), U)}.$$

Then φ is a continuous function on $E \times E$. For $(x, y) \in U$ and $x^* \in N(E, x)$,

$$\left\langle x^*, \frac{y - x}{\|x - y\|^p} \right\rangle \leq \|x^*\| \psi(x, y) = \|x^*\| \varphi(x, y).$$

For $(x, y) \in (E \times E) \setminus V$ and $x^* \in N(E, x)$,

$$\left\langle x^*, \frac{y - x}{\|x - y\|^p} \right\rangle \leq \|x^*\| \frac{1}{\|x - y\|^{p-1}} = \|x^*\| \varphi(x, y).$$

For $(x, y) \in V \setminus U$, $x^* \in N(E, x)$,

$$\left\langle x^*, \frac{y - x}{\|x - y\|^p} \right\rangle \leq \min \left(\|x^*\| \frac{1}{\|x - y\|^{p-1}}, \|x^*\| \psi(x, y) \right) \leq \|x^*\| \varphi(x, y).$$

Therefore (26) is satisfied. ■

The following statement can be considered as an illustration of the use of a global property as in the preceding definition.

PROPOSITION 24. *Let $p \geq q \in [1, \infty)$, let (S, \mathcal{S}, μ) be a measure space with a finite measure, let $c \in L_\infty(S, \mathbb{R})$ and let $F : S \rightrightarrows \mathbb{R}^n$ be a measurable multifunction such that*

$$\forall s \in S, \forall x, y \in F(s), \forall t \in [0, 1] : d_{F(s)}((1-t)x + ty) \leq c(s)t(1-t)\|x - y\|^{p/q}.$$

Then the set E of measurable selections of $F(\cdot)$ which are in $L_p(S, \mathbb{R}^n)$ considered as a subset of $X := L_q(S, \mathbb{R}^n)$ is such that

$$(28) \quad \forall u, v \in E, \forall t \in [0, 1] : d_X((1-t)u + tv, E) \leq \|c\|_\infty t(1-t)\|u - v\|^{p/q}.$$

Proof. Let $u, v \in E$, and for $t \in [0, 1]$, let $w := (1 - t)u + tv$. Then, by [55, Thm. 14.60], we can interchange minimization and integration to get

$$\begin{aligned} d_X(w, E)^q &:= \inf_{z \in E} \int_S \|w(s) - z(s)\|^q \mu(ds) = \int_S \inf_{e \in F(s)} \|w(s) - e\|^q \mu(ds) \\ &\leq \int_S c(s)^q t^q (1 - t)^q \|u(s) - v(s)\|^p \mu(ds) \\ &\leq \|c\|_\infty^q t^q (1 - t)^q \|u - v\|_p^p. \end{aligned}$$

Inequality (28) follows. ■

7. Links between p -paraconvex sets and functions. Some links between geometrical properties and analytical properties are contained in the next statements. In the present section we suppose $p > 1$. Unless otherwise specified, we endow the product space $X := W \times \mathbb{R}$ of a n.v.s. W with \mathbb{R} with a *product norm*, i.e. a norm such that the projections and the injections $w \mapsto (w, 0)$ and $r \mapsto (0, r)$ are nonexpansive. Then for each $(w, r) \in W \times \mathbb{R}$ we have

$$\max(\|w\|, |r|) \leq \|(w, r)\| \leq \|w\| + |r|.$$

PROPOSITION 25. *Let W be a normed vector space and let $f : W \rightarrow \mathbb{R}_\infty$ be a l.s.c. function which is p -paraconvex around $\bar{w} \in W$. Then for any $\bar{r} \geq f(\bar{w})$ the epigraph E of f is intrinsically p -paraconvex around $\bar{x} := (\bar{w}, \bar{r})$.*

