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Connes amenability-like properties

by

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Abstract. We introduce and study the notions of w^*-approximate Connes amenability and pseudo-Connes amenability for dual Banach algebras. We prove that the dual Banach sequence algebra ℓ^1 is not w^*-approximately Connes amenable. We show that in general the concepts of pseudo-Connes amenability and Connes amenability are distinct. Moreover the relations between these new notions are also discussed.

1. Introduction. The concept of amenability for Banach algebras was introduced by B. E. Johnson in 1972 [10]. Several modifications of this notion were introduced by relaxing some of the restrictions on the definition of amenability. Some of the most notable are the concepts of Connes amenability [12] and pseudo-amenability [9]; the former had been studied previously under different names. We recall the definitions in Definitions 1.1 and 1.2 below.

Suppose that $\mathcal{A}$ is a Banach algebra and that E is a Banach $\mathcal{A}$-bimodule. A bounded linear operator $D : \mathcal{A} \to E$ is a *derivation* if it satisfies $D(ab) = D(a) . b + a . D(b)$ for all $a, b \in \mathcal{A}$. A derivation D is *inner* if there is $x \in E$ such that $D(a) = a . x - x . a$ for $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra. It is known that the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. There is a continuous linear $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ such that $\pi(a \otimes b) = ab$ for $a, b \in \mathcal{A}$. The dual of a Banach space E is denoted by E^*. In the case where E is a Banach $\mathcal{A}$-bimodule, E^* is also a Banach $\mathcal{A}$-bimodule. The reader may see [1] for more information.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A dual Banach $\mathcal{A}$-bimodule E is *normal* if the module actions of $\mathcal{A}$ on E are w^*-continuous. A Banach algebra is *dual* if it is dual as a Banach

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$\mathcal{A}$-bimodule. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$.

DEFINITION 1.1 ([12]). A dual Banach algebra $\mathcal{A}$ is *Connes amenable* if every w^*-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner.

The reader is referred to [13] for basic properties of Connes amenable dual Banach algebras. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \to E, \qquad a \mapsto \begin{cases} a \,.\, x, \\ x \,.\, a, \end{cases}$$

are w^*-weak continuous. The space $\sigma wc(E)$ ia a closed submodule of E. It is shown in [15, Corollary 4.6] that $\pi^*(\mathcal{A}_*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Taking adjoints, we can extend π to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$. A *σwc-virtual diagonal* for a dual Banach algebra $\mathcal{A}$ is an element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \,.\, M = M \,.\, a$ and $a\pi_{\sigma wc}(M) = a$ for $a \in \mathcal{A}$. It is known that Connes amenability of $\mathcal{A}$ is equivalent to existence of a σwc-virtual diagonal for $\mathcal{A}$ [15].

DEFINITION 1.2 ([9]). A Banach algebra $\mathcal{A}$ is *pseudo-amenable* if there is a net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$, called an *approximate diagonal* for $\mathcal{A}$, such that $a \,.\, m_\alpha - m_\alpha \,.\, a \to 0$ and $\pi(m_\alpha)a \to a$ for each $a \in \mathcal{A}$.

In this paper we introduce and investigate two notions of amenability for dual Banach algebras; w^*-approximate Connes amenability and pseudo-Connes amenability. The former is based on a property of derivations, while the latter consists in existence of a kind of diagonal (not necessarily bounded) with certain properties.

The organization of the paper is as follows. Firstly, in Section 2 we study basic properties of w^*-approximate Connes amenability. We give some characterizations of both Connes amenability and w^*-approximate Connes amenability in terms of diagonals.

In Section 3 we show that the dual Banach sequence algebra $\ell^1 = \ell^1(\mathbb{N})$ is not w^*-approximately Connes amenable. The same result holds for the standard dual Banach sequence algebra $\ell^1(\omega)$, where ω is a weight on $\mathbb{N}$.

In Section 4 we introduce the notion of pseudo-Connes amenability. We show that the ℓ^1-direct sum of two pseudo-Connes amenable dual Banach algebras is pseudo-Connes amenable. We show that in general $\mathcal{A}$ being pseudo-Connes amenable is weaker than $\mathcal{A}^\sharp$ being pseudo-Connes amenable. But if $\mathcal{A}$ has an identity, then they are equivalent.

In Section 5 we study the relations between pseudo-Connes amenability and w^*-approximate Connes amenability. We show that a dual Banach algebra $\mathcal{A}$ is w^*-approximately Connes amenable if and only if its unitization

$\mathcal{A}^\sharp$ is pseudo-Connes amenable. We examine pseudo-Connes amenability of a dual Banach algebra with identity and in particular we show that pseudo-Connes amenability and w^*-approximate Connes amenability are the same in this case. We show that every w^*-approximately Connes amenable, commutative dual Banach algebra is pseudo-Connes amenable. We prove that any w^*-approximately Connes amenable dual Banach algebra having a central w^*-approximate identity is pseudo-Connes amenable. We also describe w^*-continuous derivations from a pseudo-Connes amenable dual Banach algebra $\mathcal{A}$ into certain normal, dual Banach $\mathcal{A}$-bimodules.

Finally, in Section 6 we verify these various notions of Connes amenability for certain dual Banach algebras. In particular, we see that pseudo-Connes amenability is weaker than Connes amenability.

2. w^*-Approximate Connes amenability. We start with basic constructions. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to E^*$ is *w^*-approximately inner* if there exists a net $(\phi_i) \subseteq E^*$ such that $D(a) = w^*\text{-}\lim_i(a \,.\, \phi_i - \phi_i \,.\, a)$ for $a \in \mathcal{A}$, the limit being in the w^*-topology on E^*.

Definition 2.1. A dual Banach algebra $\mathcal{A}$ is *w^*-approximately Connes amenable* if for every normal, dual Banach $\mathcal{A}$-bimodule E every w^*-continuous derivation $D : \mathcal{A} \to E$ is w^*-approximately inner.

We remark that, in [6], the notion of approximate Connes amenability was also introduced: a dual Banach algebra $\mathcal{A}$ is said to be *approximately Connes amenable* if, for each normal dual Banach $\mathcal{A}$-bimodule E, each w^*-continuous derivation $D : \mathcal{A} \to E$ is the limit of a sequence of inner derivations in the strong-operator topology of $\mathcal{B}(\mathcal{A}, E)$. Of course, each approximately Connes amenable dual Banach algebra is w^*-approximately Connes amenable. However, in contrast to [8, Theorem 2.1], we are not able to prove (or disprove) the converse.

Throughout, if $\mathcal{A}$ is a Banach algebra we shall write $\mathcal{A}^\sharp$ for the forced unitization of $\mathcal{A}$. The adjoined identity element will usually be denoted by e.

The proofs of the following two propositions are analogous to those of [6, Propositions 2.2 and 2.3].

Proposition 2.2. *Suppose that $\mathcal{A}$ is a w^*-approximately Connes amenable dual Banach algebra. Then $\mathcal{A}$ has a left and right w^*-approximate identity. In particular $\mathcal{A}^2$ is w^*-dense in $\mathcal{A}$.*

Proposition 2.3. *A dual Banach algebra $\mathcal{A}$ is w^*-approximately Connes amenable if and only if $\mathcal{A}^\sharp$ is w^*-approximately Connes amenable.*

Proposition 2.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras and that $\theta : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism which is w^*-continuous. If $\mathcal{A}$ is w^*-approximately Connes amenable, then so is $\mathcal{B}$.*

Proof. Use the argument of [7, Proposition 2.2]. ∎

PROPOSITION 2.5. *Suppose that $\mathcal{A}$ is a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is w^*-approximately Connes amenable.*

(ii) *There is a net $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that*

$$a \,.\, M_\alpha - M_\alpha \,.\, a \xrightarrow{w^*} 0 \text{ in } \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^* \quad (a \in \mathcal{A}^\sharp),$$

and $\pi_{\sigma wc}(M_\alpha) \xrightarrow{w^*} e$ in $\mathcal{A}^\sharp$.

(iii) *There is a net $(M'_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that*

$$a \,.\, M'_\alpha - M'_\alpha \,.\, a \xrightarrow{w^*} 0 \text{ in } \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^* \quad (a \in \mathcal{A}^\sharp),$$

and $\pi_{\sigma wc}(M'_\alpha) = e$ for all α.

