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Time regularity and functions of the Volterra operator

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Abstract. Our aim is to prove that for any fixed $1/2 < \alpha < 1$ there exists a Hilbert space contraction T such that $\sigma(T) = \{1\}$ and

$$\|T^{n+1} - T^n\| \asymp n^{-\alpha} \quad (n \geq 1).$$

This answers Zemánek's question on the time regularity property.

1. Introduction. Let T denote a bounded linear operator on a Banach space $\mathcal{X}$. One of the basic asymptotical properties of the discrete semigroup $(T^n)_{n\geq 0}$ is the time regularity property. That is, we are interested in estimates that have the form

$$\|T^{n+1} - T^n\| \leq \text{const} \cdot n^{-\alpha}, \quad n \in \mathbb{N}, \alpha > 0, \tag{1.1}$$

and lower estimates as well. The above property, known as the time regularity of the operator T, readily implies the norm stability of the differences $T^{n+1} - T^n$ $(n \geq 1)$, which means that the spectral inclusion $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ holds. The converse statement is a well-known result in operator theory. The Esterle–Katznelson–Tzafriri theorem states that if T is a power-bounded operator on $\mathcal{X}$ such that $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ then $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$ ([6], [8]). The time regularity and uniform stability of the differences $T^{n+1} - T^n$ have been considered by various authors over a long time. We refer the reader to [3]–[6], [8], [14], [16], [23], and the references therein.

For any fixed $1/2 \leq \alpha \leq 1$, it is not difficult to construct an operator T with a large spectrum, i.e., with 1 being a nonisolated spectrum point, where (1.1) holds and the estimate is sharp (e.g. [16, Example 4.5.2] and Remark 3.5 at the end of Section 3 below). Not long ago N. Dungey gave a characterization of the time regularity property by making an analogous estimation for the operator semigroup $(e^{-t(I-T)})_{t\geq 0}$, and by applying the

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uniform boundedness of the semigroup $e^{-z(I-T)}$ on special domains of the complex plane (see [4]). Yu. Lyubich [10] and independently B. Nagy and J. Zemánek [15] characterized the operators for which $\|T^{n+1}-T^n\| = O(1/n)$ by means of Ritt's resolvent condition. However, Esterle's result [6, Corollary 9.5] shows that the fastest decay of the differences $T^{n+1} - T^n$ is just $O(1/n)$ whenever $T \neq I$ and $\sigma(T) = \{1\}$. Moreover, the authors of [7], [13] and [5] were able to determine the sharp constant in Esterle's result using different methods.

In [24] Zemánek asked the following question on the time regularity property. Let T be a bounded linear operator on a Banach space, with single-point spectrum $\{1\}$. Suppose that

$$\|T^{n+1} - T^n\| \leq \frac{\text{const}}{n^{1/2+\varepsilon}}$$

for a fixed $0 < \varepsilon < 1/2$ and all $n \in \mathbb{N}$. Does it actually follow that

$$\|T^{n+1} - T^n\| \leq \frac{\text{const}}{n}$$

for all $n \in \mathbb{N}$?

To tackle this issue, one of the most natural ways to construct operators with a single-point (or minimal) spectrum is to look at functions of the Volterra integral operator in $L^p[0,1]$, $1 \leq p \leq \infty$. We recall that V is defined by

$$(Vf)(x) = \int_0^x f(s)\,ds \qquad (0 \leq x \leq 1).$$

A detailed study of the operator $I - V$ was made in [14], where Montes-Rodríguez, Sánchez-Álvarez and Zemánek proved for instance that $I - V$ is power-bounded if and only if $p = 2$. Using the same technique, they were also able to show that the exact order of decay of the consecutive powers of $I - V$ is $1/\sqrt{n}$ (up to constant factors) in $L^2[0,1]$; for their general result in L^p, see [14, Theorem 2.5]. Quite recently Yu. Lyubich [12] remarked that if ϕ is a holomorphic function around 0, $\phi(0) = 1$ and $\phi \not\equiv 1$ then the exact time regularity of $\phi(V)$ is also $1/\sqrt{n}$ (up to constant factors) whenever $\phi(V)$ is power-bounded. However, the operator $I - V^\alpha$ for $0 < \alpha < 1$ fails to satisfy this condition, but it satisfies Ritt's resolvent condition in any $L^p[0,1]$ (see [11]), hence the time regularity is $O(1/n)$ here.

In this paper our goal is to supply a negative answer to Zemánek's question in the case of Hilbert spaces. We shall construct Hilbert space contractions with any possible time regularity under the minimal spectral assumption. Quite recently in [9], we gave a new proof of the regularity of differences of $(I - V)^n$ in $L^2[0,1]$. Exploiting this method, we can construct operators with various regularity conditions. Instead of using functional calculus for V, we shall consider holomorphic functions of $I - 2V$. In the next section we determine the regularity properties of differences of powers of bounded holomorphic functions on the unit disk, and address the same problem in a factor algebra as well. Once this is done, we have enough preliminary knowledge to handle the problem in the operator algebra. As regards the time regularity of bounded linear operators, (local) smoothness properties on the boundaries of holomorphic functions turn out to be the major tool here. The operator construction is provided in the last section.

