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Characterization of the convolution operators on quasianalytic classes of Beurling type that admit a continuous linear right inverse

by

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Abstract. Extending previous work by Meise and Vogt, we characterize those convolution operators, defined on the space $\mathcal{E}_{(\omega)}(\mathbb{R})$ of (ω)-quasianalytic functions of Beurling type of one variable, which admit a continuous linear right inverse. Also, we characterize those (ω)-ultradifferential operators which admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}[a,b]$ for each compact interval $[a,b]$ and we show that this property is in fact weaker than the existence of a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$.

1. Introduction. For a weight function ω let $\mathcal{E}_{(\omega)}(\mathbb{R})$ denote the space of all (ω)-ultradifferentiable functions of Beurling type on $\mathbb{R}$. Then each $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ induces a convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$. If ω is non-quasianalytic, i.e., if $\mathcal{E}_{(\omega)}(\mathbb{R})$ admits non-trivial functions with compact support, then Meise and Vogt [21] characterized by various equivalences those convolution operators T_μ on $\mathcal{E}_{(\omega)}(\mathbb{R})$ that admit a continuous linear right inverse. The arguments which they apply in their proofs use the non-quasianalyticity in an essential way, in particular through the existence of cut-off functions and fundamental solutions.

In the present paper we first show that for each quasianalytic weight function ω which satisfies condition (α_1), a convolution operator T_μ on $\mathcal{E}_{(\omega)}(\mathbb{R})$ admits a continuous linear right inverse only if its Fourier–Laplace transform $\widehat{\mu}$ is (ω)-slowly decreasing and satisfies $|\operatorname{Im} a| = O(\omega(a))$ for $a \in V(\widehat{\mu})$ as $|a|$ tends to infinity. Conversely, if these two conditions hold and if ω is a (DN)-weight function, then T_μ admits a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$, even without the assumption that ω satisfies condition (α_1). Accordingly, when ω is a quasianalytic (DN)-weight function satisfying condition (α_1), we obtain a characterization (see Theorem 3.4). To

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prove these results we use an idea of Langenbruch [13] who characterized the convolution operators on the real-analytic functions on $\mathbb{R}$ which admit a solution operator. Also, we work with the space of Fourier–Laplace transforms of the (ω)-quasianalytic functionals, where we apply methods that go back to Berenstein and Taylor [2], Meise [15], and Meise and Taylor [17]. An important step in proving the necessity of the conditions given above is a recent result of Vogt [30] and Bonet and Domański [4] on topological invariants of spaces of $\{\omega\}$-ultradifferentiable functions of Roumieu type.

We also investigate (ω)-ultradifferential operators T_μ on $\mathcal{E}_{(\omega)}(\mathbb{R})$ and on $\mathcal{E}_{(\omega)}[a,b]$ for compact intervals $[a,b]$. It turns out that the surjectivity of such an operator on $\mathcal{E}_{(\omega)}(\mathbb{R})$ already implies the existence of a continuous linear right inverse for T_μ, restricted to $\mathcal{E}_{(\omega)}[a,b]$. As a consequence we deduce that an analogue of a result of Domański and Vogt [9, Theorem 4.7] in the real-analytic case also holds for the class $\mathcal{E}_{(\omega)}$, provided that ω is a quasianalytic (DN)-weight function which satisfies condition (α_1). More precisely, we show that each (ω)-ultradifferential operator T_μ which is surjective on $\mathcal{E}_{(\omega)}(\mathbb{R})$ admits a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$ if and only if for each compact interval $[a,b]$ and each $f \in \mathcal{E}_{(\omega)}[a,b]$ satisfying $T_\mu(f) = 0$ there exists $g \in \mathcal{E}_{(\omega)}(\mathbb{R})$ satisfying $T_\mu(g) = 0$ and $f = g|_{[a,b]}$.

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2. Preliminaries. In this section we introduce the notation that will be used throughout the entire paper.

2.1. *Weight functions.* A function $\omega : \mathbb{R} \to [0,\infty[$ is called a *weight function* if it is continuous, even, increasing on $[0,\infty[$, satisfies $\omega(0) = 0$, and also the following conditions:

(α) There exists $K \geq 1$ such that $\omega(2t) \leq K\omega(t) + K$ for all $t \geq 0$.
(β) $\omega(t) = o(t)$ as t tends to infinity.
(γ) $\log(t) = o(\omega(t))$ as t tends to infinity.
(δ) $\varphi : t \mapsto \omega(e^t)$ is convex on $[0,\infty[$.

(a) If a weight function ω satisfies

$$\int_1^\infty \frac{\omega(t)}{t^2}\,dt = \infty, \tag{Q}$$

then it is called a *quasianalytic weight function*. Otherwise it is called *non-quasianalytic*.

(b) If a weight function ω satisfies the condition

(ε) there exists $C > 0$ such that $\int_1^\infty (\omega(yt)/t^2)\,dt \le C\omega(y) + C$ for all $y > 0$,

then ω is called a *strong weight function*. Obviously, each strong weight function is non-quasianalytic. The reverse implication does not hold.

(c) A weight function ω satisfies *condition* (α_1) if

$$\sup_{\lambda\ge 1}\limsup_{t\to\infty}\frac{\omega(\lambda t)}{\lambda\omega(t)} < \infty.$$

This condition was introduced by Petzsche and Vogt [26] and is equivalent to the existence of $C_1 > 0$ such that for each $W \ge 1$ there exists $C_2 > 0$ such that

$$\omega(Wt + W) \le WC_1\omega(t) + C_2,\ t \ge 0.$$

(d) The *radial extension* $\widetilde{\omega}$ of a weight function ω is defined as

$$\widetilde{\omega} : \mathbb{C}^n \to [0,\infty[, \qquad \widetilde{\omega}(z) := \omega(|z|).$$

It will also be denoted by ω, by abuse of notation.

(e) The *Young conjugate* of the function $\varphi = \varphi_\omega$, which appears in (δ), is defined as

$$\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}, \qquad x \ge 0.$$

2.2. Example. The following are easily seen to be weight functions:

(1) $\omega(t) := |t|(\log(e + |t|))^{-\alpha}$, $\alpha > 0$.
(2) $\omega(t) := |t|^\alpha$, $0 < \alpha < 1$.
(3) $\omega(t) = (\max(0, \log t))^s$, $s > 1$.

2.3. *Ultradifferentiable functions defined by weight functions.* Let ω be a given weight function, let K be a compact and G be an open subset of $\mathbb{R}^N$, and denote by $C^\infty(K)$ the space of all C^∞-Whitney jets on K.

(a) The space $\mathcal{E}_{(\omega)}(G)$ of *(ω)-ultradifferentiable functions of Beurling type* on G is defined as

$$\begin{aligned}\mathcal{E}_{(\omega)}(G) := \{f \in C^\infty(G) &: \text{for each } K \subset G \text{ compact and } m \in \mathbb{N},\\ &p_{K,m}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)|\exp(-m\varphi^*(|\alpha|/m)) < \infty\}.\end{aligned}$$

It is easy to check that $\mathcal{E}_{(\omega)}(G)$ is a Fréchet space if we endow it with the locally convex topology given by the seminorms $p_{K,m}$. We also define the space

$$\mathcal{E}_{(\omega)}(K) := \{f \in C^\infty(K) : p_{K,m}(f) < \infty\ \forall m \in \mathbb{N}\}.$$

(b) For $m \in \mathbb{N}$ let

$$\mathcal{E}_{\{\omega\}}^m(K) := \Big\{ f \in C^\infty(K) : \\ \|f\|_{K,m} := \sup_{x\in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\Big(-\frac{1}{m}\varphi^*(m|\alpha|)\Big) < \infty \Big\}$$

and define the space $\mathcal{E}_{\{\omega\}}(G)$ of $\{\omega\}$*-ultradifferentiable functions of Roumieu type* on G as

$$\mathcal{E}_{\{\omega\}}(G) := \{f \in C^\infty(G) : \text{for each } K \subset G \text{ compact} \\ \text{there is } m \in \mathbb{N} \text{ so that } \|f\|_{K,m} < \infty\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(G) = \operatorname{proj}_{\leftarrow K} \operatorname{ind}_{m\rightarrow} \mathcal{E}_{\{\omega\}}^m(K),$$

where K runs over all compact subsets of G.

Note that $\mathcal{E}_{\{\omega\}}(G)$ is a countable projective limit of (DFN)-spaces, which is ultrabornological, reflexive and complete. This follows from Rösner [27, Satz 3.25] and Vogt [30, Theorem 3.4].

If a statement holds in the Beurling and the Roumieu case then we will use the notation $\mathcal{E}_*(G)$. It means that in all cases $*$ can be replaced either by (ω) or by $\{\omega\}$.

2.4. Definition. Let ω be a weight function and G an open convex set in $\mathbb{R}^N$.

(a) We define

$$A_{(\omega)} := \{f \in H(\mathbb{C}^N) : \exists n \in \mathbb{N} : \|f\|_n := \sup_{z\in\mathbb{C}} |f(z)| \exp(-n\omega(z)) < \infty\}.$$

Endowed with its natural (LB)-topology, $A_{(\omega)}$ is a (DFN)-space.

(b) For each compact set K in G, the *support functional* of K is

$$h_K : \mathbb{R}^N \to \mathbb{R}, \qquad h_K(x) := \sup\{\langle x, y\rangle : y \in K\}.$$

(c) For K as in (b) and $\lambda > 0$ let

$$A(K,\lambda) := \{f \in H(\mathbb{C}^N) : \\ \|f\|_{K,\lambda} := \sup_{z\in\mathbb{C}^N} |f(z)| \exp(-h_K(\operatorname{Im} z) - \lambda\omega(|z|)) < \infty\}$$

and define

$$A_{(\omega)}(\mathbb{C}^N, G) := \operatorname{ind}_{K,n\rightarrow} A(K,n),$$
$$A_{\{\omega\}}(\mathbb{C}^N, G) := \operatorname{ind}_{K\rightarrow} \operatorname{proj}_{\leftarrow n} A(K,1/n).$$

It is easy to check that $A(K,\lambda)$ is a Banach space, that $A_{(\omega)}(\mathbb{C}^N,G)$ is an (LB)-space, and that $A_{\{\omega\}}(\mathbb{C}^N,G)$ is an (LF)-space.

2.5. *The Fourier–Laplace transform.* Let ω be a weight function and let G be an open convex set in $\mathbb{R}^N$. For each $u\in\mathcal{E}_*(G)'$ it is easy to check that

$$\widehat{u}:\mathbb{C}^N\to\mathbb{C},\qquad \widehat{u}(z):=u_x(e^{-i\langle x,z\rangle}),$$

is an entire function which belongs to $A_*(\mathbb{C}^N,G)$ and that

$$\mathcal{F}:\mathcal{E}_*'(G)\to A_*(\mathbb{C}^N,G),\qquad \mathcal{F}(u):=\widehat{u},$$

is linear and continuous.

The following result was proved for $N=1$ by Meyer [23] and for $N\geq 1$ in the Roumieu case by Rösner [27]. For a unified proof we refer to Heinrich and Meise [10, Theorems 3.6 and 3.7].

2.6. Theorem. *For each weight function ω and each convex open set $G\subset\mathbb{R}^N$ the Fourier–Laplace transform*

$$\mathcal{F}:\mathcal{E}_*'(G)\to A_*(\mathbb{C}^N,G)$$

is a linear topological isomorphism.