Proof. (i)⇒(iii): By Proposition 2.3, $\mathcal{A}^\sharp$ is w^*-approximately Connes amenable. Canonically $\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ is a submodule of $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$. Because of the normality of the dual module $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$, the derivation

$$D : \mathcal{A}^\sharp \to \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*, \qquad a \mapsto a \otimes e - e \otimes a,$$

is w^*-continuous. Also D attains its values in the w^*-closed submodule $\ker \pi_{\sigma wc}$. Therefore, there exists a net $(N_\alpha)_\alpha \subseteq \ker \pi_{\sigma wc}$ such that $Da = w^*\text{-}\lim_\alpha (a \,.\, N_\alpha - N_\alpha \,.\, a)$ for each $a \in \mathcal{A}^\sharp$. Letting $M'_\alpha := e \otimes e - N_\alpha$, we obtain a net as required in (iii).

(iii)⇒(ii): It is immediate.

(ii)⇒(i): Let E be a normal, dual Banach $\mathcal{A}^\sharp$-bimodule which we may suppose is *unital*, i.e., $x \,.\, e = e \,.\, x = x$ for every $x \in E$. Let $D : \mathcal{A}^\sharp \to E$ be a w^*-continuous derivation. Define $\phi : \mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp \to E$ by $a \otimes b \mapsto a \,.\, Db$. In [15, Theorem 4.8], it is shown that ϕ^* maps the predual E_* of E into $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)$, so that the w^*-continuous map $\Phi := (\phi^*|_{E_*})^*$ maps $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ into E. Let $x_\alpha := \Phi(M_\alpha)$. Since $\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ is w^*-dense in $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$, there is a net $m_\alpha^{(i)} := \sum_{n=1}^\infty a_{n,\alpha}^{(i)} \otimes b_{n,\alpha}^{(i)} \in \mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ such that $M_\alpha = w^*\text{-}\lim_i m_\alpha^{(i)}$ in $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$. Then, for each $\psi \in E_*$ and $a \in \mathcal{A}^\sharp$, we have

$$\begin{aligned}
\langle \psi, a \,.\, x_\alpha \rangle &= \lim_i \Big\langle \psi, \sum_{n=1}^\infty a \; a_{n,\alpha}^{(i)} \,.\, D(b_{n,\alpha}^{(i)}) \Big\rangle \\
&= \lim_i \Big\langle \psi, \sum_{n=1}^\infty a_{n,\alpha}^{(i)} \,.\, D(b_{n,\alpha}^{(i)} a) \Big\rangle \\
&= \lim_i \Big\langle \psi, \sum_{n=1}^\infty a_{n,\alpha}^{(i)} b_{n,\alpha}^{(i)} \,.\, Da \Big\rangle + \lim_i \Big\langle \psi, \sum_{n=1}^\infty a_{n,\alpha}^{(i)} \,.\, D(b_{n,\alpha}^{(i)}) \,.\, a \Big\rangle \\
&= \langle \psi, \pi_{\sigma wc} M_\alpha \,.\, Da \rangle + \langle \psi, x_\alpha \,.\, a \rangle.
\end{aligned}$$

Hence $\langle \psi, a\,.\,x_\alpha - x_\alpha.a\,\rangle \to \langle \psi, Da\rangle$ so that by Proposition 2.3, $\mathcal{A}$ is w^*-approximately Connes amenable. ∎

The proof of the following lemma is elementary.

LEMMA 2.6. *Let $\mathcal{A}$ be a dual Banach algebra and let E and F be Banach $\mathcal{A}$-bimodules. Then $\sigma wc(E \oplus F) = \sigma wc(E) \oplus \sigma wc(F)$.*

Reformulating Proposition 2.5 to avoid using an adjoined identity we have the following.

THEOREM 2.7. *Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is w^*-approximately Connes amenable if and only if there are nets $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $(U_\alpha)_\alpha, (V_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$ such that for all $a \in \mathcal{A}$:*

(i) $a\,.\,M_\alpha - M_\alpha\,.\,a + U_\alpha \otimes a - a \otimes V_\alpha \xrightarrow{w^*} 0$ *in* $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$*;*
(ii) $a\,.\,U_\alpha - a \xrightarrow{w^*} 0$ *and* $V_\alpha\,.\,a - a \xrightarrow{w^*} 0$ *in* $\sigma wc(\mathcal{A}^*)^*$*;*
(iii) $\pi_{\sigma wc}(M_\alpha) - U_\alpha - V_\alpha \xrightarrow{w^*} 0$ *in* $\mathcal{A}$.

Proof. Suppose $\mathcal{A}$ is w^*-approximately Connes amenable, and take the net $(N_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ given in Proposition 2.5(ii). By Lemma 2.6, since $(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* = (\mathcal{A} \hat{\otimes} \mathcal{A})^* \oplus (\mathcal{A}^* \otimes e) \oplus (e \otimes \mathcal{A}^*) \oplus (\mathbb{C}e \otimes e)$, we have

$$\sigma wc(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* = \sigma wc(\mathcal{A} \hat{\otimes} \mathcal{A})^* \oplus (\sigma wc(\mathcal{A}^*) \otimes e) \oplus (e \otimes \sigma wc(\mathcal{A}^*)) \oplus (\mathbb{C}e \otimes e).$$

Therefore

$$\begin{aligned}&\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^* \\ &\quad = \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^* \oplus (\sigma wc(\mathcal{A}^*)^* \otimes e) \oplus (e \otimes \sigma wc(\mathcal{A}^*)^*) \oplus (\mathbb{C}e \otimes e).\end{aligned}$$

Thus we can write $N_\alpha = M_\alpha - U_\alpha \otimes e - e \otimes V_\alpha + c_\alpha e \otimes e$, where $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $(U_\alpha)_\alpha, (V_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$ and $(c_\alpha)_\alpha \subseteq \mathbb{C}$.

Applying $\pi_{\sigma wc}$, we observe that

$$\pi_{\sigma wc}(M_\alpha) - U_\alpha - V_\alpha + c_\alpha \xrightarrow{w^*} e,$$

whence $c_\alpha \to 1$ and $\pi_{\sigma wc}(M_\alpha) - U_\alpha - V_\alpha \xrightarrow{w^*} 0$, that is, we have (iii).

Next, for $a \in \mathcal{A}$,

$$\begin{aligned}a\,.\,N_\alpha - N_\alpha\,.\,a &= a\,.\,M_\alpha - M_\alpha\,.\,a + U_\alpha \otimes a - a \otimes V_\alpha + e \otimes V_\alpha\,.\,a \\ &\quad - a\,.\,U_\alpha \otimes e + a \otimes e - e \otimes a \xrightarrow{w^*} 0,\end{aligned}$$

whence we conclude that

$$\lim_\alpha (a\,.\,M_\alpha - M_\alpha\,.\,a + U_\alpha \otimes a - a \otimes V_\alpha) = 0 \quad \text{and} \quad \lim_\alpha a\,.\,U_\alpha = \lim_\alpha V_\alpha\,.\,a = a,$$

in the w^*-topology of $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $\sigma wc(\mathcal{A}^*)^*$, respectively, as required.

Conversely, given $(M_\alpha)_\alpha$, $(U_\alpha)_\alpha$ and $(V_\alpha)_\alpha$, set $c_\alpha = 1$ and define $N_\alpha := M_\alpha - U_\alpha \otimes e - e \otimes V_\alpha + e \otimes e$. Then it is easy to check that, for all $a \in \mathcal{A}^\sharp$,

$$a \,.\, N_\alpha - N_\alpha \,.\, a \xrightarrow{w^*} 0 \quad \text{and} \quad \pi_{\sigma wc}(N_\alpha) \xrightarrow{w^*} e,$$

so that, by Proposition 2.5, $\mathcal{A}$ is w^*-approximately Connes amenable. ∎

We write $\mathcal{F}(X)$ for the family of finite subsets of a given set X.

Theorem 2.8. *A dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ is Connes amenable if and only if there is a constant $C > 0$ such that for each $\varepsilon > 0$ and any finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, there exists $F \in \mathcal{A} \otimes \mathcal{A}$ with $\|F\| \le C$ such that*

(i) $|\langle \phi, a \,.\, F - F \,.\, a\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \phi \in \mathcal{H})$;
(ii) $|\langle \psi, a - a\pi(F)\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \psi \in \mathcal{K})$.