2. Time regularity and holomorphic functions. First we shall study differences of the powers $(1-f)^n$ $(n \geq 1)$ of holomorphic functions. The disk algebra $A(\mathbb{D})$ is the algebra of holomorphic functions on the unit open disk $\mathbb{D}$ which can be extended continuously to $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$. We also address the question in $H^\infty(\mathbb{D})$, the algebra of bounded, holomorphic functions, and in the factor algebra $H^\infty(\mathbb{D})/\psi H^\infty(\mathbb{D})$, where ψ stands for the singular inner function

$$\psi(z) := \exp\left(-\frac{1+z}{1-z}\right)$$

on $\mathbb{D}$. To give the operator construction in Section 3, the role of the Banach algebra $H^\infty/\psi H^\infty$ and the related estimates are crucial. In fact, the essential part of the construction is based on function-theoretical estimates which has nothing to do with operator theory. Applying an earlier result of Sarason on the factor algebra, which provides a link to operator algebra, we shall obtain analogous estimates for operators on Hilbert space. Actually, in the disk algebra or in $H^\infty(\mathbb{D})$ it is quite straightforward to produce functions with various time regularity. However, to get analogous results in the factor algebra as well, we will need a much more detailed analysis of the previous cases.

For a continuous function f on $[0, 2\pi]$, the *continuity modulus* of f is the increasing function

$$\omega(f;t) := \sup\{|f(x) - f(y)| : |x-y| \leq t \text{ and } 0 \leq x, y \leq 2\pi\}.$$

A *Dini-continuous function* is a continuous function whose continuity modulus $\omega(f;t)$ satisfies the inequality

$$\int_0^{2\pi} \frac{\omega(f;t)}{t}\, dt < \infty.$$

We say that a curve C in the complex plane is *Dini-smooth* if it has a parametrization $w(\tau), 0 \leq \tau \leq 2\pi$, such that $w'(\tau)$ is Dini-continuous and $\neq 0$. Obviously, the definition does not depend on the parameter interval. One can easily check that a $C^{1,\varepsilon}$ curve (i.e. w' is Hölder continuous with exponent ε) is Dini-smooth. Conformal maps onto domains with a Dini-smooth boundary have nice properties on the boundary, as the following Kellogg–Warschawski theorem shows. The result has many general forms and we refer the reader to [18, pp. 48–49] for the proof of each one.

THEOREM 2.1 (Kellogg–Warschawski). *Let f map $\mathbb{D}$ conformally onto the inner domain of a Dini-smooth Jordan curve C. Then f' has a continuous extension to $\overline{\mathbb{D}}$ and*

$$\frac{f(\zeta)-f(z)}{\zeta - z} \to f'(z) \neq 0 \quad \text{as } \zeta \to z,\ \zeta, z \in \overline{\mathbb{D}}.$$

This theorem is one of the major tools that we require in order to have our operator construction in the regularity problem. Actually, we shall prove that if the range of a function $f \in A(\mathbb{D})$ has a $C^{1,\varepsilon}$ boundary satisfying a few reasonable conditions, then the operator $I - f(I-2V)$ in $L^2[0,1]$ has time regularity $n^{-1/(1+\varepsilon)}$ up to constant factors, that is,

$$\|I - f(I-2V)\|_2 \asymp n^{-1/(1+\varepsilon)}.$$

The construction is based on a few preliminary lemmas. Here is the first one.

LEMMA 2.2. *Let $0 < \varepsilon < 1$. Then the closed curve*

$$w_\varepsilon \colon t \mapsto |t^3/3 - t|^{1+\varepsilon} + i(t^3/3 - t)(1-t^2)$$

is Dini-smooth on $[-1,1]$.

Proof. Since the function $s \mapsto |s|^\varepsilon$ is ε-Hölder continuous on $[-1,1]$, a straightforward calculation shows that $w_\varepsilon' \neq 0$ is ε-Hölder continuous as well. Hence w_ε is a $C^{1,\varepsilon}$ smooth curve, so it is Dini as well. ∎

From now on we shall denote the inner domain of the Jordan curve w_ε by Ω_α, where $\alpha := 1/(1+\varepsilon)$. Next we provide elements of $A(\mathbb{D}) \subseteq H^\infty(\mathbb{D})$ with sharp regularity properties.