2.7. *Convolution operators.* For $\mu\in\mathcal{E}_*(\mathbb{R})'$, $\mu\neq 0$, and $\varphi\in\mathcal{E}_*(\mathbb{R})$ we define

$$\check{\mu}(\varphi):=\mu(\check{\varphi}),\qquad \check{\varphi}(x):=\varphi(-x),\qquad x\in\mathbb{R}.$$

The convolution operator $T_\mu:\mathcal{E}_*(\mathbb{R})\to\mathcal{E}_*(\mathbb{R})$ is defined by

$$T_\mu(f):=\check{\mu}*f,\qquad (\check{\mu}*f)(x):=\check{\mu}(f(x-\cdot)),\qquad x\in\mathbb{R}.$$

It is a well-defined, linear, continuous operator; see Meyer [23] and [24]. For $g\in A_*(\mathbb{C},\mathbb{R})$ we define the multiplication operator $M_g:A_*(\mathbb{C},\mathbb{R})\to A_*(\mathbb{C},\mathbb{R})$ by $M_g(f)=gf$. It is well-known that for $\mu\in\mathcal{E}_*(\mathbb{R})$ we have $\mathcal{F}\circ T_\mu^t=M_{\widehat{\mu}}\circ\mathcal{F}$ on $\mathcal{E}_*'(\mathbb{R})$.

By the work of S. Momm [25, Proposition 2 and Corollary 1] (see also [5, Proposition 2.6 and Theorem 2.7]) we have

2.8. Theorem. *For each weight function ω the following conditions are equivalent for $\mu\in\mathcal{E}_{(\omega)}'(\mathbb{R})$, $\mu\neq 0$:*

(1) *$T_\mu:\mathcal{E}_{(\omega)}(\mathbb{R})\to\mathcal{E}_{(\omega)}(\mathbb{R})$ is surjective.*
(2) *The principal ideal $\widehat{\mu}A_{(\omega)}(\mathbb{C},\mathbb{R})$ is closed in $A_{(\omega)}(\mathbb{C},\mathbb{R})$.*
(3) *$\widehat{\mu}$ is (ω)-slowly decreasing in the sense of Ehrenpreis, i.e., there exist $k,x_0>0$ such that for each $x\in\mathbb{R}$ with $|x|\geq x_0$ there exists $t\in\mathbb{R}$ with $|t-x|\leq k\omega(x)$ such that*

$$|\widehat{\mu}(t)|\geq\exp(-k\omega(t)).$$

(4) $\widehat{\mu}$ *is* (ω)*-slowly decreasing*, *i.e.*, *there exists* $C > 0$ *such that for each* $x \in \mathbb{R}$ *with* $|x| \geq C$ *there exists* $\xi \in \mathbb{C}$ *such that*

$$|x - \xi| \leq C\omega(x), \qquad |\widehat{\mu}(\xi)| \geq \exp(-C|\operatorname{Im}\xi| - C\omega(\xi)).$$

(5) $\widehat{\mu}$ *is slowly decreasing for* $A_{(\omega)}(\mathbb{C}, \mathbb{R})$ *in the sense of Berenstein and Taylor*, *i.e.*, *for* $p(z) := |\operatorname{Im} z| + \omega(z)$, $z \in \mathbb{C}$, *there exist* $\varepsilon, C, D > 0$ *such that each connected component* S *of*

$$S(\widehat{\mu}, \varepsilon, C) := \{z \in \mathbb{C} : |\widehat{\mu}(z)| < \varepsilon \exp(-Cp(z))\}$$

satisfies

$$\sup_{z \in S} p(z) \leq D(1 + \inf_{z \in S} p(z)).$$

(6) *There exist* $k \in \mathbb{N}$, $m \in \mathbb{N}$, *and* $R > 0$ *such that for each* $z \in \mathbb{C}$ *with* $|z| \geq R$ *there is a circle* T *surrounding* z *with diameter* $d(T) \leq |\operatorname{Im} z| + k\omega(z)$ *and* $|F(w)| \geq \exp(-mp(w))$ *for each* $w \in T$.

In the next section we will use the following definitions.

2.9. Definition. Let $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ be an increasing, unbounded sequence in $[0, \infty[$. For $R \in \{0, \infty\}$ the *power series spaces* $\Lambda_R(\alpha)$ are defined as

$$\Lambda_R(\alpha) := \Big\{x = (x_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_r := \sum_{j=1}^{\infty} |x_j| \exp(r\alpha_j) < \infty \ \forall r < R\Big\}.$$

$\Lambda_\infty(\alpha)$ is called a *power series space of infinite type*, while $\Lambda_0(\alpha)$ is said to be of *finite type*. Note that $\Lambda_R(\alpha)$ is a Fréchet–Schwartz space for each α and each R.

The following linear topological invariants are related to power series spaces of infinite type. For details we refer to Meise and Vogt [22, Section 29].

2.10. Definition. Let E be a Fréchet space and let $(\|\cdot\|_j)_{j \in \mathbb{N}}$ be a fundamental sequence of seminorms for E.

(a) E has *property* (DN) if there exists $p \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_k^2 \leq C\|x\|_p\|x\|_n, \qquad x \in E.$$

(b) E has *property* (Ω) if for each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ and $0 < \Theta < 1$ there exists $C > 0$ such that

$$\|y\|_q^* \leq C\|y\|_p^{*1-\Theta}\|y\|_k^{*\Theta}, \qquad y \in E',$$

where for $y \in E'$,

$$\|y\|_k^* := \sup\{|y(x)| : \|x\|_k \leq 1\} \in \mathbb{R} \cup \{+\infty\}.$$

Note that each subspace of a power series space of infinite type has (DN), while each quotient space has (Ω).

From Meise and Taylor [18], we recall the following definition.

2.11. Definition. A weight function ω is called a (DN)-*weight function* if it satisfies:

(2.1) for each $C > 1$ there exist $R_0 > 0$ and $0 < \delta < 1$ such that for each $R \geq R_0$,
$$\omega^{-1}(CR)\omega^{-1}(\delta R) \leq (\omega^{-1}(R))^2.$$

For the significance of (DN)-weight functions we refer to Meise and Taylor [18, Theorem 3.4].

2.12. Example. Let ω be a weight function for which there exists $A > 0$ such that
$$2\omega(t) \leq \omega(At) + A, \quad t \geq 0.$$
Then ω is a (DN)-weight function by Meise and Taylor [18, Example 3.5(4)]. In particular, the following functions are quasianalytic (DN)-weight functions which also satisfy (α_1):

(1) $\omega(t) := |t|(\log(e + |t|))^{-\alpha}$, $0 < \alpha \leq 1$.
(2) $\omega(t) := |t|(\log(e + \log(e + |t|)))^{-1}$.

3. The global case. In this section we characterize when a convolution operator T_μ admits a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$. To do this we will use the following lemma.

3.1. Lemma. *Let ω be a weight function which satisfies condition (α_1). Suppose that there is a sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers with $0 < |a_j| < |a_{j+1}|$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty} |a_j| = \infty$ such that $\omega(a_j) \leq |\operatorname{Im} a_j|/j^2$ for each $j \in \mathbb{N}$. Then there is a weight function σ satisfying condition (α_1) and $\omega(t) = o(\sigma(t))$ as $t \to \infty$, and there is a subsequence $(a_{j(k)})_{k\in\mathbb{N}}$ of $(a_j)_{j\in\mathbb{N}}$ such that $\sigma(a_{j(k)}) = O(|\operatorname{Im} a_{j(k)}|)$ as $k \to \infty$.*

Proof. Case 1: $|\operatorname{Im} a_j| = o(|a_j|)$ as $j \to \infty$. Then we write $a_j = \alpha_j + i\beta_j$, $j \in \mathbb{N}$, and choose $j(1) \in \mathbb{N}$ such that $|\beta_j| \leq |\alpha_j|/2$ and $\omega(\alpha_j) \geq 1$ if $j \geq j(1)$. By the properties of ω, there exists $D \geq 1$ such that for all $j \geq j(1)$,
$$\omega(\alpha_j) \leq \omega(|a_j|) \leq \omega(|\alpha_j| + |\beta_j|) \leq \omega\big(\tfrac{3}{2}|\alpha_j|\big) \leq D\omega(|\alpha_j|) + D \leq 2D\omega(\alpha_j).$$
Without restriction we may assume that $\alpha_j > 0$ for $j \geq j(1)$. Then we define $x_1 := 0$ and $x_2 := \alpha_{j(1)}$. Proceeding by induction suppose that $j(n) > \max(n, j(n-1))$ and $x_{n+1} := \alpha_{j(n)} > 2x_n$ are already selected satisfying
$$\omega(x_{n+1}) \geq 2^{n-i+1}\omega(x_i),\ 1 \leq i \leq n, \quad \text{and} \quad \omega(x) \leq x/n^2 \quad \text{if } x \geq x_n.$$
Since $(|\alpha_j|)_{j\in\mathbb{N}}$ tends to infinity and ω satisfies condition 2.1(γ) and $\lim_{t\to\infty}\omega(t)/t = 0$, we can find $j(n+1) > \max(n+1, j(n))$ such that $\alpha_{j(n+1)} > 2x_{n+1}$, $\omega(\alpha_{j(n+1)}) \geq 2^{n+2-i}\omega(x_i)$, $1 \leq i \leq n+1$, and $\omega(x) \leq x/(n+1)^2$ if $x \geq \alpha_{j(n+1)}$. Now define $x_{n+2} := \alpha_{j(n+1)}$.

As in Braun, Meise, and Taylor [7, Lemma 1.6], define

$$\text{(3.1)} \quad \sigma : [0,\infty[\, \to [0,\infty[, \qquad \sigma(x) := n\omega(x) - \sum_{i=1}^{n} \omega(x_i), \quad x \in [x_n, x_{n+1}[.$$

Then it follows as in the proof of [7, Lemma 1.6] that σ is continuous, has properties (α) and (δ) of 2.1, and $\omega(t) = o(\sigma(t))$ as $t \to \infty$, which implies that σ has property (γ) as well. On the other hand, for each $t \in [x_n, x_{n+1}]$, we get $\sigma(t) \le n\omega(t) \le t/n$, and therefore $\sigma(t) = o(t)$ as $t \to \infty$. Moreover, since ω satisfies condition (α_1) it follows from [7, Lemma 1.7] that σ also satisfies (α_1).

To show that $\sigma(a_{j(k)}) = O(|\mathrm{Im}\, a_{j(k)}|)$ as $k \to \infty$, we first fix $k(0) \in \mathbb{N}$ such that $\sigma(a_{j(k)}) \ge 1$ if $k \ge k(0)$. Then we apply property 2.1(α) for the weight function σ to find $L \ge 1$ such that for all $k \ge k(0)$,

$$\begin{aligned}\sigma(a_{j(k)}) &\le \sigma(3|\alpha_{j(k)}|/2) \le L\sigma(\alpha_{j(k)}) = L\sigma(x_{k+1}) \le Lk\omega(x_{k+1}) \\ &\le Lj(k)\omega(a_{j(k)}) \le Lj(k)\,\frac{1}{j(k)^2}\,|\mathrm{Im}\, a_{j(k)}| \le L|\mathrm{Im}\, a_{j(k)}|.\end{aligned}$$

Case 2: $\liminf_{j\to\infty} |\mathrm{Im}\, a_j|/|a_j| > 0$. Then there are $\delta > 0$ and a subsequence $(a_{j(k)})_{k\in\mathbb{N}}$ such that $|\mathrm{Im}\, a_{j(k)}| \ge \delta |a_{j(k)}|$ for each $k \in \mathbb{N}$. Next choose inductively a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,\infty[$ which satisfies

$$x_1 = 0, \qquad x_{n+1} \ge 2x_n, \qquad \omega(x_2) > 0, \qquad \omega(t)/t \le 1/n^2 \quad \text{if } t \ge x_n,$$

and

$$\omega(x_{n+1}) \ge 2^{n+1-i}\omega(x_i), \qquad 1 \le i \le n,$$

and define σ as in (3.1). Then σ is a weight function which satisfies $\omega(t) = o(\sigma(t))$, $\sigma(t) = o(t)$ as $t \to \infty$, and condition (α_1). Next choose $k_0 \in \mathbb{N}$ such that $\sigma(a_{j(k)}) \ge 1$ if $k \ge k_0$. By the property 2.1(β) of the weight σ, we find $C \ge 1$ such that, for $k \ge k_0$,

$$\sigma(a_{j(k)}) \le C|a_{j(k)}| \le \frac{C}{\delta}\,|\mathrm{Im}\, a_{j(k)}|. \ \blacksquare$$

3.2. Proposition. *Let ω be a weight function which satisfies (α_1) and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$. If the convolution operator*

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

admits a continuous linear right inverse, then the following two conditions are satisfied:

(a) $\widehat{\mu}$ *is (ω)-slowly decreasing.*
(b) *There exists $C > 0$ such that*

$$|\mathrm{Im}\, a| \le C(1 + \omega(a)), \qquad a \in \mathbb{C},\ \widehat{\mu}(a) = 0.$$

Proof. If $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ admits a continuous linear right inverse, then T_μ is surjective. Hence Momm's Theorem 2.8 implies that $\widehat{\mu}$ is (ω)-slowly decreasing, which proves (a).