Proof. Suppose that $\mathcal{A}$ is Connes amenable. By Proposition 4.2 below there are $C > 0$ and a net $(m_\alpha)_\alpha$ in $\mathcal{A} \hat{\otimes} \mathcal{A}$ with $\|m_\alpha\| \le C$ such that

$$a \,.\, m_\alpha - m_\alpha \,.\, a \xrightarrow{w^*} 0 \quad \text{and} \quad a\pi(m_\alpha) - a \xrightarrow{w^*} 0$$

in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $\mathcal{A}$, respectively. Then, for given $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, there exists $F \in \mathcal{A} \hat{\otimes} \mathcal{A}$ with $\|F\| \le C$ such that clauses (i) and (ii) of the theorem are satisfied. By modifying F slightly, we may suppose that $F \in \mathcal{A} \otimes \mathcal{A}$.

Conversely, suppose that the condition in the theorem is satisfied. Consider the set

$$I = (0,1) \times \mathcal{F}(\mathcal{A}) \times \mathcal{F}(\mathcal{A}_*) \times \mathcal{F}(\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)).$$

We order I by setting $(\varepsilon_1, \mathcal{S}_1, \mathcal{K}_1, \mathcal{H}_1) \preceq (\varepsilon_2, \mathcal{S}_2, \mathcal{K}_2, \mathcal{H}_2)$ whenever $\varepsilon_1 \ge \varepsilon_2$, $\mathcal{S}_1 \subseteq \mathcal{S}_2$, $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $\mathcal{H}_1 \subseteq \mathcal{H}_2$. Then I is a directed set. The conditions show that there exists a net $(F_\alpha)_{\alpha \in I} \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ such that $\|F_\alpha\| \le C$ and

$$\langle \phi, a \,.\, F_\alpha - F_\alpha \,.\, a\rangle \to 0 \quad \text{and} \quad \langle \psi, a - a\pi(F_\alpha)\rangle \to 0$$

for all $a \in \mathcal{A}$, $\psi \in \mathcal{A}_*$ and $\phi \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. Therefore, by Proposition 4.2, $\mathcal{A}$ is Connes amenable. ∎

For comparison, we recall [11] that a Banach algebra $\mathcal{A}$ is *amenable* if and only if there is a constant $C > 0$ such that, for each $\varepsilon > 0$ and each finite subset $\mathcal{S} \subseteq \mathcal{A}$, there exists $F \in \mathcal{A} \otimes \mathcal{A}$ with $\|F\| \le C$ such that, for each $a \in \mathcal{S}$, we have

(i) $\|a \,.\, F - F \,.\, a\| < \varepsilon$;
(ii) $\|a - a\pi(F)\| < \varepsilon$.

We can extend Theorem 2.8 to the approximate case as follows.

Theorem 2.9. *Suppose that $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra. Then $\mathcal{A}$ is w^*-approximately Connes amenable if and only if, for each $\varepsilon > 0$ and* *any finite subsets* $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$ *and* $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, *there exist* $F \in \mathcal{A} \otimes \mathcal{A}$ *and* $u, v \in \mathcal{A}$ *such that* $\pi(F) = u + v$ *on* $\mathcal{K}$ *and*

(i) $|\langle \phi, a \,.\, F - F \,.\, a + u \otimes a - a \otimes v\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \phi \in \mathcal{H})$;
(ii) $|\langle \psi, au - a\rangle| < \varepsilon$ *and* $|\langle \psi, va - a\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \psi \in \mathcal{T})$.

Proof. Suppose that $\mathcal{A}$ is w^*-approximately Connes amenable and let (M_α), (U_α) and (V_α) be the nets given in Theorem 2.7. Then, for the specified $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$, $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, there is $\alpha = \alpha(\varepsilon, \mathcal{S}, \mathcal{K}, \mathcal{T}, \mathcal{H})$ such that

$$|\langle \phi, a \,.\, M_\alpha - M_\alpha \,.\, a + U_\alpha \otimes a - a \otimes V_\alpha\rangle| < \varepsilon,$$
$$|\langle \psi, a \,.\, U_\alpha - a\rangle| < \varepsilon \quad \text{and} \quad |\langle \psi, V_\alpha \,.\, a - a\rangle| < \varepsilon,$$
$$|\langle f, \pi_{\sigma wc}(M_\alpha) - U_\alpha - V_\alpha\rangle| < \varepsilon,$$

for all $a \in \mathcal{S}$, $f \in \mathcal{K}$, $\psi \in \mathcal{T}$ and $\phi \in \mathcal{H}$.

By the w^*-density of $\mathcal{A}$ and $\mathcal{A} \hat{\otimes} \mathcal{A}$ in $\sigma wc(\mathcal{A}^*)^*$ and $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$, respectively, and by the w^*-continuity of $\pi_{\sigma wc}$, there are $F \in \mathcal{A} \hat{\otimes} \mathcal{A}$ and $u, v \in \mathcal{A}$ such that clauses (i) and (ii) of the theorem are satisfied, and further such that $|\langle f, \pi(F) - u - v\rangle| < \varepsilon$ for every $f \in \mathcal{K}$. By modifying F and u slightly, we may suppose that $F \in \mathcal{A} \otimes \mathcal{A}$ and that $\langle f, \pi(F) - u - v\rangle = 0$ for all $f \in \mathcal{K}$.

Conversely, suppose that the condition in the theorem is satisfied. Similar to Theorem 2.8, we order the set

$$I = (0,1) \times \mathcal{F}(\mathcal{A}) \times \mathcal{F}(\mathcal{A}_*) \times \mathcal{F}(\sigma wc(\mathcal{A}^*)) \times \mathcal{F}\big(\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)\big).$$

The above assumptions show that there exist nets $(F_\alpha)_{\alpha \in I} \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ and $(U_\alpha)_{\alpha \in I}$, $(V_\alpha)_{\alpha \in I} \subseteq \mathcal{A}$ such that $\pi(F_\alpha) = U_\alpha + V_\alpha$ on $\mathcal{K}$ (note that α depends on $\mathcal{K}$) and such that for each $a \in \mathcal{A}$, we have

$$a \,.\, F_\alpha - F_\alpha \,.\, a + u_\alpha \otimes a - a \otimes v_\alpha \xrightarrow{w^*} 0 \quad \text{in } \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*,$$
$$au_\alpha - a \xrightarrow{w^*} 0 \quad \text{and} \quad v_\alpha a - a \xrightarrow{w^*} 0 \quad \text{in } \sigma wc(\mathcal{A}^*)^*.$$

Therefore, the conditions of Theorem 2.7 are satisfied, and so $\mathcal{A}$ is w^*-approximately Connes amenable. ■

For each Banach algebra $\mathcal{A}$, there is an isometry $\iota : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$ such that $\iota(a \otimes b) = b \otimes a$.

LEMMA 2.10. *Suppose that* $\mathcal{A}$ *is a commutative dual Banach algebra. If* ϕ *is an element of* $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, *then so is* $\iota^*(\phi)$.

Proof. Since $\mathcal{A}$ is commutative

$$\iota^*(a \,.\, \phi) = \iota^*(\phi) \,.\, a \quad \text{and} \quad \iota^*(\phi \,.\, a) = a \,.\, \iota^*(\phi) \quad (a \in \mathcal{A},\ \phi \in (\mathcal{A} \hat{\otimes} \mathcal{A})^*).$$

Let $\phi \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$ and let $(a_\alpha)_\alpha$ be a net in $\mathcal{A}$, which tends to $a \in \mathcal{A}$ in the w^*-topology. For every $\Lambda \in (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$, we have

$$\langle \Lambda, a_\alpha \cdot \iota^*(\phi)\rangle = \langle \iota^{**}(\Lambda), \phi \cdot a_\alpha\rangle \to \langle \iota^{**}(\Lambda), \phi \cdot a\rangle = \langle \Lambda, a \cdot \iota^*(\phi)\rangle$$

so that $a_\alpha \cdot \iota^*(\phi) \xrightarrow{w} a \cdot \iota^*(\phi)$ in $(\mathcal{A} \hat{\otimes} \mathcal{A})^*$. Similarly, $\iota^*(\phi) \cdot a_\alpha \xrightarrow{w} \iota^*(\phi) \cdot a$, as required. ∎

We now give a variation of Theorem 2.9 in the case where $\mathcal{A}$ is commutative.