LEMMA 2.3. *For a fixed $1/2 < \alpha < 1$ and $0 < \beta \leq 1/2$, there exists a function $f_\alpha \in A(\mathbb{D})$ such that $f_\alpha(1) = 0$, $|f_\alpha'(1)| \leq \beta$, $\|f_\alpha\|_\infty \leq 2\beta$,*

$$0 < \lim_{\theta \to 0} \frac{|f_\alpha(e^{i\theta})|^{1/\alpha}}{\operatorname{Re} f_\alpha(e^{i\theta})} < \infty$$

and

$$\|(1-f_\alpha)^n f_\alpha\|_\infty \asymp n^{-\alpha} \quad (n \geq 1).$$

Proof. From the Riemann mapping theorem and Carathéodory's extension theorem, we can find a conformal map h from $\mathbb{D}$ onto Ω_α such that $\lim_{z\to 1} h(z) = 0$. Now we shall define a modification of h. For a sufficiently small $a > 0$, we claim that the convex combination

$$f_\alpha(z) := (1-a)\beta\frac{1-z}{2} + ah(z) \quad (z \in \mathbb{D})$$

satisfies the required conditions. Obviously, $f_\alpha(1) = 0$, $0 < |f'_\alpha(1)| \le \beta$ and $\|f_\alpha\|_\infty \le 2\beta$ for small a because h and h' are bounded on $\overline{\mathbb{D}}$ from Theorem 2.1.

Now we prove that f_α has the required regularity property.

Since $\lim_{z\to 1,\,|z|\le 1} h(z) = 0$ and $\varepsilon = (1-\alpha)/\alpha \in (0,1)$, the definition of the curve w_ε and the homeomorphic extension of h onto $\overline{\Omega}_\alpha$ yield

$$\lim_{\theta\to 0} \frac{|h(e^{i\theta})|^{1+\varepsilon}}{\operatorname{Re} h(e^{i\theta})} = \lim_{\theta\to 0} \frac{|\operatorname{Im} h(e^{i\theta})|^{1+\varepsilon}}{\operatorname{Re} h(e^{i\theta})} = 1. \tag{2.1}$$

We shall see that f_α has the same property, i.e. $\lim_{\theta\to 0} |f_\alpha(e^{i\theta})|^{1+\varepsilon}/\operatorname{Re} f_\alpha(e^{i\theta}) > 0$. Set $u(\theta) = \operatorname{Im} h(e^{i\theta})$ and $v(\theta) = \operatorname{Re} h(e^{i\theta})$. Since $u \in C^1(\overline{\mathbb{D}})$, an elementary reasoning shows that

$$\lim_{\theta\to 0} \frac{u(\theta)}{2\sin(\theta/2)} = \lim_{\theta\to 0} u'(\theta) = u'(0).$$

Moreover,

$$\lim_{\theta\to 0} \frac{v(\theta)}{|\sin(\theta/2)|^{1+\varepsilon}} = |2u'(0)|^{1+\varepsilon} > 0. \tag{2.2}$$

In fact, from (2.1) we get

$$\lim_{\theta\to 0} \frac{(1+\varepsilon)|u(\theta)|^{\varepsilon} u'(\theta)}{v'(\theta)} = 1,$$

hence $v'(0) = 0$. Then $u'(0) \ne 0$ because $h'(0) = u'(0) - iv'(0) \ne 0$ from Theorem 2.1. On the other hand,

$$1 = \lim_{\theta\to 0} \frac{|\sin(\theta/2)|^{1+\varepsilon}}{|u(\theta)|^{1+\varepsilon}} \frac{v(\theta)}{|\sin(\theta/2)|^{1+\varepsilon}} = \frac{1}{|2u'(0)|^{1+\varepsilon}} \lim_{\theta\to 0} \frac{v(\theta)}{|\sin(\theta/2)|^{1+\varepsilon}},$$

which gives (2.2). Now, for small $a > 0$, we see that

$$\begin{aligned} \lim_{\theta\to 0} \frac{|f_\alpha(e^{i\theta})|^{1+\varepsilon}}{\operatorname{Re} f_\alpha(e^{i\theta})} &= \lim_{\theta\to 0} \frac{|(1-a)\beta\cos(\theta/2) - au(\theta)(\sin(\theta/2))^{-1}|^{1+\varepsilon}}{(1-a)\beta|\sin(\theta/2)|^{1-\varepsilon} + av(\theta)|\sin(\theta/2)|^{-1-\varepsilon}} \\ &= \frac{((1-a)\beta - 2au'(0))^{1+\varepsilon}}{a|2u'(0)|^{1+\varepsilon}} =: M \end{aligned}$$

exists and can be arbitrarily large.

Next we note that $|1 - f_\alpha(z)| \le 1$ $(z \in \overline{\mathbb{D}})$, and equality holds if and only if $z = 1$. Indeed, $f_\alpha(z)$ is a convex combination of $h(z), \beta(1-z)/2$, and these two functions have these properties. Since $f_\alpha(e^{i\theta}) \to 0$ as $\theta \to 0$, for every sufficiently small θ we have $|f_\alpha(e^{i\theta})|^2 < M^{-1}|f_\alpha(e^{i\theta})|^{1+\varepsilon}$. Hence, for any sufficiently small positive η and sufficiently large n,