To prove that condition (b) holds we argue by contradiction and assume that it is not satisfied. Then there exists a sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers such that $\widehat{\mu}(a_j) = 0$ and $\omega(a_j) \le j^{-2}|\operatorname{Im} a_j|$ for each $j \in \mathbb{N}$. By Lemma 3.1 we can choose a weight function σ with $\omega = o(\sigma)$, which also satisfies condition (α_1), such that by passing to a subsequence, we have $\sigma(a_j) = O(|\operatorname{Im} a_j|)$ as $j \to \infty$. Proceeding by recurrence, we extract a subsequence of $(a_j)_{j\in\mathbb{N}}$, which we denote in the same way, such that, for $n(t) := \operatorname{card}\{j \in \mathbb{N} : |a_j| \le t\}$,

(i) $|a_{j+1}| \ge 4|a_j|$,
(ii) $n(t)\log t = o(\omega(t))$ as $t \to \infty$.

As in Braun, Meise, and Vogt [8, 3.11], define

$$F(z) := \prod_{j=1}^{\infty}\left(1 - \frac{z}{a_j}\right), \qquad z \in \mathbb{C}.$$

By Rudin [28, Theorem 15.6], F is an entire function such that its set of zeros consists of the sequence $(a_j)_{j\in\mathbb{N}}$, and satisfies the following conditions:

(1) There exists $C > 0$ such that $|F(z)| \le C\exp(\omega(z))$, $z \in \mathbb{C}$.
(2) There exists $\varepsilon_0 > 0$ such that $|F(\zeta)| \ge \varepsilon_0 \exp(-\omega(\zeta))$ for all $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} B(a_j, 1)$.
(3) There exist $\varepsilon_0, K_0 > 0$ such that, if $\zeta \in \mathbb{C}$ satisfies $1 \le |\zeta - a_j| \le 2$ for some j, then
$$|F(\zeta)| \ge \varepsilon_0 \exp(-K_0\omega(a_j)).$$

This can be achieved by the arguments given in [6, proof of Lemma 3.5], based on [8, 3.11]. In particular, F is (ω)-slowly decreasing.

Since each a_j is a zero of $\widehat{\mu}$, it follows that $g := \widehat{\mu}/F$ is an entire function. Since F is (ω)-slowly decreasing, we conclude that $g \in A_{(\omega)}(\mathbb{C}, \mathbb{R})$ (notation as in 2.4). This implies, in particular, that $M_g : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})$ is continuous.

By hypothesis $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ admits a continuous linear right inverse. Therefore, $M_{\widehat{\mu}} : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})$ admits a continuous linear left inverse $L_{\widehat{\mu}}$. The operator $L_F := L_{\widehat{\mu}} \circ M_g : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})$ is continuous, and it is a left inverse of M_F since

$$L_F M_F(h) = L_{\widehat{\mu}} M_g M_F(h) = L_{\widehat{\mu}}(gFh) = L_{\widehat{\mu}}(\widehat{\mu}h) = h, \qquad h \in A_{(\omega)}(\mathbb{C}, \mathbb{R}).$$

Thus $M_F : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})$ admits a continuous linear left inverse.

Since the weight σ constructed at the beginning of the proof by the use of Lemma 3.1 satisfies $\omega(t) = o(\sigma(t))$ as $t \to \infty$, we conclude that $A_{(\omega)}(\mathbb{C},\mathbb{R}) \subset A_{\{\sigma\}}(\mathbb{C},\mathbb{R})$ and that the inclusion map is continuous. Moreover, the growth estimate for F in (1) implies that M_F also defines a continuous linear operator from $A_{\{\sigma\}}(\mathbb{C},\mathbb{R})$ into itself. Next define

$$\varrho : H(\mathbb{C}) \to \mathbb{C}^{\mathbb{N}}, \qquad \varrho(f) := (f(a_j))_{j\in\mathbb{N}}.$$

Proceeding as in the proof of [6, Lemma 3.8] (based on the method of Meise [15, Theorem 3.7]), we can conclude from the properties (1)–(3) of F that

$$\begin{aligned} M_F A_{(\omega)}(\mathbb{C},\mathbb{R}) &= \{f \in A_{(\omega)}(\mathbb{C},\mathbb{R}) : f(a_j) = 0\ \forall j \in \mathbb{N}\} \\ &= \ker \varrho \cap A_{(\omega)}(\mathbb{C},\mathbb{R}), \\ M_F A_{\{\sigma\}}(\mathbb{C},\mathbb{R}) &= \{f \in A_{\{\sigma\}}(\mathbb{C},\mathbb{R}) : f(a_j) = 0\ \forall j \in \mathbb{N}\} \\ &= \ker \varrho \cap A_{\{\sigma\}}(\mathbb{C},\mathbb{R}). \end{aligned}$$

Moreover, the map ϱ defined above induces an isomorphism between the quotient $A_{(\omega)}(\mathbb{C},\mathbb{R})/M_F A_{(\omega)}(\mathbb{C},\mathbb{R})$ and the sequence space

$$E := \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \exists n : \|x\|_n := \sup_{j\in\mathbb{N}} |x_j| \exp(-n|\operatorname{Im} a_j| - n\omega(a_j)) < \infty\},$$

as well as an isomorphism between $A_{\{\sigma\}}(\mathbb{C},\mathbb{R})/M_F A_{\{\sigma\}}(\mathbb{C},\mathbb{R})$ and the (LF)-sequence space

$$G := \operatorname{ind}_{\to n} \operatorname{proj}_{\leftarrow k} K(n,k),$$

where

$$K(n,k) := \left\{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{n,k} := \sup_{j\in\mathbb{N}} |x_j| \exp\left(-n|\operatorname{Im} a_j| - \frac{1}{k}\,\sigma(a_j)\right) < \infty\right\}.$$

We then obtain the diagram

$$\begin{array}{ccccccc} 0 \to & A_{(\omega)}(\mathbb{C},\mathbb{R}) & \xrightarrow{M_F} & A_{(\omega)}(\mathbb{C},\mathbb{R}) & \xrightarrow{\varrho_1} & E \to 0 \\ & \cap & & \cap & & \\ 0 \to & A_{\{\sigma\}}(\mathbb{C},\mathbb{R}) & \xrightarrow{M_F} & A_{\{\sigma\}}(\mathbb{C},\mathbb{R}) & \xrightarrow{\varrho_2} & G \to 0 \end{array}$$

where ϱ_1 and ϱ_2 are the restrictions of ϱ. Since M_F has a continuous linear left inverse, ϱ_1 has a continuous linear right inverse $R_1 : E \to A_{(\omega)}(\mathbb{C},\mathbb{R})$.

Next note that $\omega(a_j) \le j^{-2}|\operatorname{Im} a_j|$, $j \in \mathbb{N}$, implies, for each $n \in \mathbb{N}$,

$$(3.2) \quad 0 < n|\operatorname{Im} a_j| \le n|\operatorname{Im} a_j| + n\omega(a_j) \le \left(n + \frac{n}{j^2}\right)|\operatorname{Im} a_j| \le 2n|\operatorname{Im} a_j|,$$

while the choice of σ implies the existence of $C \in \mathbb{N}$ such that $\sigma(a_j) \le$

$C|\mathrm{Im}\, a_j| + C$, $j \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$ we have

$$(3.3)\quad \begin{aligned} 0 < n|\mathrm{Im}\, a_j| &\le n|\mathrm{Im}\, a_j| + \frac{1}{k}\,\sigma(a_j) \le n|\mathrm{Im}\, a_j| + \frac{C}{k}\,|\mathrm{Im}\, a_j| + \frac{C}{k} \\ &\le (n+C)|\mathrm{Im}\, a_j| + C. \end{aligned}$$

From (3.2) and (3.3) it follows easily that the sequence spaces E and G coincide algebraically and topologically with the dual of $P := \Lambda_\infty((|\mathrm{Im}\, a_j|)_{j\in\mathbb{N}})$. If we identify E and G and use the fact that $A_{(\omega)}(\mathbb{C},\mathbb{R}) \subset A_{\{\sigma\}}(\mathbb{C},\mathbb{R})$, then the map $R_1 : E = G \to A_{\{\sigma\}}(\mathbb{C},\mathbb{R})$ is a continuous linear right inverse for ϱ_2. Hence $G = P'$ is isomorphic to a complemented subspace of $A_{\{\sigma\}}(\mathbb{C},\mathbb{R}) = \mathcal{E}'_{\{\sigma\}}(\mathbb{R})$. Since σ satisfies condition (α_1), a result of Vogt [30, Theorem 1.8], and Bonet and Domański [4, Theorem 6.2 in connection with Propositions 5.3(a) and 5.4(b)] implies that each Fréchet quotient of $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ and hence P has the topological invariant $(\overline{\overline{\Omega}})$. However, this is a contradiction since no power series space satisfies $(\overline{\overline{\Omega}})$. Hence condition (b) is satisfied. ∎

Remark. For $\omega(t) = t$ we get $\mathcal{E}_{(\omega)}(\mathbb{R}) = H(\mathbb{C})$. By Taylor [29, Theorem 5.1] or Meise [16, Theorem 3.5], each convolution operator on $H(\mathbb{C})$ admits a continuous linear right inverse. This is the reason why we require in 2.1(β) that $\omega(t) = o(t)$ as t tends to infinity.

Next we show that the necessary conditions in Proposition 3.2 are sufficient if ω is a (DN)-weight function.

3.3. Proposition. *Let ω be a* (DN)*-weight function. Then for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ the convolution operator*

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

admits a continuous linear right inverse if conditions (a) *and* (b) *in Proposition* 3.2 *hold.*

Proof. Obviously, it is no restriction to assume that $V(\widehat{\mu}) := \{a \in \mathbb{C} : \widehat{\mu}(a) = 0\}$ is infinite. To simplify the notation we let

$$F := \widehat{\mu} \in A_{(\omega)}(\mathbb{C},\mathbb{R}), \qquad p(z) := |\mathrm{Im}\, z| + \omega(z).$$

Note that then $A_{(\omega)}(\mathbb{C},\mathbb{R})$ coincides with the algebra $A_p(\mathbb{C})$ of Berenstein and Taylor [2] and Meise [15]. Since $F = \widehat{\mu}$ is (ω)-slowly decreasing, it follows from Theorem 2.8 that there are $k, m \in \mathbb{N}$ and $R > 0$ such that for each $a \in V(F)$, $|a| \ge R$, there is a circle T_a surrounding a with diameter $d(T_a) \le k\omega(a) + |\mathrm{Im}\, a|$ and $|F(w)| \ge \exp(-mp(w))$ for each w in T_a. By condition (b), $d(T_a) \le (k+C)\omega(a) + C$.