THEOREM 2.11. *Suppose that $\mathcal{A} = (\mathcal{A}_*)^*$ is a commutative dual Banach algebra. Then $\mathcal{A}$ is w^*-approximately Connes amenable if and only if, for each $\varepsilon > 0$ and any finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, there exist $F \in \mathcal{A} \otimes \mathcal{A}$ with $\iota(F) = F$ and $u \in \mathcal{A}$ such that $\pi(F) = 2u$ on $\mathcal{K}$ and*

(i) $|\langle \phi, a \cdot F - F \cdot a + u \otimes a - a \otimes u\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \phi \in \mathcal{H})$;
(ii) $|\langle \psi, au - a\rangle| < \varepsilon$ $(a \in \mathcal{S},\ \psi \in \mathcal{T})$.

Proof. By the commutativity of $\mathcal{A}$,

$$\iota(a \cdot F) = \iota(F) \cdot a \quad \text{and} \quad \iota(F \cdot a) = a \cdot \iota(F) \qquad (a \in \mathcal{A},\ F \in \mathcal{A} \hat{\otimes} \mathcal{A}).$$

Suppose that $\mathcal{A}$ is w^*-approximately Connes amenable, and take $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. Put $\mathcal{H}^* = \{\iota^*(\phi) : \phi \in \mathcal{H}\}$. Then, by Lemma 2.10, $\mathcal{H}^*$ is a finite subset of $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. By Theorem 2.9, there exist $G \in \mathcal{A} \otimes \mathcal{A}$ and $v, w \in \mathcal{A}$ with $\pi(G) = v + w$ on $\mathcal{K}$, satisfying

$$|\langle \Lambda, a \cdot G - G \cdot a + v \otimes a - a \otimes w\rangle| < \varepsilon \qquad (a \in \mathcal{S},\ \Lambda \in \mathcal{H} \cup \mathcal{H}^*),$$
$$|\langle \psi, av - a\rangle| < \varepsilon \quad \text{and} \quad |\langle \psi, wa - a\rangle| < \varepsilon \qquad (a \in \mathcal{S},\ \psi \in \mathcal{T}).$$

For each $a \in \mathcal{S}$ and $\phi \in \mathcal{H}$, we have

$$\begin{aligned} &|\langle \phi, \iota(G) \cdot a - a \cdot \iota(G) + a \otimes v - w \otimes a\rangle| \\ &\qquad = |\langle \phi, \iota(a \cdot G - G \cdot a + v \otimes a - a \otimes w)\rangle| \\ &\qquad = |\langle \iota^*(\phi), a \cdot G - G \cdot a + v \otimes a - a \otimes w\rangle| < \varepsilon. \end{aligned}$$

Set $F = \frac{1}{2}(G + \iota(G))$ and $u = \frac{1}{2}(v + w)$. Then $\iota(F) = F$ and $\pi(F) = 2u$ on $\mathcal{K}$. Further

$$|\langle \phi, a \cdot F - F \cdot a + u \otimes a - a \otimes u\rangle| < \varepsilon \quad \text{and} \quad |\langle \psi, au - a\rangle| < \varepsilon$$

for all $a \in \mathcal{S}$, $\psi \in \mathcal{T}$ and $\phi \in \mathcal{H}$.

The converse is immediate. ∎

3. (Non-) w^*-approximate Connes amenability for ℓ^1. We consider the spaces ℓ^∞, c_0 and ℓ^p, $1 \le p < \infty$, in the standard way. These algebras are discussed in [1, Example 4.1.42]. It is shown in [4] that the

Banach sequence algebra ℓ^p, $1 \le p < \infty$, and its weighted variant $\ell^p(\omega)$, for any weight ω on $\mathbb{N}$, are not approximately amenable. In this section we show that these algebras are not w^*-approximately Connes amenable in the case $p = 1$.

As usual c_{00} will be the subalgebra of $\mathbb{C}^{\mathbb{N}}$ consisting of all sequences having finite support. We write δ_n for the *characteristic function* of the singleton $\{n\}$, for every $n \in \mathbb{N}$. It is known that ℓ^1 is a Banach algebra under pointwise multiplication. It is routine to check that c_0 is a closed ℓ^1-subbimodule of ℓ^∞, and so ℓ^1 is a dual Banach algebra. We set $e_n = \delta_1 + \cdots + \delta_n$ for every $n \in \mathbb{N}$. Then the sequence $(e_n)_n$ is an approximate identity for ℓ^1.

We now recall some preliminaries and further notations from [4]. A (*dual*) *Banach sequence algebra* on $\mathbb{N}$ is a (dual) Banach algebra $\mathcal{A}$ which is a subalgebra of $\mathbb{C}^{\mathbb{N}}$ such that $c_{00} \subseteq \mathcal{A}$. Our specific algebra ℓ^1 is a dual Banach sequence algebra, and c_{00} is dense in ℓ^1. Let $\mathcal{A}$ be a Banach sequence algebra, and let e_n be as before, for all $n \in \mathbb{N}$. Then $(e_n)_n \subseteq c_{00} \subseteq \mathcal{A}$. Each element $a \in \mathcal{A}$ is considered both as the sequence $(a_i)_i$ and the formal sum $\sum_{i=1} a_i\delta_i$. We may identify an element $a \otimes b$ in $\mathcal{A} \otimes \mathcal{A}$ as a function on $\mathbb{N} \times \mathbb{N}$ by setting $(a \otimes b)(i, j) = a_i b_j$, $i, j \in \mathbb{N}$. In particular, $\delta_i \otimes \delta_j = \delta_{(i,j)}$, the characteristic function of (i, j). For an element $F = \sum_{i,j} F(i, j)\delta_{(i,j)}$ in $c_{00} \otimes c_{00}$, note that

$$(a \,.\, F)(i, j) = a_i F(i, j) \quad \text{and} \quad (F \,.\, a)(i, j) = a_j F(i, j) \qquad (i, j \in \mathbb{N})$$

and that $\pi(F) = \sum_i F(i, i)\delta_i$. For $F \in c_{00} \otimes c_{00}$ and for $a \in \mathcal{A}$, we set

$$\varDelta_a(F) = a \,.\, F - F \,.\, a + \pi(F) \otimes a - a \otimes \pi(F).$$

It is clear that $\varDelta_a(F) \in c_{00} \otimes c_{00}$ whenever $a \in c_{00}$.

THEOREM 3.1. *Suppose that $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach sequence algebra with c_{00} dense in $\mathcal{A}$. Then $\mathcal{A}$ is w^*-approximately Connes amenable if and only if, for each $\varepsilon > 0$ and any finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, there exists $F \in c_{00} \otimes c_{00}$ with $\iota(F) = F$ such that*

(i) $|\langle \phi, \varDelta_a(F)\rangle| < \varepsilon \quad (a \in \mathcal{S},\ \phi \in \mathcal{H})$;
(ii) $|\langle \psi, a\pi(F) - a\rangle| < \varepsilon \quad (a \in \mathcal{S},\ \psi \in \mathcal{T})$.

Proof. Suppose that $\mathcal{A}$ is w^*-approximately Connes amenable, and take $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$, $\mathcal{T} \subseteq \sigma wc(\mathcal{A}^*)$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. Let F and u be given by Theorem 2.11. Since $c_{00} \otimes c_{00}$ is dense in $\mathcal{A} \otimes \mathcal{A}$, we can replace F by an element $G \in c_{00} \otimes c_{00}$ such that (i) and (ii) of Theorem 2.11 remain true with $v = \pi(G)/2$ replacing u. Set

$$B = G + \sum_i (v_i - \pi(G)_i)\delta_i \otimes \delta_i \in c_{00} \otimes c_{00}.$$

Clearly $\pi(B) = v$ on $\mathcal{K}$. We replace G by B and since

$$a \,.\, \Big(\sum_i (v_i - \pi(G)_i)\delta_i \otimes \delta_i\Big) = \Big(\sum_i (v_i - \pi(G)_i)\delta_i \otimes \delta_i\Big) \,.\, a \qquad (a \in \mathcal{A}),$$

this does not affect clauses (i) and (ii) of Theorem 2.11. Therefore, conditions (i) and (ii) of the current theorem are satisfied.