Proof. Let $c, \varrho > 0$ be such that

$$f((1 - t)w_1 + tw_2) \leq (1 - t)f(w_1) + tf(w_2) + ct(1 - t)\|w_1 - w_2\|^p$$

for any $w_1, w_2 \in B(\bar{w}, \varrho)$, $t \in [0, 1]$. Let $x_i := (w_i, r_i)$ ($i = 1, 2$) be elements of the epigraph E of f in $B(\bar{x}, \varrho)$ and let $t \in [0, 1]$, $w := (1 - t)w_1 + tw_2$, $r := (1 - t)r_1 + tr_2$ and $x := (w, r)$. Then, as $w_1, w_2 \in B(\bar{w}, \varrho)$ and $(w, f(w)) \in E$, one has $d_E(x) = 0$ if $f(w) \leq r$ and $d_E(x) \leq \|(w, r) - (w, f(w))\| \leq f(w) - r$ if $f(w) > r$, so that

$$\begin{aligned} d_E(x) &\leq \max(0, f(w) - r) \\ &\leq \max(0, (1 - t)(f(w_1) - r_1) + t(f(w_2) - r_2) + ct(1 - t)\|w_1 - w_2\|^p) \\ &\leq ct(1 - t)\|w_1 - w_2\|^p \leq ct(1 - t)\|x_1 - x_2\|^p. \end{aligned}$$

Thus E is intrinsically p -paraconvex around \bar{x} . ■

Let us complete the preceding result with the following one.

PROPOSITION 26. *Let $f : W \rightarrow \mathbb{R}$ be a function which is Lipschitzian with constant $\ell > 0$ on some ball $B(\bar{w}, \varrho)$. Suppose $X := W \times \mathbb{R}$ is endowed with the norm given by $\|(w, r)\| = \ell\|w\| + |r|$. If f is p -paraconvex around \bar{w} , then, for any $\bar{r} \geq f(\bar{w})$, the epigraph E of f is p -paraconvex around $\bar{x} := (\bar{w}, \bar{r})$.*

Proof. We may suppose $\bar{r} = f(\bar{w})$ since otherwise \bar{x} is in the interior of E . By [23], [26] one can find $\varrho' \in]0, \varrho[$ such that

$$d_E(w, r) = (f(w) - r)_+$$

for $(w, r) \in B(\bar{x}, \varrho')$ with $\bar{x} := (\bar{w}, f(\bar{w}))$, X being endowed with the norm described in the statement; here, for $t \in \mathbb{R}$, t_+ stands for $\max(t, 0)$. Let $c > 0$ and $\delta \in]0, \varrho'[$ be such that

$$f((1-t)w_1 + tw_2) \leq (1-t)f(w_1) + tf(w_2) + ct(1-t)\|w_1 - w_2\|^p$$

for any $w_1, w_2 \in B(\bar{w}, \delta)$, $t \in [0, 1]$. Let $x_i := (w_i, r_i) \in B(\bar{x}, \delta)$ for $i = 1, 2$ and let $w := (1-t)w_1 + tw_2$, $r := (1-t)r_1 + tr_2$ and $x := (w, r)$. Then we have $w_1, w_2 \in B(\bar{w}, \delta)$ and

$$f(w) - r \leq (1-t)(f(w_1) - r_1)_+ + t(f(w_2) - r_2)_+ + ct(1-t)\|w_1 - w_2\|^p,$$

hence

$$d_E(x) \leq (1-t)d_E(x_1) + td_E(x_2) + c\ell^{-p}t(1-t)\|x_1 - x_2\|^p. \blacksquare$$

Let us give a kind of converse to the preceding propositions.