$$\begin{aligned}
\|(1-f_\alpha)^n f_\alpha\|_\infty &= \sup_{|\theta|\le\eta} |(1-f_\alpha(e^{i\theta}))^n f_\alpha(e^{i\theta})| \\
&= \sup_{|\theta|\le\eta} \left(1-2\frac{\operatorname{Re} f_\alpha(e^{i\theta})}{|f_\alpha(e^{i\theta})|^{1+\varepsilon}}|f_\alpha(e^{i\theta})|^{1+\varepsilon}+|f_\alpha(e^{i\theta})|^2\right)^{n/2}|f_\alpha(e^{i\theta})| \\
&\le \sup_{|\theta|\le\eta} (1-M^{-1}|f_\alpha(e^{i\theta})|^{1+\varepsilon})^{n/2}|f_\alpha(e^{i\theta})| \\
&\le \sup_{0\le t\le 1} (1-M^{-1}t^{1+\varepsilon})^{n/2}t < (2eMn^{-1})^{1/(1+\varepsilon)}.
\end{aligned}$$

In a similar way, for all sufficiently large n we get

$$\|(1-f_\alpha)^n f_\alpha\|_\infty \ge \sup_{|\theta|\le\eta} \left|1-2M^{-1}|f_\alpha(e^{i\theta})|^{1+\varepsilon}\right|^{n/2}|f_\alpha(e^{i\theta})| \ge \frac{1}{e^{1/M}}\frac{1}{n^\alpha},$$

and the lemma follows. ■

Now we shall prove that the sequence $\{(1-f_\alpha)^n\}_n$ in Lemma 2.3 has the same time regularity in the factor algebra $H^\infty/\psi H^\infty$ as in $A(\mathbb{D})$ or $H^\infty(\mathbb{D})$; that is, the smoothness of f_α at 1 determines the regularity in the factor algebra as well. At this point we will utilize and develop our earlier method presented in [9], which provides a geometric picture of the values of the above function sequence. Intuitively speaking, we can exploit the fact that 1 is a singular point of the map ψ (the argument of ψ is changing rapidly), and that $|(1-f_\alpha)^n f_\alpha|$ can attain its maximum arbitrarily close to 1 if n is sufficiently large. These observations make it possible to prove that no nontrivial approximation of $(1-f_\alpha)^n f_\alpha$ over ψH^∞ is possible, because the argument of ψh_n, for any $0\ne h_n\in H^\infty$, increases rapidly compared with that of $(1-f_\alpha)^n f_\alpha$.

Let $P_r(\theta)$ denote the Poisson kernel on the unit disk, and $Q_r(\theta)$ its harmonic conjugate:

$$P_r(\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}, \qquad Q_r(\theta)=\frac{2r\sin\theta}{1-2r\cos\theta+r^2}.$$

Then a simple calculation shows that the following properties hold.

Lemma 2.4. *For any positive constant* c,

(i) $P_{1-t}(\sqrt{ct}+O(t))\to 2/c$ *as* $t\downarrow 0$;

(ii) $\left.\dfrac{\partial Q_{1-t}}{\partial\theta}\right|_{\theta=\sqrt{ct}+O(t)} \sim -\dfrac{2}{ct}$ *as* $t\downarrow 0$.

We recall that $H^\infty(\mathbb{D})$ is the dual of $L^1(\mathbb{T})/H^1_0(\mathbb{D})$, where $H^1_0(\mathbb{D})$ denotes the elements of the Hardy space $H^1(\mathbb{D})$ which vanish at 0. For simplicity, we shall introduce a notation for the quantity from the previous lemma,

$$M := \lim_{\theta\to 0} \frac{|f_\alpha(e^{i\theta})|^{1/\alpha}}{\operatorname{Re} f_\alpha(e^{i\theta})}.$$

We shall also use the upper estimate for every large n from the proof of Lemma 2.3:

$$\|(1-f_\alpha)^n f_\alpha\|_\infty < (2eM/n)^\alpha. \tag{2.3}$$

Now we can prove the main statement of this section.

THEOREM 2.5. *In the factor algebra $H^\infty/\psi H^\infty$, we have*

$$M_1 n^{-\alpha} \le \|(1-f_\alpha)^{n+1} - (1-f_\alpha)^n\| \le M_2 n^{-\alpha} \qquad (n \ge 1)$$

with positive constants M_1 and M_2.

Proof. The right inequality follows immediately from Lemma 2.3. For the lower bound, we shall provide a simple geometric description of the values of the difference $(1-f_\alpha)^{n+1} - (1-f_\alpha)^n$ on the set

$$I_n := \{(1-K(M/n)^{2\alpha})e^{i\theta}\}, \text{ where } \theta \in \left[\frac{M^\alpha}{n^\alpha} - \frac{M^{2\alpha}\pi}{n^{2\alpha}}, \frac{M^\alpha}{n^\alpha} + \frac{M^{2\alpha}\pi}{n^{2\alpha}}\right] =: S_n,$$

where the value of the constant $K > 0$ will be determined later. We claim that $((1-f_\alpha)^n f_\alpha)(I_n)$ lies in one half of an annulus D_n, for any sufficiently large n. To verify this, first we shall estimate the modulus on I_n. Since $f_\alpha \in C^1(\overline{\mathbb{D}})$, for any $z \in I_n$ we have