We apply again the fact that F is (ω)-slowly decreasing to find $0 < \varepsilon_0 < 1$, $C_0 > 0$, $D_0 > 0$ such that every component of $S(F, \varepsilon_0, C_0) := \{z \in \mathbb{C} : |F(z)| < \varepsilon_0 \exp(-C_0 p(z))\}$ (is bounded and) satisfies $\sup_{z\in S} p(z) \le D_0(1 + \inf_{z\in S} p(z))$. Without restriction, we may assume that $C_0 > m$.

Let S be a connected component of $S(F,\varepsilon_0,C_0)$ which has a non-empty intersection with $V(F)$ and let $a\in S\cap V(F)$. If $|a|\ge R$, then S is contained in the disc D_a limited by the circle T_a. In particular, if $z\in S$, then

$$|\operatorname{Im} z|\le|\operatorname{Im} a|+d(T_a)\le 2|\operatorname{Im} a|+k\omega(a)\le(k+2C)\omega(a)+2C.$$

On the other hand, if $z\in D_a$, $|z-a|\le d(T_a)$, and if K_0 is chosen according to condition 2.1(α) such that $\omega(2t)\le K_0(\omega(t)+1)$ for $t\ge 0$, then

$$\begin{aligned}\omega(a)&\le K_0\omega(z)+K_0\omega(d(T_a))+K_0^2\\&\le K_0\omega(z)+K_0\omega(C+(k+C)\omega(a))+K_0^2\\&\le K_0\omega(z)+K_0^2\omega(C)+K_0^2\omega((k+C)\omega(a))+2K_0^2.\end{aligned}$$

Thus

$$\omega(a)\left(1-(k+C)K_0^2\,\frac{\omega((k+C)\omega(a))}{(k+C)\omega(a)}\right)\le K_0\omega(z)+K_0^2\omega(C)+2K_0^2.$$

Since $\omega(t)=o(t)$, if $|a|\ge R_1>R$, we get $\omega(a)\le 2K_0\omega(z)+2K_0^2\omega(C)+4K_0^2$, and the existence of $A_1>0$ such that $|\operatorname{Im} z|\le 2(k+2C)K_0\omega(z)+A_1$ for each $z\in S$.

Since there are only finitely many components S which meet $V(F)$ at a point a with $|a|\le R_1$, and they are bounded, we conclude that there is $A_2\ge 1$ such that for each component S of $S(F,\varepsilon_0,C_0)$ which satisfies $S\cap V(F)\neq\emptyset$ we have

$$|\operatorname{Im} z|\le A_2\omega(z)+A_2,\qquad z\in S.$$

Denote by $I_{\mathrm{loc}}(F)$ the ideal of $A_{(\omega)}(\mathbb{C},\mathbb{R})$ of all $g\in A_{(\omega)}(\mathbb{C},\mathbb{R})$ such that $V(g)\supset V(F)$ and at each $a\in V(F)$ the order of vanishing of g is at least as high as the one of F at a, i.e., $\operatorname{ord}(g,a)\ge\operatorname{ord}(F,a)$.

Since F is (ω)-slowly decreasing, $I_{\mathrm{loc}}(F)=FA_{(\omega)}(\mathbb{C},\mathbb{R})$ and it is closed; see [2] and [15, 3.5]. We select the connected components of $S(F,\varepsilon_0,C_0)$ which intersect $V(F)$ and label them as $(S_j)_{j\in\mathbb{N}}$ so that $\gamma_j:=\sup_{z\in S_j}p(z)$, $j\in\mathbb{N}$, is increasing. Recall that there is $D_1>0$ such that

$$\omega(z)\le p(z)\le D_1+D_1\omega(z),\qquad z\in S_j,\ j\in\mathbb{N}. \tag{3.4}$$

Put $\gamma=(\gamma_j)_{j\in\mathbb{N}}$. Following the proof of [15, Theorem 3.7, pp. 77–78] we let

$$E_j:=\prod_{b\in S_j\cap V(F)}\mathbb{C}^{\operatorname{ord}(F,b)},\qquad j\in\mathbb{N}, \tag{3.5}$$

and we define $\varrho_j: H^\infty(S_j)\to E_j$ by

$$\varrho_j(f):=\left(\left(\frac{1}{k!}\,f^{(k)}(b)\right)_{0\le k<\operatorname{ord}(F,b)}\right)_{b\in S_j\cap V(F)}. \tag{3.6}$$

We endow E_j with the quotient norm

$$\|\varrho_j(g)\| := \inf\{\|h\|_{H^\infty(S_j)} : \varrho_j(h) = \varrho_j(g)\}, \quad g \in H^\infty(S_j).$$

Then ϱ_j is linear, continuous and surjective. Now, for each $j \in \mathbb{N}$ there is $R_j : E_j \to H^\infty(S_j)$ continuous and linear such that $\varrho_j R_j = \mathrm{id}_{E_j}$ and $\|R_j\| \le 2\dim E_j$. If $f \in A_{(\omega)}(\mathbb{C},\mathbb{R})$, then $\|f\|_n := \sup_{z\in\mathbb{C}} |f(z)| \exp(-np(z)) < \infty$ for some $n \in \mathbb{N}$. This implies

$$\|f|_{S_j}\|_{H^\infty(S_j)} \le e^{n\gamma_j}\|f\|_n.$$

Therefore, $\|\varrho_j(f|_{S_j})\|_j \le e^{n\gamma_j}\|f\|_n$ and consequently $(\varrho_j(f|_{S_j}))_{j\in\mathbb{N}}$ is in $K_\infty(\gamma,(E_j)_j)$, where

$$K_\infty(\gamma,(E_j)_j) := \Big\{(x_j)_j \in \prod_{j\in\mathbb{N}} E_j : \exists n : |||x|||_n := \sup_{j\in\mathbb{N}} \|x_j\|_j e^{-n\gamma_j} < \infty\Big\}.$$

By the above, the map

$$\varrho : A_{(\omega)}(\mathbb{C},\mathbb{R}) \to K_\infty(\gamma,(E_j)_{j\in\mathbb{N}}), \quad \varrho(g) := (\varrho_j(g|_{S_j}))_{j\in\mathbb{N}}$$

is linear and continuous. The arguments of Meise [15, pp. 77–78], which involve the semilocal to global extension theorem of Berenstein and Taylor [3, 2.2] (see also [2, p. 110]), show that ϱ is surjective and $\ker\varrho = I_{\mathrm{loc}}(F)$. In particular, $K_\infty(\gamma,(E_j)_{j\in\mathbb{N}})$ is nuclear, so we can apply [15, 1.3] to conclude that for each $n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $d > 0$ such that

$$2(\dim E_j)e^{n\gamma_j} \le de^{k\gamma_j}, \quad j \in \mathbb{N}. \tag{3.7}$$

We want to show that ϱ has a continuous linear right inverse. To do this, fix $y = (y_j)_{j\in\mathbb{N}} \in K_\infty(\gamma,(E_j)_j)$. Then there is $n \in \mathbb{N}$ with $|||y|||_n := \sup_{j\in\mathbb{N}} \|y_j\|_j e^{-n\gamma_j} < \infty$. Now $\lambda_j := R_j y_j$ is in $H^\infty(S_j)$ and satisfies $\|\lambda_j\|_{H^\infty(S_j)} \le 2(\dim E_j)\|y_j\|_j \le 2(\dim E_j)|||y|||_n e^{n\gamma_j}$. By (3.7) this implies

$$\|\lambda_j\|_{H^\infty(S_j)} \le d|||y|||_n e^{k\gamma_j} \le d|||y|||_n \exp(kD_0(1 + \inf_{z\in S_j} p(z))).$$

Therefore, we deduce by (3.4) that for each $z \in S_j$ we have

$$|\lambda_j(z)| \le de^{kD_0}|||y|||_n \exp(kD_0 p(z)) \le de^{kD_0+kD_0D_1}|||y|||_n \exp(kD_0D_1\omega(z)).$$

We define $P(y) \in H(S(F,\varepsilon_0,C_0))$ by $P(y)(z) = R_j y_j(z)$ if $z \in S_j$ and $P(y)(z) = 0$ otherwise. Clearly, $P : K_\infty(\gamma,(E_j)_j) \to H(S(F,\varepsilon_0,C_0))$ is well-defined and linear. Moreover,

$$|P(y)(z)| \le de^{k(D_0+D_0D_1)}|||y|||_n \exp(kD_0D_1\omega(z)), \quad z \in S(F,\varepsilon_0,C_0). \tag{3.8}$$

As in the proof of Bernstein and Gay [1, Theorem 2.1.4], it follows that there exist $0 < \varepsilon_1 < \varepsilon_0$, $C_1 > C$, $A_0 > 0$, $B_0 > 0$, and $\chi \in C^\infty(\mathbb{C})$ satisfying $0 \le \chi \le 1$, $\chi \equiv 1$ on $S(F,\varepsilon_1,C_1)$, $\operatorname{supp}\chi \subset S(F,\varepsilon_0,C_0)$ and

$$\Big|\frac{\partial\chi}{\partial\bar z}(z)\Big| \le A_0 \exp(B_0 p(z)), \quad z \in \mathbb{C}.$$

Then $\chi P(y)$ is in $C^\infty(\mathbb{C})$ and is holomorphic on $S(F,\varepsilon_1,C_1)$. This implies $\frac{\partial}{\partial\overline{z}}(\chi P(y)) = \frac{\partial\chi}{\partial\overline{z}}P(y)$. We set

$$v := -\frac{1}{F}\,\frac{\partial}{\partial\overline{z}}(\chi P(y)) = -\frac{1}{F}\,\frac{\partial\chi}{\partial\overline{z}}\,P(y).$$

Clearly, v is a C^∞-function on $\mathbb{C}$ which vanishes on $S(F,\varepsilon_1,C_1)$. To derive an estimate for v, note that $|F(z)| \ge \varepsilon_1 \exp(-C_1 p(z))$ whenever $z \notin S(F,\varepsilon_1,C_1)$ for some $j \in \mathbb{N}$. From this estimate and (3.8) as well as (3.4) we get the existence of $A_3, A_4, B_1, B_2 > 0$ such that

$$|v(z)| \le A_3 e^{B_1 p(z)}\,\frac{1}{\varepsilon_1}\,e^{C_1 p(z)} A_0 e^{B_0 p(z)} \le A_4 e^{B_2\omega(z)}. \tag{3.9}$$

To use this estimate we let

$$K(\omega) := \ \{f \in C^\infty(\mathbb{C}) : \|f\|_k := \sup_{|\alpha|\le k}\sup_{z\in\mathbb{C}} |f^{(\alpha)}(z)|e^{k\omega(z)} < \infty\ \forall k \in \mathbb{N}\}.$$

Then $K(\omega)$ is a nuclear Fréchet space. Since ω is a (DN)-weight function, it follows from Meise and Taylor [18, Proposition 1.9 and Theorem 2.17] that

$$0 \to A_{(\omega)} \to K'(\omega) \xrightarrow{\overline{\partial}} K'(\omega) \to 0 \tag{3.10}$$

is an exact sequence which splits. Hence there exists a continuous linear operator $L : K'(\omega) \to K'(\omega)$ which satisfies $\overline{\partial} \circ L = \mathrm{id}_{K'(\omega)}$. Now note that the estimate (3.9) and the fact that ω satisfies condition 2.1(γ) imply that v defines an element of $K'(\omega)$ by

$$v(f) := \int_{\mathbb{C}} v(z)f(z)\,dz, \qquad f \in K(\omega).$$

Next we define

$$Q : K_\infty(\gamma,(E_j)_{j\in\mathbb{N}}) \to K'(\omega), \qquad Q(y) := \chi P(y) + FL(v).$$