THEOREM 27. *Let W be a Banach space and let $f : W \rightarrow \mathbb{R}$ be a function which is locally Lipschitzian around $\bar{w} \in W$ and such that the epigraph E of f is an intrinsically p -paraconvex subset of $X := W \times \mathbb{R}$ around $\bar{x} := (\bar{w}, f(\bar{w}))$. Then f is p -paraconvex around \bar{w} .*

Proof in the case W is an Asplund space. In view of the characterization of p -paraconvexity of a function given in Theorem 8, it suffices to prove that $\partial^F f$ is p -paramonotone around \bar{w} . Let ℓ be the Lipschitz constant of f on some ball $B(\bar{x}, \varrho_0)$. By Corollary 19 there exist $c > 0$ and $\varrho \in]0, \varrho_0[$ such that for any $x_1, x_2 \in E \cap B(\bar{x}, \varrho)$ and any $x_1^* \in N^F(E, x_1)$, $x_2^* \in N^F(E, x_2)$,

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -c\|x_1 - x_2\|^p.$$

Let $\varrho' := \varrho/(\ell + 1)$. Then, for $w_1, w_2 \in B(\bar{w}, \varrho')$, $w_i^* \in \partial^F f(w_i)$ for $i = 1, 2$, setting $x_i := (w_i, f(w_i))$, $x_i^* := (w_i^*, -1)$, one has $x_i \in B(\bar{x}, \varrho)$ and $x_i^* \in N^F(E, x_i)$, hence

$$\begin{aligned} \langle w_1^* - w_2^*, w_1 - w_2 \rangle &= \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -c\|x_1 - x_2\|^p \\ &\geq -c(\ell + 1)^p\|w_1 - w_2\|^p. \end{aligned}$$

Thus f is p -paraconvex around \bar{w} . \blacksquare

The proof in the general case relies on the following lemma extracted from the proof of [48, Prop. 10 c]; it is close to previous results of that kind due to F. H. Clarke [13] and to A. D. Ioffe [27] in the case $\bar{x} = (\bar{w}, f(\bar{w}))$.

LEMMA 28. *Let $f : W \rightarrow \mathbb{R}$ be Lipschitzian with constant ℓ on a ball $B(\bar{w}, \varrho)$ of W . Then for $\sigma \in]0, \varrho[$ small enough and for any $w \in B(\bar{w}, \sigma)$ and any $w^* \in \partial^C f(w)$ one has $(w^*, -1) \in \partial^C d_E(x)$, where E is the epigraph*

of f and $x := (w, f(w))$, $X := W \times \mathbb{R}$ being endowed with the norm given by $\|(w, r)\| = \ell\|w\| + |r|$.

Proof of the theorem in the general case. Since intrinsic p -paraconvexity is preserved when using an equivalent norm, we may use the norm described in the lemma and take $\varrho, \sigma > 0$ as there. We use the implication (a) \Rightarrow (c) of Corollary 21: there exist $c > 0$, $\gamma \in]0, \sigma[$ such that for any $x, x' \in E \cap B(\bar{x}, \gamma)$ and $x^* \in \partial^C d_E(x)$,

$$(29) \quad \langle x^*, x' - x \rangle \leq c\|x - x'\|^p.$$

Let $\delta := \gamma/4\ell$. Now, by the preceding lemma, for $w \in B(\bar{w}, \delta)$, $u \in \delta\bar{B}_W$, $w' := w + u \in B(\bar{w}, 2\delta)$ and $w^* \in \partial^C f(w)$ we have $x := (w, f(w)) \in E \cap B(\bar{x}, \gamma)$, $x' := (w', f(w')) \in E \cap B(\bar{x}, \sigma)$ and $(w^*, -1) \in \partial^C d_E(x)$. Then, by (29),

$$\langle w^*, u \rangle - (f(w') - f(w)) \leq c(\ell\|w' - w\| + |f(w') - f(w)|)^p \leq c2^p\ell^p\|u\|^p.$$

Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $w \in B(\bar{w}, \delta)$, any $w^* \in \partial^C f(w)$ and $u \in \bar{B}(0, \delta)$ one has $f(w+u) - f(w) \geq \langle w^*, u \rangle - c2^p\ell^p\|u\|^p$, so that f is p -paraconvex around \bar{w} by Theorem 8. ■

Finally, let us turn to sublevel sets.