$$f_\alpha(z) = f_\alpha(\exp(i(M/n)^\alpha)) + \nu(z),$$

where $|\nu(z)| = O(n^{-2\alpha})$. From the inequality $|f_\alpha(e^{i\theta})|^2 < M^{-1}|f_\alpha(e^{i\theta})|^{1+\varepsilon}$, if θ is sufficiently small, we find as in the proof of Lemma 2.3 that

$$|1 - f_\alpha(\exp(i(M/n)^\alpha))|^2 < 1 - M^{-1}|f_\alpha(\exp(i(M/n)^\alpha))|^{1+\varepsilon}$$

for n large enough. Moreover, from Theorem 2.1 we note that $|f_\alpha(e^{i\theta})| \sim |\theta f'_\alpha(1)|$ if $\theta \to 0$. Then for any large n (recall that $\varepsilon = (1-\alpha)/\alpha$)

$$\begin{aligned}
|(1-f_\alpha(z))^n f_\alpha(z)| &= \big(|1-f_\alpha(\exp(i(M/n)^\alpha))|^2 \\
&\qquad - 2\operatorname{Re}(1-f_\alpha(\exp(i(M/n)^\alpha)))\overline{\nu(z)} + |\nu(z)|^2\big)^{n/2}|f_\alpha(z)| \\
&< \big(1 - M^{-1}|f_\alpha(\exp(i(M/n)^\alpha))|^{1+\varepsilon} + O(n^{-2\alpha})\big)^{n/2}|f_\alpha(z)| \\
&\sim \left(1 - M^{-1}\frac{|M^\alpha f'_\alpha(1)|^{1+\varepsilon}}{n}\right)^{n/2}|f_\alpha(\exp(i(M/n)^\alpha))| \\
&\sim \exp\big(-|f'_\alpha(1)|^{1+\varepsilon}/2\big)|f'_\alpha(1)|M^\alpha n^{-\alpha} =: C_1 n^{-\alpha}.
\end{aligned}$$

In a similar way, for an appropriate $\delta > 0$ and any $z \in I_n$ we get

$$|1-f_\alpha(z)|^2 > 1 - 2(M^{-1}+\delta)|f_\alpha(\exp(i(M/n)^\alpha))|^{1+\varepsilon} + O(n^{-2\alpha})$$

and

$$|(1-f_\alpha(z))^n f_\alpha(z)| > \tfrac{2}{3}\exp(-|f'_\alpha(1)|^{1+\varepsilon})|f'_\alpha(1)|M^\alpha n^{-\alpha} =: C_2 n^{-\alpha}.$$

Moreover, one can easily check that

$$\max_{z_1,z_2\in I_n} |\arg[(1-f_\alpha(z_1))^n f_\alpha(z_1)] - \arg[(1-f_\alpha(z_2))^n f_\alpha(z_2)]| \le O(n\cdot n^{-2\alpha}+n^{-\alpha})$$

which tends to 0 as $n\to\infty$. In fact, the above arguments readily imply that there indeed exists a $\beta_{\alpha,n}\in[-\pi,\pi]$ such that

(2.4)

$$((1-f_\alpha)^n f_\alpha)(I_n) \subseteq D_n := \{re^{i\theta}\colon |\theta-\beta_{\alpha,n}|\le \pi/2 \text{ and } C_2 n^{-\alpha}\le r\le C_1 n^{-\alpha}\}$$

for any sufficiently large n.

Now we are going to calculate the regularity of the norm of the sequence $\{(1-f_\alpha)^{n+1}-(1-f_\alpha)^n\}_n$ in $H^\infty(\mathbb{D})/\psi H^\infty(\mathbb{D})$. From a standard weak-$*$ compactness argument, there exist functions $h_n\in H^\infty=(L^1/H^1_0)^*$ such that $\|(1-f_\alpha)^n f_\alpha-\psi h_n\|_\infty = \|(1-f_\alpha)^n f_\alpha+\psi H^\infty\|$. Recall that ψh_n is the best approximation of the function $(1-f_\alpha)^n f_\alpha$ over ψH^∞, hence we can assume for instance that $.75\|(1-f_\alpha)^n f_\alpha\|_\infty \le \|\psi h_n\|_\infty = \|h_n\|_\infty \le 1.25\|(1-f_\alpha)^n f_\alpha\|_\infty$, as otherwise the proof is done. From now on we will discuss two separate cases for every large n. The first is when h_n has a relatively large or small modulus on I_n, and in this case one can readily get estimates for the norm in the factor algebra. The second case deals with the remaining situation.

Case 1a. Let us assume that either

$$\|\chi_{I_n}\cdot h_n\|_\infty \ge (5e^{2K}/4)\|\chi_{I_n}(1-f_\alpha)^n f_\alpha\|_\infty \quad\text{or}$$
$$\min_{z\in I_n}|h_n(z)| \le (3e^{2K}/4)\min_{z\in I_n}|((1-f_\alpha)^n f_\alpha)(z)|$$

where χ_{I_n} denotes the characteristic function of I_n.