Then it follows from (3.8) and an easy computation that Q is well-defined, linear and continuous. Moreover,

$$\frac{\partial}{\partial\overline{z}}Q(y) = \frac{\partial\chi}{\partial\overline{z}}\,P(y) + F\,\frac{\partial}{\partial\overline{z}}L(v) = \frac{\partial\chi}{\partial\overline{z}}\,P(y) + Fv = 0.$$

Hence $Q(y) \in \ker\overline{\partial}$ and by (3.10) this implies $Q(y) \in A_{(\omega)}$. Since the sequence (3.10) is exact and consists of (DFN)-spaces, it follows from Meise and Vogt [22, Propositions 26.4 and 26.24] that it is topologically exact. Hence Q is in fact a continuous linear map from $K_\infty(\gamma,(E_j)_{j\in\mathbb{N}})$ into $A_{(\omega)}$. Since $A_{(\omega)} \subset A_{(\omega)}(\mathbb{C},\mathbb{R})$ with continuous inclusion,

$$Q : K_\infty(\gamma,(E_j)_j) \to A_{(\omega)}(\mathbb{C},\mathbb{R})$$

is linear, continuous, and satisfies $\varrho(Q(y)) = y$ for each y in $K_\infty(\gamma,(E_j)_j)$. In particular, $\ker\varrho = FA_{(\omega)}(\mathbb{C},\mathbb{R})$ is complemented in $A_{(\omega)}(\mathbb{C},\mathbb{R})$.

To complete the proof, we must show that T_μ has a continuous linear right inverse. It is enough to see that $T_\mu^t : \mathcal{E}'_{(\omega)}(\mathbb{R}) \to \mathcal{E}'_{(\omega)}(\mathbb{R})$ has a continuous linear left inverse. Since $\mathcal{F} \circ T_\mu^t = M_{\widehat{\mu}} \circ \mathcal{F}$ for $M_{\widehat{\mu}} g := \widehat{\mu} g$, we have proved that the image of $M_{\widehat{\mu}}$ is $\widehat{\mu} A_{(\omega)}(\mathbb{C}, \mathbb{R}) = F A_{(\omega)}(\mathbb{C}, \mathbb{R})$, and that it is complemented in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$. This yields the conclusion. ∎

3.4. Theorem. *Let ω be a quasianalytic* (DN)*-weight function which satisfies condition* (α_1). *Then for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ the convolution operator*

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

admits a continuous linear right inverse if and only if conditions (a) *and* (b) *in Proposition* 3.2 *hold.*

Proof. This follows immediately from Propositions 3.2 and 3.3. ∎

Remark. Note that Theorem 3.4 extends the main result of Meise and Vogt [21] from the non-quasianalytic to the quasianalytic case. In the non-quasianalytic case, however, it was not required that ω is a (DN)-weight function, nor that ω satisfies (α_1). In fact, the theorem holds for $\omega(t) = (\log(1+t))^\alpha$, $\alpha > 1$, which is not a (DN)-weight function, however, it satisfies condition (α_1).

4. Ultradifferential operators on compact intervals. In this section we prove that (ω)-ultradifferential operators can behave differently on the real line than on compact intervals. Also, we show that for them the characterization in Theorem 3.4 can be complemented by a condition which is similar to the one which was given by Domański and Vogt [9, Theorem 4.7]. To do so we need two lemmas and more notation.

4.1. Definition. Let ω be a weight function and assume that for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ its Fourier–Laplace transform $\widehat{\mu}$ is in $A_{(\omega)}$. Then the operator T_μ will be called an (ω)-*ultradifferential operator* since for each $f \in \mathcal{E}_{(\omega)}(\mathbb{R})$ we have

$$T_\mu(f) = \sum_{j=0}^{\infty} i^j \frac{\widehat{\mu}^{(j)}(0)}{j!} f^{(j)}.$$

4.2. Lemma. *Let ω be a weight function. Assume that $F \in A_{(\omega)}$ satisfies*

$$|F(z)| \le B \exp(B\omega(z)), \qquad z \in \mathbb{C}, \tag{4.1}$$

for some $B > 0$ and that F is (ω)-slowly decreasing. Then there exist $m > 0$ and $R_0 > 0$ such that for each $z \in \mathbb{C}$ with $|z| \ge R_0$ there exists $\zeta \in \mathbb{C}$ with $|z - \zeta| \le m\omega(z)$ such that

$$|F(\zeta)| \ge \exp(-m\omega(\zeta)).$$

Proof. Since F is slowly decreasing, it follows from Theorem 2.8(3) that there exist $k, x_0 > 0$ such that for each $x \in \mathbb{R}$ with $|x| \geq x_0$ there exists $t \in \mathbb{R}$ with $|t - x| \leq k\omega(x)$ such that

$$|F(t)| \geq \exp(-k\omega(t)).$$

Next we fix $z = x + iy$ in $\mathbb{C}$ with $|x| \geq x_0$ and consider two cases:

Case 1: $|y| \leq k\omega(x)$. Since $|x| \geq x_0$ we can choose $t \in \mathbb{R}$ according to 2.8(3). Then $\zeta := t$ satisfies

$$|\zeta - z| = |t - x - iy| \leq |t - x| + |y| \leq 2k\omega(x) \leq 2k\omega(z).$$

Hence the assertion follows for $m = 2k$ in this case.

Case 2: $|y| > k\omega(x)$. Select $n > 2$ (depending on z) with $|y|/n < k\omega(x)$ and $t \in \mathbb{R}$ according to 2.8(3). We apply the minimum-modulus theorem [14, Chap. I, Theorem 11] with

$$\eta := \frac{1 - \left(1 + \frac{1}{\sqrt{2}\,n}\right)^{-1}}{16}$$

to find $\varrho > 0$ with $|t - z| < \varrho < \left(1 + \frac{1}{\sqrt{2}\,n}\right)|t - z| =: R$ such that for $H = 2 + \log(3e/2\eta)$ we have

$$|F(\xi)| \geq |F(t)|^{H+1}\big(\sup_{|w-t|=2eR} |F(w)|\big)^{-H} \tag{4.2}$$

for each $\xi \in \mathbb{C}$ with $|\xi - t| = \varrho$. We now choose $\zeta \in \mathbb{C}$ with $|\zeta - t| = \varrho$ and $|\zeta - z| < |t - z|/(\sqrt{2}\,n)$. Then

$$\begin{aligned}|\zeta - z| &< \frac{|t - z|}{\sqrt{2}\,n} = \frac{(|t - x|^2 + |y|^2)^{1/2}}{\sqrt{2}\,n} \leq \frac{((k\omega(x))^2 + |y|^2)^{1/2}}{\sqrt{2}\,n} < \frac{|y|}{n} \\ &\leq k\omega(x) \leq k\omega(z).\end{aligned}$$

To derive the desired lower bound for $F(\zeta)$ we estimate $\omega(\xi)$ for $|\xi - t| \leq 2eR$ (in particular for $\xi = t$). To do this, we note first that by 2.1(α) there exists $K \geq 1$ such that for each $a, b \in \mathbb{R}$ we have

$$\omega(a + b) \leq K(\omega(a) + \omega(b) + 1), \qquad \omega(2a) \leq K\omega(a) + K. \tag{4.3}$$

This implies

$$\omega(\xi) \leq \omega(|\xi - t| + |t|) \leq K\omega(t) + K\omega(|\xi - t|) + K \leq K\omega(t) + K\omega(2eR) + K.$$

On the other hand, it follows from (4.3) that $\omega(t) \leq K\omega(|t - z|) + K\omega(z) + K$. From the hypotheses in the present case we get

$$|t - z| \leq |t - x| + |y| \leq k\omega(x) + |y| \leq 2|y|$$

and hence

$$\omega(|t - z|) \leq \omega(2|y|) \leq K\omega(y) + K. \tag{4.4}$$

This implies

$$\omega(t) \le K^2\omega(y) + K\omega(z) + K^2 + K \le 2K^2\omega(z) + 2K^2 \tag{4.5}$$

and consequently

$$\omega(\xi) \le 2K^3\omega(z) + 2K^3 + K\omega(2eR) + K.$$

From $2eR \le 12|t-z|$ and (4.4) we get

$$\omega(2eR) \le \omega(12|t-z|) \le \omega(2^4|t-z|) \le K^4\omega(|t-z|) + 4K^4 \le K^5\omega(z) + 5K^5.$$

Therefore,

$$\omega(\xi) \le 2K^3\omega(z) + K^6\omega(z) + 2K^3 + 5K^6 + K \le 3K^6\omega(z) + 8K^6. \tag{4.6}$$

On the other hand, $|z-\zeta| \le |y|/n$ and

$$|y| = |\operatorname{Im} z| \le |\operatorname{Im}\zeta| + |\operatorname{Im}(z-\zeta)| \le |\operatorname{Im}\zeta| + |z-\zeta| \le |\operatorname{Im}\zeta| + |y|/n$$

imply

$$|y| \le \frac{n}{n-1}\,|\operatorname{Im}\zeta| \le 2|\operatorname{Im}\zeta|,$$

and consequently $|y|/n \le 2|\operatorname{Im}\zeta|/n \le |\operatorname{Im}\zeta|$. Therefore, we have

$$\begin{aligned}\omega(z) &\le K\omega(\zeta) + K\omega(|\zeta - z|) + K \le K\omega(\zeta) + K\omega(|y|/n) + K\\ &\le K\omega(\zeta) + K\omega(|\operatorname{Im}\zeta|) + K \le 2K\omega(\zeta) + K.\end{aligned} \tag{4.7}$$

Since $|\zeta - t| = \varrho$, it follows from (4.2) together with (4.5)–(4.7) that

$$\begin{aligned}|F(\zeta)| &\ge |F(t)|^{H+1}\big(\max_{|\xi-t|=2eR}|F(\xi)|\big)^{-H}\\
&\ge \exp(-(H+1)k\omega(t))\big[\max_{|\xi-t|=2eR} B\exp(B\omega(\xi))\big]^{-H}\\
&\ge B^{-H}\exp(-(H+1)k\omega(t))[\exp(3BK^6\omega(z) + 8BK^6)]^{-H}\\
&\ge B^{-H}\exp(-H8BK^6)\exp(-(H+1)k\omega(t) - 3BK^6H\omega(z))\\
&\ge B^{-H}\exp(-8HBK^6)\\
&\quad\times\exp(-2(H+1)kK^2\omega(z) - 3BK^6H\omega(z) - 2(H+1)kK^2)\\
&\ge B^{-H}\exp(-8HBK^6 - 2(H+1)kK^2)\\
&\quad\times\exp(-(2(H+1)kK^2 + 3BK^6H)\omega(z))\\
&\ge B^{-H}\exp(-8HBK^6 - 2(H+1)kK^2\\
&\quad - [2(H+1)kK + 3BK^6H](2k\omega(\zeta) + K)).\end{aligned}$$

Hence there exists d_1 independent of z (very small) and l large, depending on B, k, and K, but not on z, such that $|F(\zeta)| \ge d_1\exp(-l\omega(\zeta))$. Selecting $R_1 > 0$ such that $\exp(-\omega(\zeta)) < d_1$ if $|\zeta| \ge R_1$ we conclude that

$$|F(\zeta)| \ge \exp(-(l+1)\omega(\zeta)).$$

Hence the assertion follows for $m = \max(k, l+1)$ in this case.