PROPOSITION 29. *Let X be an Asplund space and let $f : X \rightarrow \mathbb{R}_\infty$. Suppose f is continuous and p -paraconvex around $\bar{x} \in S := \{x \in X : f(x) \leq 0\}$ and there exist $b, r > 0$ such that $\|x^*\| \geq b$ for all $x \in (X \setminus S) \cap B(\bar{x}, r)$ and $x^* \in \partial^F f(x)$. Then S is intrinsically p -paraconvex around \bar{x} .*

Proof. Without loss of generality we may suppose f takes the value $+\infty$ on $X \setminus U$, where $U := B(\bar{x}, r)$. Then, by [74], [16], [29], [46, Thm. 9.1], [41, Thm. 3.2] and several other contributions, we have $f_+(x) \geq bd_S(x)$ for $x \in U$ for $f_+ := \max(f, 0)$. Let $\varepsilon > 0$ be given. Using Theorem 8, we can find $c > 0$ and $\delta \in]0, r[$ such that

$$\langle x^*, x' - x \rangle \leq f(x') - f(x) + c\|x' - x\|^p$$

for any $x, x' \in B(\bar{x}, \delta)$ and $x^* \in \partial^F f(x)$. Given $x, x' \in S \cap B(\bar{x}, \delta)$ and $x^* \in \partial^F d_S(x)$, using [41, Cor. 4.1] we can find sequences (λ_n) , (x_n) , (x_n^*) in $[0, 1]$, X , X^* respectively such that $(x_n) \rightarrow x$, $(\lambda_n x_n^*) \rightarrow x^*$ and $x_n^* \in \partial^F f(x_n)$ for each $n \in \mathbb{N}$. Suppose first $x^* \neq 0$. Then $f(x) = 0$ since f is continuous at x and x cannot belong to the interior of S . For n so large that $x_n \in B(\bar{x}, \delta)$, and for $x' \in S \cap B(\bar{x}, \delta)$, we have

$$\langle x_n^*, x' - x_n \rangle \leq f(x') - f(x_n) + c\|x' - x_n\|^p.$$

Thus

$$\begin{aligned} \langle x^*, x' - x \rangle &= \lim_n \langle \lambda_n x_n^*, x' - x_n \rangle \\ &\leq \limsup_n \lambda_n (f(x') - f(x_n) + c\|x' - x_n\|^p) \leq c\|x' - x\|^p \end{aligned}$$

since $f(x') \leq 0$, $\lambda_n \in [0, 1]$ and $(f(x_n)) \rightarrow 0$. When $x^* = 0$, the inequality $\langle x^*, x' - x \rangle \leq c\|x' - x\|^p$ is obvious. Thus, assertion (c) of Corollary 19 is satisfied for $E := S$ and S is intrinsically p -paraconvex around \bar{x} . ■

8. A counter-example. Because the concepts of paraconvexity and intrinsic paraconvexity seem to be so close, one may wonder whether they coincide. The following counter-example shows that this is not the case.

Given $p = 3/2$, let E be the hypograph of the function $f : r \mapsto |r|^p$ from \mathbb{R} to \mathbb{R} :

$$E := \{(r, s) \in \mathbb{R}^2 : s \leq |r|^p\}.$$

Let us endow \mathbb{R}^2 with the Euclidean norm and show that E is intrinsically paraconvex around $(0, 0)$, but not paraconvex around $(0, 0)$. Since E is the image under the isometry $(r, s) \mapsto (r, -s)$ of the epigraph of the function $-f$ whose derivative is $(p-1)$ -Hölderian, it is intrinsically p -paraconvex around $(0, 0)$. Let $(r, s) \in \mathbb{R}^2$ be close to $(0, 0)$, with $r \geq 0$, $s > r^p$. In order to compute $d_E^2(r, s)$ we have to minimize the function $x \mapsto (x-r)^2 + (x^p - s)^2$. The necessary condition is