From Lemma 2.4(i), we have

$$|\psi(z)| = \exp(-P_{1-K(M/n)^{2\alpha}}(\theta)) \sim e^{-2K}$$

uniformly for any $z\in I_n$. The assumption here and the inclusion (2.4) tell us that we can pick a z' in I_n such that

$$\begin{aligned}|((1-f_\alpha)^n f_\alpha)(z')-\psi(z')h_n(z')| &\gtrsim e^{-2K}|h_n(z')| - |((1-f_\alpha)^n f_\alpha)(z')|\\ &\sim \frac{1}{4}\|\chi_{I_n}(1-f_\alpha)^n f_\alpha\|_\infty \ge \frac{C_2}{4}n^{-\alpha},\end{aligned}$$

respectively

$$|((1-f_\alpha)^n f_\alpha)(z')-\psi(z')h_n(z')| \gtrsim \frac{1}{4}\min_{z\in I_n}|((1-f_\alpha)^n f_\alpha)(z)| \ge \frac{C_2}{4}n^{-\alpha}.$$

That is,

$$\|(1-f_\alpha)^n f_\alpha - \psi H^\infty\| \ge \frac{C_2}{4}n^{-\alpha}.$$

Case 1b. Let us assume that

$$\|\chi_{I_n} \cdot h_n\|_\infty \leq (3e^{2K}/4)\|\chi_{I_n}(1-f_\alpha)^n f_\alpha\|_\infty \quad \text{or}$$
$$\min_{z\in I_n} |h_n(z)| \geq (5e^{2K}/4) \min_{z\in I_n} |((1-f_\alpha)^n f_\alpha)(z)|.$$

As in Case 1a we can get

$$\|(1-f_\alpha)^n f_\alpha - \psi H^\infty\| \geq \frac{C_2}{4} n^{-\alpha}.$$

Case 2. In this remaining case, we see from the definition of D_n that

$$\Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \|\chi_{I_n} h_n\|_\infty < \frac{5C_1}{3C_2} = \frac{5}{2}\exp(|f'_\alpha(1)|^{1+\varepsilon}/2).$$

Next, let us estimate the derivative of $\arg h_n$ on I_n. We recall that if $z = re^{i\theta}$,

$$\frac{d}{d\theta} \arg h_n(z) = \operatorname{Re}\left[z \frac{h'_n(z)}{h_n(z)}\right].$$

Now let γ denote a positively oriented circle around any $z_0 \in I_n$ with radius $r = .75K(M/n)^{2\alpha}$. From Cauchy's integral formula, our assumptions, the inequality (2.3), the inclusion (2.4) and the inequality $|f'_\alpha(1)| \leq 1/2$ in Lemma 2.3 it follows that

$$\begin{aligned}
\left|\frac{d}{d\theta} \arg h_n(z_0)\right| &\leq \Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \frac{1}{2\pi}\left|\int_\gamma \frac{h_n(z)}{(z-z_0)^2}\, dz\right| \\
&\leq \Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \max_{z\in I_n} |h_n(z)| r^{-1} \\
&\leq \frac{4}{3}\Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \|(1-f_\alpha)^n f_\alpha\|_\infty r^{-1} \\
&< \frac{4(2eM)^\alpha}{3C_2}\Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \|\chi_{I_n}(1-f_\alpha)^n f_\alpha\|_\infty r^{-1} \\
&< \frac{32e^2}{3e^{2K}|f'_\alpha(1)|}\Big(\min_{z\in I_n} |h_n(z)|\Big)^{-1} \max_{z\in I_n} |h_n(z)| r^{-1} \\
&< \frac{320e^2 M^{-2\alpha}}{9Ke^{2K}|f'_\alpha(1)|} \exp(|f'_\alpha(1)|^{1+\varepsilon}/2) n^{2\alpha}.
\end{aligned}$$

Now let us choose K such that the coefficient of $(n/M)^{2\alpha}$ above is 1. Moreover, for the phase of the singular inner function ψ, we have

$$\arg\left[\exp\left(-\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\right] = -Q_r(\theta),$$

and Lemma 2.4(ii) tells us that

$$-\frac{d}{d\theta} Q_{1-K(M/n)^{2\alpha}}(\theta) \sim 2(n/M)^{2\alpha} \qquad \text{if } \theta \in S_n \text{ and } n \to \infty.$$