For $z = x + iy \in \mathbb{C}$ satisfying $|x| \le x_0$ let $\widetilde{z} := x_0 + iy$ and choose $\zeta \in \mathbb{C}$ for $\widetilde{z}$ according to what we proved so far. Then

$$|\zeta - z| \le |\zeta - \widetilde{z}| + |\widetilde{z} - z| \le m\omega(\widetilde{z}) + |x_0 - x| \le m\omega(\widetilde{z}) + 2|x_0|.$$

Note that by (4.3) we also have

$$\omega(\widetilde{z}) \le K\omega(z) + K\omega(2x_0) + K.$$

Since $\lim_{t\to\infty} \omega(t) = \infty$ this shows that there is $m' > m$ such that the statement of the lemma holds with m replaced by m'. ∎

4.3. LEMMA. *Let ω be a weight function and assume that $F \in A_{(\omega)}$ satisfies the hypotheses of Lemma 4.2. Then there exist positive numbers ε_0, C_0, and D such that each component S of*

$$S_\omega(F, \varepsilon_0, C_0) := \{z \in \mathbb{C} : |F(z)| < \varepsilon_0 \exp(-C_0\omega(z))\}$$

satisfies

$$\operatorname{diam} S \le D \inf_{z\in S} \omega(z) + D.$$

Proof. By Lemma 4.2 there exist $m > 0$ and $R_0 > 0$ such that the conclusion of that lemma holds. We fix $z \in \mathbb{C}$ with $|z| \ge R_0$ and choose $\zeta = \zeta(z) \in \mathbb{C}$ with $|\zeta - z| \le m\omega(z)$ such that

$$|F(\zeta)| \ge \exp(-m\omega(\zeta)).$$

In order to apply the minimum-modulus theorem we let

$$\eta := \frac{1}{32}, \qquad H := 2 + \log\left(\frac{3e}{2\eta}\right), \qquad R := 2|\zeta - z|.$$

Then there exists ϱ with $|\zeta - z| < \varrho < 2|\zeta - z|$ such that

$$|F(\xi)| \ge |F(\zeta)|^{H+1}\big(\sup_{|w-\zeta|=2eR} |F(w)|\big)^{-H} \tag{4.8}$$

for each $\xi \in \mathbb{C}$ with $|\xi - \zeta| = \varrho$. Next choose $\delta > 0$ so small that $\delta \le 1/28m$. Since $\omega(t) = o(t)$ by 2.1 (β), we may assume that R_0 is so large that

$$\omega(t) \le \delta t \quad \text{ for } t \ge R_0.$$

Now fix any $\xi \in \mathbb{C}$ with $|\xi - \zeta| = \varrho$ and note that by our choices we have

$$|z| \le |z - \zeta| + |\zeta - \xi| + |\xi| \le 3m\omega(z) + |\xi| \le \tfrac{1}{2}|z| + |\xi|$$

and hence $|z| \le 2|\xi|$. This implies

$$|\zeta| \le |\xi| + |\zeta - \xi| \le |\xi| + 2m\omega(z) \le |\xi| + \tfrac{1}{4}|z| \le 2|\xi|.$$

If we assume that $K \ge 1$ is a constant for which (4.3) holds, then it follows from this that

$$\omega(\zeta) \le K\omega(\xi) + K, \qquad \xi \in \mathbb{C},\ |\zeta - \xi| = \varrho. \tag{4.9}$$

Similarly we get for any $w \in \mathbb{C}$ with $|w-\zeta| = 2eR$ the estimate

$$\begin{aligned}|w| &\le |\zeta| + 2eR \le |\zeta| + 12m\omega(z) \le |\xi| + |\zeta-\xi| + 12m\omega(z)\\ &\le |\xi| + 14m\omega(z) \le |\xi| + |z|/2 \le 2|\xi|\end{aligned}$$

and hence

$$\omega(w) \le K\omega(\xi) + K$$

for each $w, \xi \in \mathbb{C}$ satisfying $|w-\zeta| = 2eR$, $|\xi-\zeta| = \varrho$. From this estimate together with (4.9) and (4.8) it follows that for each $\xi \in \mathbb{C}$ with $|\xi-\zeta| = \varrho$ we have the estimate

$$\begin{aligned}|F(\xi)| &\ge \exp(-m(H+1)(K\omega(\xi)+K))\exp(-H(K\omega(\xi)+K))\\ &= \exp(-[m(H+1)+H]K)\exp(-K(m(H+1)+1)\omega(\xi)).\end{aligned}$$

If we let $\varepsilon_0 := \exp(-[m(H+1)+H]K)$ and $C_0 := K(m(H+1)+1)$ then it follows that for each $z_0 \in S_\omega(F,\varepsilon_0,C_0)$ with $|z_0| \ge R_0$ the connected component S of $S_\omega(F,\varepsilon_0,C_0)$ which contains z_0 is contained in the open disk $B(\zeta(z_0), 2m\omega(z_0))$. In particular, we have

$$\operatorname{diam} S \le 4m\omega(z_0).$$

By our choice of δ, we have $4m\omega(z_0) \le |z_0|/7$. Now we choose $z_1 \in \overline{S}$ such that $\inf_{z\in S}\omega(z) = \omega(z_1)$. Then we get

$$|z_0| \le |z_1| + |z_0 - z_1| \le |z_1| + \operatorname{diam} S \le |z_1| + \tfrac{1}{7}|z_0|$$

and consequently $|z_0| \le \frac{7}{6}|z_1| \le 2|z_1|$. This implies

$$\operatorname{diam} S \le 4m\omega(z_0) \le 4m\omega(2z_1) \le 4mK \inf_{z\in S}\omega(z) + 4mK.$$

If we choose $D = \max(r_0, 4mK)$, then we deduce the estimate $\operatorname{diam} S \le D\inf_{z\in S}\omega(z) + D$. ■

4.4. Definition. For a weight function ω and $R > 1$ we define the space

$$A_{(\omega,R)} := \operatorname{ind}_{n\to} A([-R,R], n).$$

4.5. Lemma. *Let ω be a* (DN)*-weight function or a strong weight function. Then for each $R > 0$ the following assertions hold*:

(a) $\mathcal{F} : \mathcal{E}'_{(\omega)}[-R,R] \to A_{(\omega,R)}$ *is a linear topological isomorphism.*
(b) $A_{(\omega,R)}$ *is a* (DFN)*-space.*
(c) $A'_{(\omega,R)}$ *has properties* (DN) *and* (Ω).

Proof. (a) This follows from the proof of Meise and Taylor [17, Proposition 3.6], since the proof of the surjectivity of the map G defined in that proposition does not use the assumption that ω is a non-quasianalytic weight function.

(b) The properties of a weight function and well-known results imply that the (LB)-space $A_{(\omega,R)}$ is in fact a (DFN)-space.

(c) $A'_{(\omega,R)}$ has (Ω) by Meise and Taylor [19, Lemma 1.10(c)]. If ω is a (DN)-weight function, the proof of [19, Proposition 5.3] shows that $\mathcal{E}_{(\omega)}[-R,R]$ and hence $A'_{(\omega,R)}$ has (DN). If ω is a strong weight function, then $\mathcal{E}_{(\omega)}[-R,R]$ has (DN) by [19, Corollary 5.6]. ∎

4.6. Lemma. *Let ω be a weight function. Then for $F \in A_{(\omega)}$ the following conditions are equivalent*:

(1) *F is (ω)-slowly decreasing.*
(2) *For each $R > 0$ the multiplication operator*

$$M_F : A_{(\omega,R)} \to A_{(\omega,R)}, \qquad M_F(g) := Fg,$$

is an injective topological homomorphism.
(3) *There exist positive numbers ε_0, C_0, and D such that for $S_\omega(F,\varepsilon_0,C_0)$ the conclusion of Lemma* 4.3 *holds.*

Proof. (1)⇒(2). Fix $R > 0$ and note first that $A_{(\omega,R)}$ is a (DFN)-space by Lemma 4.5. Hence (2) follows from the Baernstein lemma (see Meise and Vogt [22, Proposition 26.26]) if we show that for each bounded set B in $A_{(\omega,R)}$ the set $M_F^{-1}(B)$ is bounded. Since the sets

$$B_n := \{f \in A_{(\omega,R)} : \sup_{z\in\mathbb{C}} |f(z)|e^{-R|\operatorname{Im} z|-n\omega(z)} \le 1\}, \qquad n \in \mathbb{N},$$

form a fundamental sequence of bounded sets in $A_{(\omega,R)}$, it suffices to show that $M_F^{-1}(B_n)$ is bounded for each $n \in \mathbb{N}$. To do so, fix $n \in \mathbb{N}$. Since F is (ω)-slowly decreasing by hypothesis, it follows from Lemma 4.2 that there exist $m, R_0 > 0$ such that for each $z \in \mathbb{C}$, $|z| \ge R_0$, there exists $\zeta \in \mathbb{C}$ such that $|\zeta - z| \le m\omega(z)$ and $|F(\zeta)| \ge \exp(-m\omega(\zeta))$. Since F is in $A_{(\omega)}$, there exists $A > 0$ such that

$$|F(z)| \le A\exp(A\omega(z)), \qquad z \in \mathbb{C}. \tag{4.10}$$

Next fix $h \in M_F^{-1}(B_n)$ and let $g := Fh \in B_n$. Then

$$|g(z)| \le \exp(R|\operatorname{Im} z| + n\omega(z)), \qquad z \in \mathbb{C}. \tag{4.11}$$

Now we apply Hörmander [11, Lemma 3.2] for $r = m\omega(z) > 0$ to get, for $|z| \ge R_0$,

$$|h(z)| = \left|\frac{g(z)}{F(z)}\right| \le \frac{\sup_{|w-z|\le 4m\omega(z)} |g(w)| \sup_{|w-z|\le 4m\omega(z)} |F(w)|}{(\sup_{|w-z|\le m\omega(z)} |F(w)|)^2} \tag{4.12}$$

To derive further estimates from (4.12), note that $\lim_{t\to\infty} \omega(t)/t = 0$ implies the existence of $t_0 > 0$ such that $\omega(t + 4m\omega(t)) \le \omega(2t)$ for $t \ge t_0$. Since ω satisfies 2.1(α), this implies the existence of $K_1 \ge K$ such that

$$\omega(t + 4m\omega(t)) \le K\omega(t) + K_1, \qquad t > 0. \tag{4.13}$$

Next note that for $\zeta = \zeta(z)$ we similarly get the existence of $C_1 \geq 1$ such that

$$|\zeta| \leq |z| + |\zeta - z| \leq |z| + m\omega(z) \leq 2|z| + C_1.$$

By (4.3), this implies

$$2m\omega(\zeta) \leq 2m\omega(2|z| + C_1) \leq 2m(K^2\omega(z) + K(K + 1 + \omega(C_1))). \tag{4.14}$$

Now from (4.12) by (4.10), (4.11), (4.13), and (4.14) we get

$$\begin{aligned} |h(z)|/A &\leq \exp(R|\mathrm{Im}\, z| + (4Rm + nK + AK)\omega(z) + K_1(n + A)) \\ &\quad \times \exp(2m\omega(\zeta)) \\ &\leq \exp(R|\mathrm{Im}\, z| + (4Rm + nK + AK + 2mK^2)\omega(z)) \\ &\quad \times \exp(K_1(n + A) + 2mK(K + 1 + \omega(C_1))). \end{aligned}$$

Hence there exist $D_1 \geq 1$ and $D_2 \geq 1$, not depending on h, such that

$$|h(z)| \leq D_1 \exp(R|\mathrm{Im}\, z| + D_2\omega(z)), \qquad z \in \mathbb{C},\ |z| \geq R_0.$$

This estimate implies that we can enlarge the constant D_1 to have the inequality for all $z \in \mathbb{C}$. Since h was an arbitrary element of $M_F^{-1}(B_n)$ we have proved that $M_F^{-1}(B_n)$ is bounded in $A_{(\omega,R)}$.