$$(30) \quad x - r + px^{p-1}(x^p - s) = 0.$$

Let $y := x^{1/2}$. The preceding equation is equivalent to

$$(31) \quad 2y^2 - 2r + 3y(y^3 - s) = 0.$$

Fixing s , this equation defines $y := x^{1/2}$ as an implicit function of r ; moreover, $y(0) = y_0$, where $y_0 > 0$ is the unique solution of the equation $3y^3 + 2y - 3s = 0$. The implicit function theorem asserts that $y'(0)$ is determined by the relation

$$y'(0)[4y_0 + 12y_0^3 - 3s] = 2$$

obtained by differentiating (31). Thus, since $x'(0) = 2y'(0)y_0$ and $3s = 3y_0^3 + 2y_0$ by (31),

$$x'(0) = \frac{4}{2 + 9x(0)}.$$

Therefore, for some function $r \mapsto q(r)$ with limit $x'(0)$ as $r \rightarrow 0_+$, we have

$$\begin{aligned} d_E^2(r, s) &= (x_0 + q(r)r - r)^2 + ((x_0 + q(r)r)^{3/2} - s)^2 \\ &= x_0^2 + 2x_0(q(r) - 1)r + (q(r) - 1)^2 r^2 \\ &\quad + (x_0 + q(r)r)^3 - 2s(x_0 + q(r)r)^{3/2} + s^2 \\ &= (x_0^2 + x_0^3 - 2sx_0^{3/2} + s^2) + 2x_0(q(r) - 1)r \\ &\quad + 3x_0^2 q(r)r - 3sx_0^{1/2} q(r)r + o(r) \end{aligned}$$

and since, by (30), $3sx_0^{1/2} = 3x_0^2 + 2x_0$, taking limits as $r \rightarrow 0$, we get

$$\begin{aligned} r^{-1}(d_E^2(r, s) - d_E^2(0, s)) &\rightarrow 2x_0(x'(0) - 1) + 3x_0^2x'(0) - 3sx_0^{1/2}x'(0) \\ &= -2x_0. \end{aligned}$$

Thus, given $a \in]0, 2x_0[$, for $r > 0$ small enough, we have

$$(32) \quad d_E^2(r, s) - d_E^2(0, s) \leq -ar.$$

On the other hand, if d_E is p -paraconvex, one can find $c, \delta > 0$ such that for $\|(r, s)\| \leq \delta$,

$$d_E(0, s) \leq \frac{1}{2}d(r, s) + \frac{1}{2}d(-r, s) + \frac{1}{4}c(2r)^p.$$

Now, since E is invariant under the symmetry $(r, s) \mapsto (-r, s)$, we have $d_E(r, s) \geq d_E(0, s) - 2^{p-2}cr^p$ and for r sufficiently small the right hand side of this inequality is positive, so that

$$\begin{aligned} d_E^2(r, s) &\geq (d_E(0, s) - 2^{p-2}cr^p)^2, \\ d_E^2(r, s) - d_E^2(0, s) &\geq -2^{p-1}cr^pd_E(0, s) + 2^{2p-4}c^2r^{2p}, \end{aligned}$$

a contradiction with (32) when $\|(r, s)\|$ is sufficiently small.

9. Paraconvex sets and projections. The following result shows that, in the context of superreflexive spaces, p -paraconvexity of a distance function is equivalent to its continuous differentiability with a Hölderian derivative. It is reminiscent of [14, Thm. 4.8] which is set in a Hilbert space. There, $p = 2$ and a Lipschitz property is given instead of a Hölderian property. Moreover, here U is not a uniform enlargement of E ; it may be small (or large) and far from E .