It then follows that the function

$$\gamma_n\colon \theta \mapsto \arg[\psi((1-n^{-2\alpha})e^{i\theta})h_n((1-n^{-2\alpha})e^{i\theta})]$$

is strictly increasing on the interval S_n. Actually, the increment of γ_n is asymptotically greater than

$$\frac{2\pi M^{2\alpha}}{n^{2\alpha}}\gamma_n'(\xi) \geq \frac{2\pi M^{2\alpha}}{n^{2\alpha}}\left(\frac{2n^{2\alpha}}{M^{2\alpha}} - \frac{n^{2\alpha}}{M^{2\alpha}}\right) = 2\pi,$$

for some $\xi \in S_n$, relying on the condition on K. Hence we infer that the graph of $\psi(z)h_n(z)$, $z \in I_n$, meets the line $s_n\colon r \mapsto re^{i(\beta_{\alpha,n}+\pi)}$, $r \geq 0$. This gives a lower bound for the norms of $(1-f_\alpha)^n f_\alpha$ in $H^\infty/\psi H^\infty$:

$$\begin{aligned}\|(1-f_\alpha)^n f_\alpha - \psi h_n\|_\infty &\geq \sup_{z\in I_n} |((1-f_\alpha)^n f_\alpha)(z) - \psi(z)h_n(z)| \\ &\geq \operatorname{dist}(((1-f_\alpha)^n f_\alpha)(I_n), s_n) \\ &\geq \operatorname{dist}(D_n, s_n) = C_2/n^\alpha.\end{aligned}$$

We conclude that

$$\|(1-f_\alpha)^{n+1} - (1-f_\alpha)^n + \psi H^\infty\| \geq \frac{C_2}{4n^\alpha}$$

for any sufficiently large n, which is what we intended to show. ∎

3. Time regularity of the operators $I - f(I-2V)$. We recall that the shift operator is the operator U on $L^2(\mathbb{T})$ defined by $(Uf)(z) = zf(z)$, and ψ is the singular inner function

$$\psi(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

on $\mathbb{D}$. The factor algebra $H^\infty/\psi H^\infty$ is closely related to holomorphic functions of the compressed shift operator on a proper subspace. In fact, let P_ψ denote the orthogonal projection from $H^2(\mathbb{D})$ onto $H^2(\mathbb{D})\ominus\psi H^2(\mathbb{D})$, and let T stand for the compression of U into $P_\psi H^2(\mathbb{D})$, that is, $T := P_\psi U P_\psi$. A natural bounded H^∞-calculus of T arises from the definition $\phi(T) := P_\psi \phi(U) P_\psi$ for $\phi \in H^\infty(\mathbb{D})$ (see [20], [21]). From Sarason's theorem we know that the algebra $H^\infty(T)$ is actually isometrically isomorphic to the algebra $H^\infty/\psi H^\infty$ [20, Proposition 2.1]. Sarason also proved that $(I+V)^{-1}$ is unitarily equivalent to the operator $(T+I)/2$ [19, Theorem 1]. Moreover, from Pedersen's similarity relation we have $D^{-1}(I+V)^{-1}D = I-V$, where D denotes the multiplication operator $(Df)(x) = e^{-x}f(x)$ on $L^2[0,1]$. Hence the preceding considerations tell us that $I-2V$ is similar to the contraction T. We note that T is actually the Sz.-Nagy–Foiaş function model of the Cayley transform $(I-V)(I+V)^{-1}$ (see [17, 1.4.12 Theorem]) and the latter is similar to $I-2V$ [23, p. 61]. Now let $J\colon L^2[0,1] \to H^2\ominus\psi H^2$ denote the bounded linear bijection for which $J^{-1}TJ = I - 2V$. The similarity between the two also enables us to define an H^∞-calculus for $I-2V$, namely $\phi(I-2V) := J^{-1}\phi(T)J$ for $\phi \in H^\infty(\mathbb{D})$. We also immediately see that $\sigma(T) = \{1\}$.

Now let us choose a function $f_\alpha \in A(\mathbb{D})$ whose existence is guaranteed by Lemma 2.3 (with a fixed β). One can also assume that $(1-f_\alpha)(\mathbb{D}) \subseteq \overline{\mathbb{D}}$, hence $I - f_\alpha(T)$ is a contraction; that relies on von Neumann's inequality. Then we arrive at the main theorem of this section, which proves the existence of contractions with a small spectrum and various time regularity.

THEOREM 3.1. *Fix an $1/2 < \alpha < 1$. Then there exist positive constants M_1, M_2 such that*

$$M_1 n^{-\alpha} \le \|(I - f_\alpha(T))^{n+1} - (I - f_\alpha(T))^n\| \le M_2 n^{-\alpha}$$

and $\sigma(f_\alpha(T)) = \{0\}$.

Proof. To get the upper bound, we note that

$$\|(I - f_\alpha(T))^{n+1} - (I - f_\alpha(T))^n\| \le \|(1 - f_\alpha)^{n+1} - (1 - f_\alpha)^n\|_\infty \le M_2 n^{-\alpha}$$

from Lemma 2.3. To see that the estimate is indeed sharp, we recall that

$$\|(I-f_\alpha(T))^{n+1}-(I-f_\alpha(T))^n\| = \|(1-f_\alpha)^{n+1}-(1-f_\alpha)^n+\psi H^\infty\| \ge M_1 n^{-\alpha},$$

relying on [20, Proposition 2.1] and Theorem 2.5.