The converse is similar. ∎

THEOREM 3.2. *The dual Banach sequence algebra ℓ^1 is not w^*-approximately Connes amenable.*

Proof. Following the proof of [4, Theorem 4.1], we may choose two suitable elements $a, b \in \ell^1$ and a certain $\varepsilon > 0$ such that there is no element $F \in c_{00} \otimes c_{00}$ satisfying both the following inequalities:

(1) $\|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon$;
(2) $\|a - a\pi(F)\| + \|b - b\pi(F)\| < \varepsilon$.

We say that if (1) is not true, then there are elements ϕ_1 and ϕ_2 in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$ such that the inequality

$$|\langle \phi_1, \Delta_a(F)\rangle| + |\langle \phi_2, \Delta_b(F)\rangle| < \varepsilon$$

does not hold. Similarly, whenever (2) is not true, there are elements ψ_1 and ψ_2 in $\sigma wc(\mathcal{A}^*)$ for which the inequality

$$|\langle \psi_1, a - a\pi(F)\rangle| + |\langle \psi_2, b - b\pi(F)\rangle| < \varepsilon$$

does not hold.

All in all, by Theorem 3.1, we conclude that ℓ^1 is not w^*-approximately Connes amenable. ∎

We conclude by looking at a weighted variant of the ℓ^1 algebra. Let $\omega : \mathbb{N} \to [1, \infty)$ be any function. We consider the spaces $c_0(1/\omega)$ and $\ell^1(\omega)$ in the standard way (see for instance [1, 2]). It is known that $\ell^1(\omega)$ is a dual Banach algebra under pointwise operations with predual $c_0(1/\omega)$.

THEOREM 3.3. *The dual Banach sequence algebra $\ell^1(\omega)$ is not w^*- approximately Connes amenable for any weight ω.*

Proof. Using the argument of [4, Theorem 4.2], the proof is similar to that of Theorem 3.2. ∎

4. Pseudo-Connes amenable dual Banach algebras. Let $\mathcal{A}$ be a dual Banach algebra. We notice that $\mathcal{A} \hat{\otimes} \mathcal{A}$ is canonically mapped into $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$. Therefore there is an inherited w^*-topology on $\mathcal{A} \hat{\otimes} \mathcal{A}$ from $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$. Therefore we can always speak of the w^*-topology on $\mathcal{A} \hat{\otimes} \mathcal{A}$ without ambiguity. This is the motivation of the basic definition for the current section.

DEFINITION 4.1. Suppose that $\mathcal{A}$ is a dual Banach algebra. A net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is an *approximate σwc-diagonal* for $\mathcal{A}$ if for every $a\in\mathcal{A}$,

(i) $a\,.\,m_\alpha - m_\alpha\,.\,a \xrightarrow{w^*} 0$ in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, and
(ii) $a\pi_{\sigma wc}(m_\alpha) \xrightarrow{w^*} a$ in $\mathcal{A}$.

The following is a characterization of Connes amenability in terms of virtual and approximate diagonals.

PROPOSITION 4.2. *Suppose that $\mathcal{A}$ is a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is Connes amenable.*
(ii) *There exists a σwc-virtual diagonal for $\mathcal{A}$.*
(iii) *There exists a bounded approximate σwc-diagonal for $\mathcal{A}$.*

Proof. The equivalence of (i) and (ii) is just [15, Theorem 4.8].

(ii)⇒(iii): Let M be a σwc-virtual diagonal for $\mathcal{A}$. Since $\mathcal{A}\hat{\otimes}\mathcal{A}$ is w^*-dense in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, there is a bounded net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ which tends to M in the w^*-topology. It is easy to check that (m_α) is an approximate σwc-diagonal for $\mathcal{A}$.

(iii)⇒(ii): Let $M\in\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ be a w^*-accumulation point of the given bounded approximate σwc-diagonal (m_α) for $\mathcal{A}$. We may assume that $M = w^*\text{-}\lim_\alpha m_\alpha$. Then it is readily seen that M is a σwc-virtual diagonal for $\mathcal{A}$. ■

DEFINITION 4.3. A dual Banach algebra $\mathcal{A}$ is *pseudo-Connes amenable* if there exists an approximate σwc-diagonal for $\mathcal{A}$.

As a consequence of Proposition 4.2, we see that Connes amenability implies pseudo-Connes amenability. It is obvious that every pseudo-Connes amenable dual Banach algebra has a w^*-approximate identity. For dual Banach algebras, clearly pseudo-amenability implies pseudo-Connes amenability.

Let E and F be Banach spaces and let $\theta : E\to F$ be a linear map. We write $\theta\otimes\theta$ for the linear map from $E\hat{\otimes}E$ into $F\hat{\otimes}F$ given by $x\otimes y\mapsto \theta(x)\otimes\theta(y)$ for $x\in E$ and $y\in F$.

LEMMA 4.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras, and that $\theta:\mathcal{A}\to\mathcal{B}$ is a continuous epimorphism which is also w^*-continuous. Then $(\theta\otimes\theta)^*(\sigma wc((\mathcal{B}\hat{\otimes}\mathcal{B})^*))\subseteq\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$.*

Proof. This is straightforward: just notice that for every $a\in\mathcal{A}$ and $\phi\in(\mathcal{B}\hat{\otimes}\mathcal{B})^*$,

$$a\,.\,(\theta\otimes\theta)^*(\phi) = (\theta\otimes\theta)^*(\theta(a)\,.\,\phi),\qquad (\theta\otimes\theta)^*(\phi)\,.\,a = (\theta\otimes\theta)^*(\phi\,.\,\theta(a)).\ ■$$

PROPOSITION 4.5. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ dual Banach algebra, and $\theta:\mathcal{A}\to\mathcal{B}$ a continuous epimorphism. Then:*

(i) *If $\mathcal{A}$ is pseudo-amenable, then $\mathcal{B}$ is pseudo-Connes amenable.*
(ii) *If $\mathcal{A}$ is dual and pseudo-Connes amenable and if θ is w^*-continuous, then $\mathcal{B}$ is pseudo-Connes amenable.*

Proof. The cases are completely analogous and we shall only prove (ii). Let $(m_\alpha)_\alpha \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ be an approximate σwc-diagonal for $\mathcal{A}$. We will show that $(\theta \otimes \theta)(m_\alpha) \subseteq \mathcal{B} \hat{\otimes} \mathcal{B}$ is an approximate σwc-diagonal for $\mathcal{B}$. Let $b \in \mathcal{B}$, and take $a \in \mathcal{A}$ such that $b = \theta(a)$. It is readily seen that

$$b \,.\, (\theta \otimes \theta)(m_\alpha) = (\theta \otimes \theta)(a \,.\, m_\alpha) \quad \text{and} \quad (\theta \otimes \theta)(m_\alpha) \,.\, b = (\theta \otimes \theta)(m_\alpha \,.\, a).$$

For every $\phi \in \sigma wc((\mathcal{B} \hat{\otimes} \mathcal{B})^*)$, using Lemma 4.4, we have

$$\begin{aligned} \langle \phi, b \,.\, (\theta \otimes \theta)(m_\alpha) - (\theta \otimes \theta)(m_\alpha) \,.\, b \rangle &= \langle \phi, (\theta \otimes \theta)(a \,.\, m_\alpha - m_\alpha \,.\, a) \rangle \\ &= \langle (\theta \otimes \theta)^*(\phi), a \,.\, m_\alpha - m_\alpha \,.\, a \rangle \\ &\to 0, \end{aligned}$$

so that $w^*\text{-}\lim_\alpha (b \,.\, (\theta \otimes \theta)(m_\alpha) - (\theta \otimes \theta)(m_\alpha) \,.\, b) = 0$ in $\sigma wc((\mathcal{B} \hat{\otimes} \mathcal{B})^*)^*$.