THEOREM 30. *Let X be a superreflexive Banach space, let E be a closed subset of X and let U be an open subset of X . The following assertions relative to some choice of $p \in]1, 2]$ and of an equivalent norm on X are equivalent:*

- (a) *Each $w \in U$ has a unique metric projection $P_E(w)$ in E and the mapping $w \mapsto P_E(w)$ is locally Hölderian on $U \setminus E$.*
- (b) *$d_E(\cdot)$ is differentiable with a locally Hölderian derivative on $U \setminus E$.*
- (c) *For each $\bar{w} \in U \setminus E$, $d_E(\cdot)$ is p -paraconvex around \bar{w} for some $p \in]1, 2]$.*

Proof. (a) \Rightarrow (b). As in [40, Thm. 21], for any $w \in U \setminus E$, we can show that d_E is differentiable and

$$d_E'(w) = j(w - P_E(w)),$$

where $j := (\|\cdot\|)'$. Let us prove that $d_E'(\cdot)$ is locally $(p-1)$ -Hölderian on $U \setminus E$, taking on X an equivalent norm of type p , i.e. such that its modulus

of smoothness ϱ_X satisfies (for some $k > 0$) $\varrho_X(t) \leq kt^p$ for $t \in \mathbb{R}_+$, which is possible by [6, Thm. A.6]. Then by (15), for any $s > r > 0$ there exists some $c > 0$ such that

$$\|j(x) - j(y)\| \leq c\|x - y\|^{p-1} \quad \forall x, y \in s\overline{B}_X \setminus r\overline{B}_X.$$

Let $\overline{w} \in U \setminus E$ be given. By assumption, there exist $\gamma \in]0, 1]$ and $\ell, \delta > 0$ such that

$$\|P_E(u) - P_E(w)\| \leq \ell\|u - w\|^\gamma \quad \forall u, w \in B(\overline{w}, \delta).$$

By the above relations, using the fact that for w close enough to \overline{w} the number $\|w - P_E(w)\|$ lies in some interval $[r, s]$ with $s > r > 0$, there exist $c, \varrho > 0$ such that for all $u, w \in B(\overline{w}, \varrho)$,

$$\begin{aligned} \|d'_E(u) - d'_E(w)\| &= \|j(u - P_E(u)) - j(w - P_E(w))\| \\ &\leq c\|(u - P_E(u)) - (w - P_E(w))\|^{p-1} \\ &\leq c(\|u - w\| + \|P_E(u) - P_E(w)\|)^{p-1} \\ &\leq c(\|u - w\| + \ell\|u - w\|^\gamma)^{p-1} \\ &\leq c(2\varrho^{1-\gamma} + \ell)^{p-1}\|u - w\|^{\gamma(p-1)}. \end{aligned}$$

(b) \Rightarrow (c) is a consequence of Corollary 10.

(c) \Rightarrow (b). Since X is superreflexive, it can be endowed with an equivalent uniformly smooth norm with a modulus of smoothness ϱ_X satisfying for some $k > 0$ and $p > 1$ the estimate $\varrho_X(t) \leq kt^p$ for $t \geq 0$. By Corollary 13, the function $-d_E$ is p -paraconvex. Thus, with the assumption of (c), $\partial^F(-d_E)(w) \neq \emptyset$ and $\partial^F d_E(w) \neq \emptyset$ at any $w \in U \setminus E$. Therefore, $d_E(\cdot)$ is Fréchet differentiable on $U \setminus E$.

Let us prove that $d'_E(\cdot)$ is locally Hölderian on $U \setminus E$. Let $\overline{x} \in U \setminus E$ and assume that $d_E(\cdot)$ is p -paraconvex around \overline{x} . By the p -paraconvexity of $d_E(\cdot)$ and $-d_E(\cdot)$, by virtue of Theorem 8, there exist $c, \delta > 0$ such that for all $u, w \in B(\overline{x}, \delta)$, $v \in \delta\overline{B}_X$,

$$\begin{aligned} -\langle d'_E(u), v \rangle &\leq -d_E(u + v) + d_E(u) + c\|v\|^p, \\ \langle d'_E(w), v \rangle &\leq d_E(w + v) - d_E(w) + c\|v\|^p. \end{aligned}$$