Let us now prove that the spectrum of $I - f_\alpha(T)$ is minimal. Obviously, $f_\alpha(T)$ can be approximated by $p_n(T)$ in the operator norm topology, where the p_n are polynomials converging uniformly to f_α on $\overline{\mathbb{D}}$. We note that the spectrum function (as a compact set-valued function) is upper semicontinuous (see e.g. [2, Theorem 3.4.2]). Here $\sigma(T) = \{1\}$ and $f_\alpha(1) = 0$, hence $\sigma(f_\alpha(T)) = \{0\}$, that is, $\sigma(I - f_\alpha(T)) = \{1\}$. ∎

We remark here that Lemma 2.3 with small β guarantees the existence of $f_\alpha(T)$ with small norm from von Neumann's inequality. Actually, Theorem 3.1 gives that one can find operators with one-point spectrum and various time regularity as quasinilpotent perturbations of the identity. The discussion prior to the above theorem leads to the following corollary.

COROLLARY 3.2. *Fix $1/2 < \alpha < 1$. Let V denote the Volterra integral operator on $L^2[0,1]$. Then there exist positive constants M_1, M_2 such that*

$$M_1 n^{-\alpha} \le \|(I - f_\alpha(I-2V))^{n+1} - (I - f_\alpha(I-2V))^n\|_2 \le M_2 n^{-\alpha}.$$

REMARK 3.3. We note that the regularity of $f \in A(\mathbb{D})$ at 1 seems to be essential for the time regularity of the operator $I - f(I-2V)$. In fact, if we choose any $0 \not\equiv f \in A(\mathbb{D})$ such that f has a holomorphic extension around 1 and $f(1) = 0$, we can apply Lyubich's result [12, Theorem 1.3] and the remarks thereafter to show that

$$\|(I - f(I-2V))^{n+1} - (I - f(I-2V))^n\| \asymp n^{-1/2}$$

whenever $I - f(I-2V)$ is power-bounded. (Indeed, one can readily see that $f(I-2V)$ agrees with the Riesz–Dunford operator

$$\frac{1}{2\pi}\int_\gamma f(z)R(z, I-2V)\,dz,$$

where γ is any positively oriented circle around 1 and $R(z, I-2V)$ denotes the resolvent of $I-2V$ at z. By the change of variable $z \mapsto -2z+1$, we get $f(I-2V) = f_1(V)$, where $f_1(z) := f(-2z+1)$, which is holomorphic around 0.) It would be interesting to know whether the time regularity remains the same if f has only a C^2 boundary at 1 (with a few reasonable conditions).

Remark 3.4. We remark that now one can construct operators with minimal spectrum which are not power-bounded but have various time regularity. In fact, fix an $\alpha \in (1/2, 1)$ and let us choose a contraction T in $L^2[0,1]$ such that $\sigma(T) = \{1\}$ and $\|T^{n+1} - T^n\| \asymp n^{-\alpha}$. Let us consider the Tomilov–Zemánek matrix $\mathcal{T}$ of T on $L^2[0,1] \oplus L^2[0,1]$ ([22])

$$\mathcal{T} = \begin{pmatrix} T & T-I \\ 0 & T \end{pmatrix}.$$

Then straightforward calculations show that $\sigma(\mathcal{T}) = \{1\}$ and

$$\mathcal{T}^{n+1} - \mathcal{T}^n = \begin{pmatrix} T^{n+1} - T^n & nT^{n-1}(T-I)^2 + T^n(T-I) \\ 0 & T^{n+1} - T^n \end{pmatrix}$$

(see [22, Lemma 2.1]). Hence $\|\mathcal{T}^{n+1} - \mathcal{T}^n\| = O(n^{-2\alpha+1})$, where $2\alpha - 1 \in (0,1)$, but $\|\mathcal{T}^n\| \geq \mathrm{const}\cdot(n-1)^{1-\alpha}$. However, it is an open question whether there exists an operator with one-point spectrum and unbounded powers such that the norms of the differences of consecutive powers have order $1/n$. If one allows the spectrum to be large, the reader can see [7] for such an example.

Remark 3.5. Lastly, we note that one can easily construct operators with a given time regularity if one allows the spectrum to be large. We refer the reader to [4] and [16] for some examples. Using the geometric properties of Ω_α from Section 2, we can provide the following simple construction. Let $L^2_a(\overline{\Omega}_\alpha)$ denote the Bergman space over the closure of Ω_α. The operator M_z is the usual multiplication operator $f \mapsto z \cdot f$ on $L^2_a(\overline{\Omega}_\alpha)$ and $\sigma(M_z) = \overline{\Omega}_\alpha$. Then the multiplier norms (see e.g. [1, p. 21]) satisfy

$$\begin{aligned}\|(I-M_z)^{n+1} - (I-M_z)^n\| &= \|(I-M_z)^n M_z\| = \|M_{(1-z)^n z}\| \\ &= \sup_{z \in \partial\overline{\Omega}_\alpha} |1-z|^n |z| \asymp n^{-\alpha},\end{aligned}$$

following the line of reasoning of Lemma 2.3.

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