Because $\pi_{\sigma wc}((\theta \otimes \theta)(m_\alpha)) = \theta(\pi_{\sigma wc}(m_\alpha))$, and by the w^*-continuity of θ, we observe that

$$b\pi_{\sigma wc}((\theta \otimes \theta)(m_\alpha)) = \theta(a)\theta(\pi_{\sigma wc}(m_\alpha)) = \theta(a\pi_{\sigma wc}(m_\alpha)) \xrightarrow{w^*} \theta(a) = b,$$

as required. ∎

Suppose that $\mathcal{A} = (\mathcal{A}_*)^*$ and $\mathcal{B} = (\mathcal{B}_*)^*$ are dual Banach algebras. We consider the ℓ^1-direct sum $\mathcal{A} \oplus^1 \mathcal{B}$ with norm $\|a + b\| = \|a\| + \|b\|$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This is a dual Banach algebra under pointwise-defined operations and with predual the ℓ^∞-direct sum $\mathcal{A}_* \oplus^\infty \mathcal{B}_*$, where the norm $\|\cdot\|_\infty$ is defined through $\|\phi + \psi\|_\infty = \max(\|\phi\|, \|\psi\|)$ for $\phi \in \mathcal{A}_*$ and $\psi \in \mathcal{B}_*$. It is known that the duality is given by

$$\langle \phi + \psi, a + b \rangle = \langle \phi, a \rangle + \langle \psi, b \rangle \quad (a \in \mathcal{A},\ b \in \mathcal{B},\ \phi \in \mathcal{A}_*,\ \psi \in \mathcal{B}_*).$$

THEOREM 4.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are pseudo-Connes amenable dual Banach algebras. Then so is $\mathcal{A} \oplus^1 \mathcal{B}$.*

Proof. First note that we can naturally embed $\mathcal{A} \hat{\otimes} \mathcal{A}$ into $E = (\mathcal{A} \oplus^1 \mathcal{B}) \hat{\otimes} (\mathcal{A} \oplus^1 \mathcal{B})$. Then we may regard each $m \in \mathcal{A} \hat{\otimes} \mathcal{A}$ as an element in E, so that

$$(a + b) \,.\, m = a \,.\, m, \qquad m \,.\, (a + b) = m \,.\, a, \qquad \pi_{\mathcal{A} \oplus^1 \mathcal{B}}(m) = \pi_{\mathcal{A}}(m),$$

for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where π_X denotes the diagonal operator for X. The same argument works for each $u \in \mathcal{B} \hat{\otimes} \mathcal{B}$. Regarding $m + u$ as an element of E, we have

$$(a+b).(m+u) = a.m+b.u, \qquad (m+u).(a+b) = m.a+u.b \qquad (a \in \mathcal{A},\ b \in \mathcal{B}),$$

and $\pi_{\mathcal{A} \oplus^1 \mathcal{B}}(m + u) = \pi_{\mathcal{A}}(m) + \pi_{\mathcal{B}}(u)$.

Next, using Lemma 2.6, we have the decomposition

$$\sigma wc(E^*) = \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*) \oplus \sigma wc((\mathcal{B} \hat{\otimes} \mathcal{B})^*) \oplus R$$

for some Banach space R. Therefore

$$\sigma wc((E^*))^* = \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^* \oplus \sigma wc((\mathcal{B} \hat{\otimes} \mathcal{B})^*)^* \oplus R^*.$$

Now, let $(m_\alpha)_\alpha \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ and $(u_\beta)_\beta \subseteq \mathcal{B} \hat{\otimes} \mathcal{B}$ be approximate σwc-diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively. Given $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{T} \subseteq \mathcal{B}$, $\mathcal{K} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{B} \hat{\otimes} \mathcal{B})^*)$, we can choose $\alpha = \alpha(\varepsilon, \mathcal{S}, \mathcal{K})$ and $\beta = \beta(\varepsilon, \mathcal{T}, \mathcal{H})$ such that

$$|\langle \phi, a \,.\, m_\alpha - m_\alpha \,.\, a\rangle| < \varepsilon/2, \quad |\langle \phi, \pi_{\mathcal{A}}(m_\alpha)a - a\rangle| < \varepsilon/2 \quad (\phi \in \mathcal{K},\, a \in \mathcal{S}),$$

and

$$|\langle \psi, b \,.\, u_\beta - u_\beta \,.\, b\rangle| < \varepsilon/2, \quad |\langle \psi, \pi_{\mathcal{B}}(u_\beta)b - b\rangle| < \varepsilon/2 \quad (\psi \in \mathcal{H},\, b \in \mathcal{T}).$$

Then we see that

$$\big|\big\langle \phi + \psi, (a+b) \,.\, (m_\alpha + u_\beta) - (m_\alpha + u_\beta) \,.\, (a+b)\big\rangle\big| < \varepsilon$$

and

$$\big|\big\langle \phi + \psi, \pi_{\mathcal{A}\oplus^1\mathcal{B}}(m_\alpha + u_\beta)(a+b) - (a+b)\big\rangle\big| < \varepsilon.$$

This shows that $\mathcal{A} \oplus^1 \mathcal{B}$ has an approximate σwc-diagonal, and hence is pseudo-Connes amenable. ∎

To end this section we compare pseudo-Connes amenability of $\mathcal{A}$ to that of $\mathcal{A}^\sharp$, for a dual Banach algebra $\mathcal{A}$. Using the decompositions given in the proof of Theorem 2.7, it is not hard to see that the latter implies the former. The converse, however, is not true. For instance, the Banach sequence algebra ℓ^1 is pseudo-Connes amenable by Example 6.2 below. If its unitization $(\ell^1)^\sharp$ is pseudo-Connes amenable, then ℓ^1 would be w^*-approximately Connes amenable by Theorem 5.3 below and Proposition 2.3, which contradicts Theorem 3.2.

5. The relations. In this section we are concerned with relations between pseudo-Connes amenability and w^*-approximate Connes amenability.

THEOREM 5.1. *Suppose that $\mathcal{A}$ is a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is w^*-approximately Connes amenable.*
(ii) *$\mathcal{A}^\sharp$ is pseudo-Connes amenable.*

Proof. Let $\mathcal{A}$ be w^*-approximately Connes amenable. By Proposition 2.5, there is a net $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that for each $a \in \mathcal{A}$ and each α, $a.M_\alpha - M_\alpha.a \to 0$ and $\pi_{\sigma wc}(M_\alpha) \to e$ in the w^*-topology of $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ and $\mathcal{A}^\sharp$, respectively. Take $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}^\sharp$, $\mathcal{K} \subseteq (\mathcal{A}^\sharp)_*$ and

$\mathcal{H} \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)$. Then there is α for which

$$|\langle f, a \cdot M_\alpha - M_\alpha \cdot a\rangle| = |\langle a \cdot f - f \cdot a, M_\alpha\rangle| < \varepsilon$$

and $|\langle \phi, \pi_{\sigma wc}(M_\alpha) - e\rangle| < \varepsilon$ for all $a \in \mathcal{S}$, $\phi \in \mathcal{K}$ and $f \in \mathcal{H}$.

Hence, by w^*-density of $\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ in $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ and w^*-continuity of $\pi_{\sigma wc}$, there is $m \in \mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ such that

$$|\langle f, a \cdot m - m \cdot a\rangle| = |\langle a \cdot f - f \cdot a, m\rangle| < \varepsilon,$$

and $|\langle \phi, \pi(m) - e\rangle| < \varepsilon$ for all $a \in \mathcal{S}$, $\phi \in \mathcal{K}$ and $f \in \mathcal{H}$.

Therefore, there is a net (m_i) in $\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ such that for every $a \in \mathcal{A}$ and every i, $a \cdot m_i - m_i \cdot a \to 0$ and $\pi(m_i) \to e$ in the w^*-topology of $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ and $\mathcal{A}^\sharp$, respectively. Hence $\mathcal{A}^\sharp$ is pseudo-Connes amenable, as required.

The converse is immediate by Proposition 2.5. ∎

LEMMA 5.2. *Suppose that $\mathcal{A}$ is a dual Banach algebra with identity e, and a, a_α are elements of $\mathcal{A}$ for all α. Then:*

(i) *If $a_\alpha \xrightarrow{w^*} a$ in $\mathcal{A}$ and if $b \in \mathcal{A}$, then $a_\alpha \otimes b \xrightarrow{w^*} a \otimes b$ and $b \otimes a_\alpha \xrightarrow{w^*} b \otimes a$ in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$.*

(ii) *If $a_\alpha \otimes e \xrightarrow{w^*} a \otimes e$ or $e \otimes a_\alpha \xrightarrow{w^*} e \otimes a$ in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$, then $a_\alpha \xrightarrow{w^*} a$ in $\mathcal{A}$.*

Proof. (i): Let $a_\alpha \xrightarrow{w^*} a$ in $\mathcal{A}$, $b \in \mathcal{A}$ and let $\phi \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. Then $\phi \cdot a_\alpha \xrightarrow{w} \phi \cdot a$ and $a_\alpha \cdot \phi \xrightarrow{w} a \cdot \phi$ in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. In particular, $(\phi \cdot a_\alpha)(e \otimes b) \to (\phi \cdot a)(e \otimes b)$ and $(a_\alpha \cdot \phi)(b \otimes e) \to (a \cdot \phi)(b \otimes e)$. Therefore, $\phi(a_\alpha \otimes b) \to \phi(a \otimes b)$ and $\phi(b \otimes a_\alpha) \to \phi(b \otimes a)$, as required.