(2)⇒(1). By Theorem 2.8, condition (1) is equivalent to

(4.15) For each bounded set B in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$ the set $M_F^{-1}(B)$ is bounded in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$.

To show that (4.15) holds, let B be any bounded set in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$. Since $A_{(\omega)}(\mathbb{C}, \mathbb{R})$ is a (DFN)-space, it follows easily that there exists $R > 0$ such that B is contained in $A_{(\omega,R)}$ and bounded there. Now (2) implies that $M_F^{-1}(B)$ is bounded in $A_{(\omega,R)}$ and hence bounded in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$.

(1)⇒(3). This holds by Lemma 4.3.

(3)⇒(1). Obviously, (3) implies condition (3) in Theorem 2.8. Hence (1) holds. ∎

4.7. Lemma. *Let ω be a weight function and assume that $F \in A_{(\omega)}$ is (ω)-slowly decreasing. Then for each $R > 0$ the space $A_{(\omega,R)}/M_F A_{(\omega,R)}$ is either finite-dimensional or isomorphic to the strong dual of a nuclear power series space of infinite type.*

Proof. Fix $R > 0$ and note that $M_F : A_{(\omega,R)} \to A_{(\omega,R)}$ has closed range by Lemma 4.6. Note also that $A_{(\omega,R)}/M_F A_{(\omega,R)}$ is finite-dimensional if and only if F has only finitely many zeros. Therefore, we assume from now on that $V(F) := \{a \in \mathbb{C} : F(a) = 0\}$ is an infinite set. Next we choose ε_0, C_0, and D according to Lemma 4.3 and we label the components S of $S_\omega(F, \varepsilon_0, C_0)$ which satisfy $S \cap V(F) \neq \emptyset$ in such a way that the sequence α defined by

$$\alpha_j := \sup_{z \in S_j} \omega(z), \qquad j \in \mathbb{N},$$

is increasing. Also, we define the sequence β by

$$\beta_j := \sup_{z\in S_j} |\mathrm{Im}\, z|, \qquad j\in\mathbb{N}.$$

Then we define the spaces E_j and the maps $\varrho_j : H^\infty(S_j)\to E_j$ as in (3.5) and (3.6). Moreover, we let

$$K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}}) := \Big\{(x_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E_j : \\ \exists n\in\mathbb{N} : |||x|||_n := \sup_{j\in\mathbb{N}} \|x_j\|_j e^{-n\alpha_j - R\beta_j} < \infty\Big\}.$$

If $f\in A_{(\omega,R)}$ satisfies

$$\|f\|_n = \sup_{z\in\mathbb{C}} |f(z)|\exp(-R|\mathrm{Im}\, z| - n\omega(z)) < \infty$$

then for each $j\in\mathbb{N}$ we have

$$\|f|_{S_j}\|_{H^\infty(S_j)} \le \|f\|_n \exp(n\alpha_j + R\beta_j).$$

This implies that

$$\|\varrho_j(f|_{S_j})\|_j e^{-n\alpha_j - R\beta_j} \le \|f\|_n, \qquad j\in\mathbb{N}.$$

Consequently,

$$\varrho : A_{(\omega,R)} \to K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}}), \qquad \varrho(f) := (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}},$$

is a linear and continuous map.

Next we claim that $\ker\varrho = M_F A_{(\omega,R)}$. To show this, note first that by the definition of ϱ we obviously have $M_F A_{(\omega,R)} \subset \ker\varrho$. To prove the reverse inclusion, let $g\in\ker\varrho$. Then g/F is an entire function. Using Lemma 4.3 and standard arguments, it follows as in the proof of Berenstein and Taylor [2, Proposition 3] that $g/F\in A_{(\omega,R)}$ and hence $g\in M_F A_{(\omega,R)}$.

To show that ϱ is surjective, we argue similarly to the proof of Theorem 3.7 in Meise [15]. Let $y = (y_j)_{j\in\mathbb{N}} \in K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}})$ and choose $n\in\mathbb{N}$ such that $|||y|||_n < \infty$. By the definition of the norm in E_j we can choose $\lambda_j\in H^\infty(S_j)$ with $\varrho_j(\lambda_j) = y_j$ so that

$$\|\lambda_j\|_{H^\infty(S_j)} \le 2|||y|||_n e^{n\alpha_j + R\beta_j}, \qquad j\in\mathbb{N}.$$

Next define $\lambda : S_\omega(F,\varepsilon_0,C_0)\to\mathbb{C}$ by $\lambda(z) := \lambda_j(z)$ if $z\in S_j$ and $\lambda(z) := 0$ if $z\in S_\omega(F,\varepsilon_0,C_0)\setminus\bigcup_{j\in\mathbb{N}} S_j$. Then the diameter estimate from Lemma 4.3 and the definition of α and β imply the existence of $D_1>0$ such that

$$|\lambda(z)| \le 2|||y|||_n \exp(R|\mathrm{Im}\, z| + D_1 n\omega(z) + D_1). \tag{4.16}$$

If K is a constant for which the estimate (4.3) holds then it follows that

$$\left|\frac{\partial F}{\partial z}(\zeta)\right| \le e^K B\exp(KB\omega(\zeta)), \qquad \zeta\in\mathbb{C}.$$

From this and Lemma 4.3 it follows that there exist $\varepsilon_1, C_1, L > 0$ such that for each $z \in S_\omega(F, \varepsilon_1, C_1)$ the distance to $\mathbb{C} \setminus S_\omega(F, \varepsilon_0, C_0)$ is at least as large as $L^{-1}\exp(-L\omega(z))$. Therefore, there are $A_0, B_0 > 0$ and $\chi \in C^\infty(\mathbb{C})$ having the following properties:

$$\begin{gathered} 0 \le \chi \le 1, \quad \chi \equiv 1 \text{ on } S_\omega(F, \varepsilon_1, C_1), \quad \operatorname{supp}\chi \subset S_\omega(F, \varepsilon_0, C_0), \\ \left|\frac{\partial\chi}{\partial\overline{z}}(z)\right| \le A_0 \exp(B_0\omega(z)), \quad z \in \mathbb{C}. \end{gathered} \tag{4.17}$$

Next let

$$v := -\frac{1}{F}\frac{\partial}{\partial\overline{z}}(\chi\lambda) = -\frac{1}{F}\frac{\partial\chi}{\partial\overline{z}}\lambda$$

and note that v is in $C^\infty(\mathbb{C})$ and vanishes on $S(F, \varepsilon_1, C_1)$. The estimates (4.16) and (4.17) imply that

$$\begin{aligned} |v(z)| &\le \frac{1}{\varepsilon_1} A_0 \cdot 2|||y|||_n e^{D_1} \exp(R|\operatorname{Im} z| + (C_1 + B_0 + D_1 n)\omega(z)) \\ &= A_1 \exp(R|\operatorname{Im} z| + B_1\omega(z)), \quad z \in \mathbb{C}, \end{aligned}$$

for suitable numbers $A_1, B_1 > 0$. Since ω satisfies condition 2.1(γ), it follows from this that

$$\int_{\mathbb{C}} [|v(z)| \exp(-R|\operatorname{Im} z| - (B_1 + 1)\omega(z))]^2 \, dz < \infty.$$

By Hörmander [12, Theorem 4.4.2], there exists $g \in L^2_{\mathrm{loc}}(\mathbb{C})$ satisfying $\partial g/\partial\overline{z} = v$ and

$$\int_{\mathbb{C}} [|g(z)| \exp(-R|\operatorname{Im} z| - (B_1 + 1)\omega(z) - \log(1 + |z|^2))]^2 \, dz < \infty. \tag{4.18}$$

Since v is in $C^\infty(\mathbb{C})$ and since $\partial/\partial\overline{z}$ is elliptic, g is in $C^\infty(\mathbb{C})$. Hence

$$f := \chi \cdot \lambda + gF$$

is in $C^\infty(\mathbb{C})$ and satisfies

$$\frac{\partial f}{\partial\overline{z}} = \frac{\partial\chi}{\partial\overline{z}}\lambda + \frac{\partial g}{\partial\overline{z}}F = 0$$

by the choice of g. Consequently, $f \in H(\mathbb{C})$ and the estimates (4.16) and (4.18) together with well-known arguments imply the existence of $A_2 > 0$ and $B_2 > 0$ such that

$$|f(z)| \le A_2 \exp(R|\operatorname{Im} z| + B_2\omega(z)), \quad z \in \mathbb{C}.$$

This shows that $f \in A_{(\omega,R)}$. By the definition of f and λ we have

$$\varrho(f) = (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}} = (\varrho_j(\lambda_j))_{j\in\mathbb{N}} = y.$$

Therefore, $\varrho : A_{(\omega,R)} \to K(\alpha, \beta, (E_j)_{j\in\mathbb{N}})$ is surjective. Thus we have proved that $A_{(\omega,R)}/M_F A_{(\omega,R)} \cong K_\infty(\alpha, \beta, (E_j)_{j\in\mathbb{N}})$ by the open mapping theorem.

Since $A_{(\omega,R)}$ is a (DFN)-space by Lemma 4.5(a), also $K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}})$ is a (DFN)-space. It is easy to check that

$$D : K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}}) \to K_\infty(\alpha,(E_j)_{j\in\mathbb{N}}), \quad D((x_j)_{j\in\mathbb{N}}) := (e^{-\beta_j}x_j)_{j\in\mathbb{N}}$$

is a linear topological isomorphism. Hence $K_\infty(\alpha,(E_j)_{j\in\mathbb{N}})$ is a (DFN)-space. By Meise [15, Propositions 1.3 and 1.4], this implies that $K_\infty(\alpha,(E_j)_{j\in\mathbb{N}})$ and hence also $K_\infty(\alpha,\beta,(E_j)_{j\in\mathbb{N}})$ is isomorphic to the strong dual of a nuclear power series space of infinite type. ∎

4.8. Lemma. *Let ω be a* (DN)*-weight function or a strong weight function. If $F \in A_{(\omega)}$ is* (ω)*-slowly decreasing then for each $R > 0$ the map $M_F : A_{(\omega,R)} \to A_{(\omega,R)}$ admits a continuous linear left inverse.*

Proof. By Lemma 4.6, the following sequence is exact:

$$0 \to A_{(\omega,R)} \xrightarrow{M_F} A_{(\omega,R)} \xrightarrow{q} A_{(\omega,R)}/M_F(A_{(\omega,R)}) \to 0,$$

where q denotes the quotient map. Since the maps M_F and q are linear and continuous, it follows from Meise and Vogt [22, Proposition 26.4] that the dual sequence

$$0 \to (A_{(\omega,R)}/M_F(A_{(\omega;R)}))' \xrightarrow{q'} A'_{(\omega,R)} \xrightarrow{M'_F} A'_{(\omega,R)} \to 0 \tag{4.19}$$

is exact as well. By Lemma 4.5, the space $A'_{(\omega,R)}$ has (DN), while by Lemma 4.7, the space $(A_{(\omega,R)}/M_F(A_{(\omega,R)}))'$ has (Ω). Hence the exact sequence (4.19) splits by the splitting theorem of Vogt and Wagner (see [22, 30.1]), which finishes the proof. ∎

4.9. Proposition. *Let ω be a* (DN)*-weight function or a strong weight function. Assume that for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ its Fourier–Laplace transform $\widehat{\mu}$ is* (ω)*-slowly decreasing and is in $A_{(\omega)}$. Then for each $a, b \in \mathbb{R}$ with $a < b$ the sequence*

$$0 \to \ker T_{\mu,[a,b]} \to \mathcal{E}_{(\omega)}[a,b] \xrightarrow{T_{\mu,[a,b]}} \mathcal{E}_{(\omega)}[a,b] \to 0$$

is exact and splits.