Thus, for any $u, w \in B(\overline{x}, \delta)$, for all $v \in B(0, \delta)$, we have

$$\langle d'_E(w) - d'_E(u), v \rangle \leq 2\|u - w\| + 2c\|v\|^p.$$

Hence

$$\|d'_E(u) - d'_E(w)\| \leq \frac{2\|u - w\|}{t} + 2ct^{p-1}$$

for all $t \in]0, \delta[$. Taking $t = 2^{-1/p}\delta^{1-1/p}\|u - w\|^{1/p} \in]0, \delta[$, we obtain

$$\|d'_E(u) - d'_E(w)\| \leq (2^{1+1/p}\delta^{(1-p)/p} + 2^{1/p}\delta^{(p-1)^2/p}c)\|u - w\|^{(p-1)/p} \\ \forall u, w \in B(\bar{x}, \delta).$$

(b) \Rightarrow (a). By a result of Pisier (see [6, Thm. A.6], [52], [53]) there exists an equivalent norm on X and $c > 0$, $q \geq 2$ such that

$$\delta_X(t) := \inf \left\{ 1 - \frac{1}{2} \|x + y\| : x, y \in \bar{B}_X, \|x - y\| \geq t \right\} \geq ct^q \quad \forall t \in [0, 2].$$

Moreover, by [21, Prop. 5.2, p. 159] we may assume $\|\cdot\|$ is differentiable on $X \setminus \{0\}$. Now, by [70, Thm. 1], given $s > r > 0$, one can find some $b > 0$ such that for any $x, y \in s\bar{B}_X \setminus r\bar{B}_X$,

$$\langle J(x) - J(y), x - y \rangle \geq b\delta_X(\|x - y\|/2s),$$

where $J(\cdot) := (\frac{1}{2}\|\cdot\|^2)'$ is $\|\cdot\|j(\cdot)$. It follows that for such x, y ,

$$(33) \quad \|J(x) - J(y)\| \geq 2^{-q}s^{-q}bc\|x - y\|^{q-1}.$$

By [40], the metric projection mapping $P_E(\cdot)$ is single-valued and continuous on $U \setminus E$. Furthermore, $d'_E(w) = j(w - P_E(w))$ for all $w \in U \setminus E$, hence $(\frac{1}{2}d_E^2)'(w) = J(w - P_E(w))$. Let us prove that $w \mapsto P_E(w)$ is locally Hölderian on $U \setminus E$. Since d'_E is locally Hölderian, $(\frac{1}{2}d_E^2)'$ is also locally Hölderian as

$$\|(d_E^2)'(x) - (d_E^2)'(y)\| \leq 2d_E(x)\|d'_E(x) - d'_E(y)\| + 2\|d'_E(y)\|\|x - y\|.$$

Given $\bar{x} \in U \setminus E$, assumption (b) yields some $\gamma \in]0, 1]$ and $a, \delta > 0$ such that

$$\|J(u - P_E(u)) - J(w - P_E(w))\| \leq a\|u - w\|^\gamma \quad \forall u, w \in B(\bar{x}, \delta).$$

Then inequality (33) yields

$$\|(u - P_E(u)) - (w - P_E(w))\|^{q-1} \leq 2^q s^q b^{-1} c^{-1} a \|u - w\|^\gamma,$$

hence

$$\|P_E(u) - P_E(w)\| \\ \leq \|u - w\| + 2^{q/(q-1)} s^{q/(q-1)} b^{1/(1-q)} c^{1/(1-q)} a^{1/(q-1)} \|u - w\|^{\gamma/(q-1)}.$$

This shows that the metric projection $P_E(\cdot)$ is locally Hölderian around \bar{x} . ■

Acknowledgements. The authors are grateful to C. Zălinescu for a careful reading of the manuscript and to an anonymous referee who pointed out some additional references and limitations. The first author would like to thank the Department of Mathematics of the University of Pau for hospitality and support.

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Received July 1, 2006
Revised version August 2, 2007

(5962)