(ii): We shall only prove the statement in the case $a_\alpha \otimes e \xrightarrow{w^*} a \otimes e$ in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$. For each $\phi \in \mathcal{A}_*$, we know that $\pi^*(\phi) \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$. Therefore,

$$\begin{aligned}\langle a, \phi\rangle &= \langle ae, \phi\rangle = \langle \pi(a \otimes e), \phi\rangle = \langle a \otimes e, \pi^*(\phi)\rangle \\ &= \lim_\alpha \langle a_\alpha \otimes e, \pi^*(\phi)\rangle = \lim_\alpha \langle \pi(a_\alpha \otimes e), \phi\rangle \\ &= \lim_\alpha \langle a_\alpha e, \phi\rangle = \lim_\alpha \langle a_\alpha, \phi\rangle. \ \blacksquare\end{aligned}$$

Now we examine pseudo-Connes amenability of a dual Banach algebra $\mathcal{A}$ with identity.

THEOREM 5.3. *Suppose that $\mathcal{A}$ is a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ has an approximate σwc-diagonal $(m_\alpha)_\alpha$ such that $\pi_{\sigma wc}(m_\alpha)$ is the identity of $\mathcal{A}$, for each α.*

(ii) *$\mathcal{A}$ is pseudo-Connes amenable and has an identity.*

(iii) *$\mathcal{A}$ is w^*-approximately Connes amenable and has an identity.*

Proof. (i)⇒(ii): This is immediate.

(ii)⇒(iii): Let $E = (E_*)^*$ be a normal, dual Banach $\mathcal{A}$-bimodule which we may suppose without loss of generality to be unital, and let $D : \mathcal{A} \to E$ be a w^*-continuous derivation. Let $(m_\alpha)_\alpha \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ be an approximate σwc-diagonal for $\mathcal{A}$. Define a bounded linear map

$$\theta : \mathcal{A} \hat{\otimes} \mathcal{A} \to E, \qquad a \otimes b \mapsto a \,.\, Db.$$

Then for every $a \in \mathcal{A}$,

$$a \,.\, \theta(m_\alpha) = \theta(a \,.\, m_\alpha - m_\alpha \,.\, a) + \pi_{\sigma wc}(m_\alpha) \,.\, Da + \theta(m_\alpha) \,.\, a.$$

Letting $x_\alpha := \theta(m_\alpha)$, for every $a \in \mathcal{A}$ we have

$$\pi_{\sigma wc}(m_\alpha) \,.\, Da = (a \,.\, x_\alpha - x_\alpha \,.\, a) - \theta(a \,.\, m_\alpha - m_\alpha \,.\, a).$$

Since E is unital and normal, $Da = w^*\text{-}\lim_\alpha \pi_{\sigma wc}(m_\alpha) \,.\, Da$ for all $a \in \mathcal{A}$. On the other hand, $\theta^*(E_*) \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$ by [15, Lemma 4.9]. Therefore, for every $\psi \in E_*$ and $a \in \mathcal{A}$,

$$\langle \psi, \theta(a \,.\, m_\alpha - m_\alpha \,.\, a)\rangle = \langle \theta^*(\psi), a \,.\, m_\alpha - m_\alpha \,.\, a\rangle \to 0,$$

so that $w^*\text{-}\lim_\alpha \theta(a \,.\, m_\alpha - m_\alpha \,.\, a) = 0$.

All in all, $Da = w^*\text{-}\lim_\alpha (a \,.\, x_\alpha - x_\alpha \,.\, a)$ for each $a \in \mathcal{A}$, as required.

(iii)⇒(i): Let e and u be the identities of $\mathcal{A}$ and $\mathcal{A}^\sharp$, respectively. By Theorem 5.1, there is an approximate σwc-diagonal $(M_\alpha)_\alpha \subseteq \mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ for $\mathcal{A}^\sharp$. Replacing M_α with $M'_\alpha := M_\alpha - \pi_{\sigma wc}(M_\alpha) \otimes u + u \otimes u$ and using Lemma 5.2(i), if necessary, we can assume that $\pi_{\sigma wc}(M_\alpha) = u$ for all α. We write $M_\alpha = t_\alpha + F_\alpha \otimes u + u \otimes G_\alpha + c_\alpha u \otimes u$ where $t_\alpha \in \mathcal{A} \hat{\otimes} \mathcal{A}$, F_α, $G_\alpha \in \mathcal{A}$ and c_α is a constant. Applying $\pi_{\sigma wc}$, it is readily seen that $c_\alpha = 1$ and that $\pi_{\sigma wc}(t_\alpha) + F_\alpha + G_\alpha = 0$ for all α. For $a \in \mathcal{A}$, by Lemma 5.2(ii),

$$a \,.\, t_\alpha - t_\alpha \,.\, a - F_\alpha \otimes a + a \otimes G_\alpha \xrightarrow{w^*} 0, \qquad G_\alpha a \xrightarrow{w^*} -a, \qquad aF_\alpha \xrightarrow{w^*} -a.$$

Put $m_\alpha := t_\alpha + F_\alpha \otimes e + e \otimes G_\alpha + e \otimes e$. Then by the above observation, the net $(m_\alpha)_\alpha$ is an approximate σwc-diagonal for $\mathcal{A}$ and $\pi_{\sigma wc}(m_\alpha) = e$ for all α. ∎

We recall that an approximate identity $(e_\alpha)_\alpha$ for a Banach algebra $\mathcal{A}$ is *central* if $(e_\alpha)_\alpha \subseteq \mathcal{Z}(\mathcal{A})$, where $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba \text{ for all } b \in \mathcal{A}\}$ is the *centre* of $\mathcal{A}$ (see for instance [1, Definition 2.9.1]). A w^*-approximate identity for a dual Banach algebra which is also central is called a *central w^*-approximate identity.*

We do not know in general whether a w^*-approximately Connes amenable dual Banach algebra is pseudo-Connes amenable. The following shows that this is true under an additional assumption.

PROPOSITION 5.4. *Suppose that $\mathcal{A}$ is a dual Banach algebra having a central w^*-approximate identity. If $\mathcal{A}$ is w^*-approximately Connes amenable, then it is pseudo-Connes amenable.*

Proof. Let $(e_\alpha)_\alpha$ be a central w^*-approximate identity for $\mathcal{A}$. Given $\varepsilon > 0$ and finite subsets $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{K} \subseteq \mathcal{A}_*$ and $\mathcal{H} \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, we choose α_1 and α_2 such that

$$|\langle \phi, e_{\alpha_1} a - a\rangle| < \varepsilon/2, \quad |\langle \phi, e_{\alpha_2} e_{\alpha_1} a - e_{\alpha_1} a\rangle| < \varepsilon/2,$$

for $a \in \mathcal{S}$ and $\phi \in \mathcal{K}$. Define a bounded derivation $D : \mathcal{A} \to \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ by

$$Da := ae_{\alpha_1} \otimes e_{\alpha_2} - e_{\alpha_1} \otimes e_{\alpha_2} a = a \,.\, (e_{\alpha_1} \otimes e_{\alpha_2}) - (e_{\alpha_1} \otimes e_{\alpha_2}) \,.\, a.$$

Since $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ is normal, D is w^*-continuous. Clearly D attains its values in the w^*-closed submodule $\ker \pi_{\sigma wc}$. Hence, there is an element $u = u(\varepsilon, \mathcal{S}, \mathcal{K}, \mathcal{H}, \alpha_1, \alpha_2)$ in $\ker \pi_{\sigma wc}$ for which

$$|\langle f, Da - (a \,.\, u - u \,.\, a)\rangle| < \varepsilon \quad (a \in \mathcal{S},\ f \in \mathcal{H}).$$

Let $M := e_{\alpha_1} \otimes e_{\alpha_2} - u$. Then $M \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and

$$|\langle f, a \,.\, M - M \,.\, a\rangle| < \varepsilon, \quad |\langle \phi, \pi_{\sigma wc}(M) a - a\rangle| < \varepsilon \quad (a \in \mathcal{S},\ \phi \in \mathcal{K},\ f \in \mathcal{H}).$$

This shows that there is a net $(M_i)_i \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ such that $a \,.\, M_i - M_i \,.\, a \xrightarrow{w^*} 0$ and $\pi_{\sigma wc}(M_i) a \xrightarrow{w^*} a$ in $\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $\mathcal{A}$, respectively.