Proof. From Lemma 4.5 we know that for each $R>0$ the space $\mathcal{E}'_{(\omega)}[-R,R]$ is isomorphic to $A_{(\omega,R)}$ via the Fourier–Laplace transform. From this and the fact that T_μ commutes with translations it follows that for $R := (a+b)/2$ and up to isomorphism the present sequence is identical with the exact sequence (4.19). Hence it is exact and splits. ∎

Remark. If we replace in Proposition 4.9 the condition "$\widehat{\mu}$ is (ω)-slowly decreasing" by "$T_{\mu,[a,b]}$ is surjective for some interval $[a,b]$", then it follows as in the proof of Proposition 4.9 that $T_{\mu,[a,b]} : \mathcal{E}_{(\omega)}[a,b] \to \mathcal{E}_{(\omega)}[a,b]$ admits a continuous linear right inverse.

REMARK. If ω is a strong weight function then the existence of a continuous linear right inverse for T_μ as in Proposition 4.9 can also be proved in the following way. By Meise and Taylor [20, Theorem 3.1], there exists a continuous linear extension operator $E_0 : \mathcal{E}_{(\omega)}[a,b] \to \mathcal{E}_{(\omega)}(\mathbb{R})$. If we fix $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R})$ such that $\varphi \equiv 1$ in some neighborhood of $[a,b]$, then $E : f \mapsto \varphi E_0(f)$ defines a continuous linear operator from $\mathcal{E}_{(\omega)}[a,b]$ into $\mathcal{D}_{(\omega)}(\mathbb{R})$. Since T_μ admits a fundamental solution $\nu \in \mathcal{D}'_{(\omega)}(\mathbb{R})$ it is easy to check that the map

$$R : \mathcal{E}_{(\omega)}[a,b] \to \mathcal{E}_{(\omega)}[a,b], \qquad R(f) := \nu * (E(f))|_{[a,b]},$$

is a continuous linear right inverse for T_μ.

4.10. EXAMPLE. Let ω be a quasianalytic (DN)-weight function which satisfies condition (α_1). Then there exist (ω)-ultradifferential operators T_μ which admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}[a,b]$, but which do not admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$.

To show this, let $(b_j)_{j\in\mathbb{N}}$ be an increasing sequence in $]0,\infty[$ which satisfies, for $n(t) := \operatorname{card}\{j \in \mathbb{N} : b_j \le t\}$,

(i) $b_{j+1} \ge 4b_j$, $j \in \mathbb{N}$,
(ii) $n(t)\log t = o(\omega(t))$ as $t \to \infty$.

Then define

$$F(z) := \prod_{j\in\mathbb{N}} \left(1 - \frac{z}{ib_j}\right), \qquad z \in \mathbb{C}.$$

By Rudin [28, Theorem 15.6], F is an entire function. Arguing as in [6, Lemma 3.5] we get the existence of $B > 0$ such that

$$|F(z)| \le B\exp(B\omega(z)), \qquad z \in \mathbb{C},$$

and the existence of $\varepsilon_0 > 0$ and $C_0 > 0$ such that for each $w \in \mathbb{C} \setminus \bigcup_{j\in\mathbb{N}} B(ib_j, 1)$ we have

$$|F(w)| \ge \varepsilon_0 \exp(-C_0\omega(w)).$$

Therefore, F is in $A_{(\omega)}$ and it follows from Lemma 4.6 that F is (ω)-slowly decreasing. By Theorem 2.6 we can find $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ such that $\widehat{\mu} = F$. It then follows from Proposition 4.9 that T_μ admits a continuous linear right inverse on $\mathcal{E}_{(\omega)}[a,b]$ for each $a, b \in \mathbb{R}$ with $a < b$. However, since condition 3.2(b) does not hold for F, it follows from Proposition 3.2 that $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ does not admit a continuous linear right inverse.

From Theorem 3.4 we now get the following corollary.

4.11. COROLLARY. *Let ω be a quasianalytic* (DN)*-weight function which satisfies condition* (α_1). *Assume that for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ its Fourier–Laplace transform $\widehat{\mu}$ is (ω)-slowly decreasing and is in $A_{(\omega)}$. Then the following assertions are equivalent*:

(1) $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ *admits a continuous linear right inverse.*
(2) *There exists* $C > 0$ *such that* $|\operatorname{Im} a| \le C(\omega(a)+1)$ *for each* $a \in V(\widehat{\mu})$.
(3) *For each/some* $a, b \in \mathbb{R}$ *with* $a < b$ *and each* $f \in \ker T_{\mu,[a,b]}$ *there exists* $g \in \ker T_\mu$ *such that* $f = g|_{[a,b]}$.

Proof. (1)⇒(2). This holds by Proposition 3.2.

(2)⇒(3). Fix any $R > 0$ and consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \ker T_\mu \arrow[r] \arrow[d, "\varrho"] & \mathcal{E}_{(\omega)}(\mathbb{R}) \arrow[r, "T_\mu"] \arrow[d, "\varrho"] & \mathcal{E}_{(\omega)}(\mathbb{R}) \arrow[r] \arrow[d, "\varrho"] & 0 \\ 0 \arrow[r] & \ker T_{\mu,R} \arrow[r] & \mathcal{E}_{(\omega)}[-R,R] \arrow[r, "T_{\mu,R}"] & \mathcal{E}_{(\omega)}[-R,R] \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps ϱ are defined by $\varrho(f) := f|_{[-R,R]}$. They are linear and continuous. Therefore, the dual diagram is commutative as well. If we apply the Fourier–Laplace transform to it, then we get the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A_{(\omega)}(\mathbb{C},\mathbb{R}) \arrow[r, "M_{\widehat{\mu}}"] & A_{(\omega)}(\mathbb{C},\mathbb{R}) \arrow[r, "q"] & A_{(\omega)}(\mathbb{C},\mathbb{R})/\widehat{\mu}A_{(\omega)}(\mathbb{C},\mathbb{R}) \arrow[r] & 0 \\ 0 \arrow[r] & A_{(\omega,R)} \arrow[r, "M_{\widehat{\mu}}"] \arrow[u, "j"] & A_{(\omega,R)} \arrow[r, "q_R"] \arrow[u, "j"] & A_{(\omega,R)}/\widehat{\mu}A_{(\omega,R)} \arrow[r] \arrow[u, "J"] & 0 \end{tikzcd}$$

where the maps j are obvious inclusions and where J is induced by j. As we proved in Proposition 3.3 the quotient $A_{(\omega)}(\mathbb{C},\mathbb{R})/\widehat{\mu}A_{(\omega)}(\mathbb{C},\mathbb{R})$ is isomorphic to $K_\infty(\gamma, (E_j)_{j\in\mathbb{N}})$. Since condition (2) holds, we also know from the proof of this proposition that there exists $A_2 > 0$ such that for each component S of $S(\widehat{\mu}, \varepsilon_0, C_0)$ which satisfies $S \cap V(\widehat{\mu}) \neq 0$ we have $|\operatorname{Im} z| \le A_2\omega(z) + A_2$ for each $z \in S$. This implies that the space $K_\infty(\lambda, (E_j)_{j\in\mathbb{N}})$ is identical with $K_\infty(\alpha, \beta, (E_j)_{j\in\mathbb{N}})$ defined in the proof of Lemma 4.7. Moreover, the map J in the diagram above is the identity if we identify both spaces. Hence J and consequently $\varrho : \ker T_\mu \to \ker T_{\mu,R}$ is an isomorphism. Since T_μ commutes with translations, this implies (3) for each $a < b$ in $\mathbb{R}$.

(3)⇒(1). Assume that condition (3) holds for some $a, b \in \mathbb{R}$. Then let $R := (a+b)/2$ and note that the present hypothesis and Proposition 4.9 imply that $T_{\mu,R} : \mathcal{E}_{(\omega)}[-R,R] \to \mathcal{E}_{(\omega)}[-R,R]$ admits a continuous linear right inverse S_R. To use it in order to define a continuous linear right inverse for T_μ, we fix any $g \in \mathcal{E}_{(\omega)}(\mathbb{R})$. Since T_μ is surjective by hypothesis, there exists $f \in \mathcal{E}_{(\omega)}(\mathbb{R})$ with $T_\mu f = g$. We denote by $\varrho : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}[-R,R]$ the restriction map $\varrho(h) := h|_{[-R,R]}$ and we let

$$h_R := \varrho(f) - S_R(\varrho(g)).$$

Then h_R belongs to $\mathcal{E}_{(\omega)}[-R,R]$ and

$$T_{\mu,R}(h_R) = T_{\mu,R}(\varrho(f)) - T_{\mu,R} \circ S_R(\varrho(g)) = \varrho(T_\mu(f)) - \varrho(g) = \varrho(g) - \varrho(g) = 0$$

implies $h_R \in \ker T_{\mu,R}$. By the present hypothesis we can choose $H \in \ker T_\mu$ satisfying $\varrho(H) = h_R$. Now we let

$$S(g) := f - H$$

and claim that $S(g)$ is well-defined, i.e., that it does not depend on the choice of f. To show this, assume that $f_1 \in \mathcal{E}_{(\omega)}(\mathbb{R})$ satisfies $T_\mu f_1 = g$. Then $h_{1,R} := \varrho(f_1) - S_R(\varrho(g))$ is again in $\ker T_{\mu,R}$ and we can choose $H_1 \in \ker T_\mu$ satisfying $\varrho(H_1) = h_{1,R}$. Now note that our choices imply

$$\varrho(f - H) = \varrho(f) - \varrho(H) = \varrho(f) - h_R = S_R(\varrho(g)),$$
$$\varrho(f_1 - H_1) = \varrho(f_1) - \varrho(H_1) = \varrho(f_1) - h_{1,R} = S_R(\varrho(g)).$$

Hence $f - H$ and $f_1 - H_1$ coincide on the interval $[-R,R]$. Since ω is a quasianalytic weight function, this implies $f - H = f_1 - H_1$ and proves that $S(g)$ is well-defined. From this it follows easily that $S : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ is a linear map. By the definition of S we have

$$\varrho(S(g)) = \varrho(f) - \varrho(H) = \varrho(f) - h_R = S_R(\varrho(g)).$$

In order to show that S is continuous, we apply the closed graph theorem. To do so, let $(g_j)_{j\in\mathbb{N}}$ be any sequence in $\mathcal{E}_{(\omega)}(\mathbb{R})$ which satisfies $g_j \to g$ and $S(g_j) \to f$ in $\mathcal{E}_{(\omega)}(\mathbb{R})$. Then the equality $\varrho(S(g_j)) = S_R(\varrho(g_j))$, $j \in \mathbb{N}$, implies that the sequence $(\varrho(S(g_j)))_{j\in\mathbb{N}}$ converges to $\varrho(f)$ as well as to $S_R(\varrho(g)) = \varrho(S(g))$. Hence f and $S(g)$ coincide on $[-R,R]$ and consequently $f = S(g)$. Thus S has a closed graph and therefore is continuous. Since

$$T_\mu(S(g)) = T_\mu(f - H) = T_\mu(f) - T_\mu(H) = T_\mu(f) = g$$

for each $g \in \mathcal{E}_{(\omega)}(\mathbb{R})$, we proved (1). ∎

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