Now, an argument similar to that for Theorem 5.1 (with $\mathcal{A}$ in place of $\mathcal{A}^\sharp$) shows that $\mathcal{A}$ has an approximate σwc-diagonal. ∎

COROLLARY 5.5. *Any w^*-approximately Connes amenable, commutative dual Banach algebra is pseudo-Connes amenable.*

PROPOSITION 5.6. *Suppose that $\mathcal{A}$ is a pseudo-Connes amenable dual Banach algebra and that E is a normal, dual Banach $\mathcal{A}$-bimodule such that each w^*-approximate identity of $\mathcal{A}$ is also a one-sided* (*left or right*) *w^*-approximate identity for E. Then every w^*-continuous derivation $D : \mathcal{A} \to E$ is w^*-approximately inner.*

Proof. Let $(m_\alpha)_\alpha \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ be an approximate σwc-diagonal for $\mathcal{A}$ and without loss of generality, let $(\pi_{\sigma wc}(m_\alpha))_\alpha$ be a left w^*-approximate identity for E. Consider the map θ given in the proof of Theorem 5.3, so that $\theta(a \,.\, m_\alpha - m_\alpha \,.\, a) \xrightarrow{w^*} 0$. Putting $x_\alpha := \theta(m_\alpha) \in E$, we obtain

$$a \,.\, x_\alpha - x_\alpha \,.\, a - \pi_{\sigma wc}(m_\alpha) \,.\, Da \xrightarrow{w^*} 0 \quad (a \in \mathcal{A}).$$

Since $\pi_{\sigma wc}(m_\alpha) \,.\, Da \xrightarrow{w^*} Da$, $Da = w^*\text{-}\lim_\alpha (a \,.\, x_\alpha - x_\alpha \,.\, a)$, as required. ∎

To state the last result of this section, we first recall some terminology from [12]. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. The normed space of all inner derivations from $\mathcal{A}$ to E is denoted by $B^1(\mathcal{A}, E)$. Furthermore, if $\mathcal{A}$ is a dual Banach algebra and if E is a normal, dual Banach $\mathcal{A}$-bimodule, we write $Z^1_{w^*}(\mathcal{A}, E)$ for the set of all w^*-continuous derivations from $\mathcal{A}$ into E. It is known that $B^1(\mathcal{A}, E) \subseteq Z^1_{w^*}(\mathcal{A}, E)$, so that we can define $H^1_{w^*}(\mathcal{A}, \mathcal{A}) = Z^1_{w^*}(\mathcal{A}, E)/B^1(\mathcal{A}, E)$.

COROLLARY 5.7. *Suppose that $\mathcal{A}$ is a pseudo-Connes amenable dual Banach algebra. Then any w^*-continuous derivation from $\mathcal{A}$ into $\mathcal{A}$ is w^*-approximately inner. In particular, if $\mathcal{A}$ is commutative then $Z^1_{w^*}(\mathcal{A}, E) = \{0\}$ so that $H^1_{w^*}(\mathcal{A}, \mathcal{A}) = \{0\}$.*

6. Examples. In this section, we present some examples of pseudo-Connes amenable dual Banach algebras which are not Connes amenable. Moreover, we look at pseudo-amenability and w^*-approximate Connes amenability for these algebras.

EXAMPLE 6.1. Let $\mathbb{N}_\vee$ be the set $\mathbb{N}$ of natural numbers with the product $m \vee n := \max\{m, n\}$. It is known that $\mathbb{N}_\vee$ is a commutative, unital, weakly cancellative semigroup, that is, for every $s, t \in \mathbb{N}_\vee$ the set $\{x \in \mathbb{N}_\vee : sx = t\}$ is finite. By [3, Theorem 4.6], $\ell^1(\mathbb{N}_\vee)$ is a dual Banach algebra with predual $c_0(\mathbb{N}_\vee)$. By [8, Example 4.6] and [9, Proposition 3.2], $\ell^1(\mathbb{N}_\vee)$ is pseudo-amenable and whence is pseudo-Connes amenable. Thus it is w^*-approximately Connes amenable, by Theorem 5.3. However, it is not Connes amenable [5, Theorem 5.13].

EXAMPLE 6.2. Define $m_n = \sum_{i=1}^{n} \delta_i \otimes \delta_i \in \ell^1 \hat{\otimes} \ell^1$ for every $n \in \mathbb{N}$. It is readily seen that (m_n) is an approximate σwc-diagonal for ℓ^1 and hence ℓ^1 is pseudo-Connes amenable. It fails to be Connes amenable because of lack of the identity. Notice that ℓ^1 is pseudo-amenable, indeed it is pseudo-contractible in the sense of [9]. As shown earlier (Theorem 3.2), ℓ^1 is not w^*-approximately Connes amenable.

EXAMPLE 6.3. Let G be an amenable, non-discrete locally compact group. It is known that $M(G)$, the measure algebra of G, is Connes amenable [14]. Set $\mathcal{A} := \ell^1(\mathbb{N}_\vee) \oplus^1 M(G)$. By Example 6.1 and Theorem 4.6, $\mathcal{A}$ is pseudo-Connes amenable. If $\mathcal{A}$ were Connes amenable, then so would be its image under the w^*-continuous natural epimorphism $\mathcal{A} \to \ell^1(\mathbb{N}_\vee)$, which would contradict Example 6.1. Analogously, if $\mathcal{A}$ were pseudo-amenable, then so would be its homomorphic image $M(G)$ under the natural epimorphism $\mathcal{A} \to M(G)$, which is not the case by [9, Proposition 4.2]. By [6, Example 2.1(ii)], $\mathcal{A}$ is approximately Connes amenable so that it is w^*-approximately Connes amenable.

EXAMPLE 6.4. We now give an example of a dual Banach algebra which is pseudo-Connes amenable but is neither Connes amenable, w^*-approximately Connes amenable nor pseudo-amenable. Let G be an amenable, non-discrete locally compact group and let $\mathcal{A} := \ell^1 \oplus^1 M(G)$. The argument of the preceding example (with Example 6.2 in place of Example 6.1) suffices to show that $\mathcal{A}$ is pseudo-Connes amenable but is neither Connes amenable nor pseudo-amenable. The same argument, using Theorem 3.2 and Proposition 2.4, shows that $\mathcal{A}$ is not w^*-approximately Connes amenable.

References

[1] H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.

[2] H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. 177 (2005), no. 836.

[3] H. G. Dales, A. T.-M. Lau and D. Strauss, *Banach algebras on semigroups and on their compactifications*, Mem. Amer. Math. Soc. 205 (2010), no. 966.

[4] H. G. Dales, R. J. Loy and Y. Zang, *Approximate amenability for Banach sequence algebras*, Studia Math. 177 (2006), 81–96.

[5] M. Daws, *Connes-amenability of bidual and weighted semigroup algebras*, Math. Scand. 99 (2006), 217–246.

[6] G. H. Esslamzadeh, B. Shojaee and A. Mahmoodi, *Approximate Connes-amenability of dual Banach algebras*, Bull. Belgian Math. Soc. Simon Stevin. 19 (2012), 193–213.

[7] F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal. 208 (2004), 229–260.

[8] F. Ghahramani, R. J. Loy and Y. Zang, *Generalized notions of amenability, II*, J. Funct. Anal. 254 (2008), 1776–1810.

[9] F. Ghahramani and Y. Zang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc. 142 (2007), 111–123.

[10] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).

[11] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. 94 (1972), 685–698.

[12] V. Runde, *Amenability for dual Banach algebras*, Studia Math. 148 (2001), 47–66.

[13] V. Runde, *Lectures on Amenability*, Lecture Notes in Math. 1774, Springer, Berlin, 2002.

[14] V. Runde, *Connes-amenability and normal, virtual diagonals for measure algebras. I*, J. London Math. Soc. 67 (2003), 643–656.

[15] V. Runde, *Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule*, Math. Scand. 95 (2004), 124–144.

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