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Sufficient conditions for the spectrality of self-affine measures with prime determinant

by

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Abstract. Let $\mu_{M,D}$ be a self-affine measure associated with an expanding matrix M and a finite digit set D. We study the spectrality of $\mu_{M,D}$ when $|\det(M)| = |D| = p$ is a prime. We obtain several new sufficient conditions on M and D for $\mu_{M,D}$ to be a spectral measure with lattice spectrum. As an application, we present some properties of the digit sets of integral self-affine tiles, which are connected with a conjecture of Lagarias and Wang.

1. Introduction. Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, and $D \subset \mathbb{Z}^n$ be a finite digit set of cardinality $|D|$. In the case when $|\det(M)| = |D| = p$ is a prime, we investigate the spectrality of the self-affine measure $\mu_{M,D}$ as well as its application to the tile digit set D. It is known that the *self-affine measure* $\mu := \mu_{M,D}$ is the unique probability measure satisfying

$$\mu = \frac{1}{|D|} \sum_{d\in D} \mu \circ \phi_d^{-1},$$

and is supported on the compact set $T \subset \mathbb{R}^n$, where $T := T(M,D)$ is the *attractor* (or *invariant set*) of the affine iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$. This is the unique compact set satisfying $T = \bigcup_{d\in D} \phi_d(T)$. The self-affine measure $\mu_{M,D}$ is called *spectral* if there exists a subset $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e_\lambda(x) = e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu_{M,D})$. The set Λ is then called a *spectrum* for $\mu_{M,D}$; we also say that $(\mu_{M,D}, \Lambda)$ is a *spectral pair*. A spectral measure is a natural generalization of the spectral set. The notion of spectral set was introduced by Fuglede [5], whose famous spectrum-tiling conjecture has attracted much interest, although no desired result in higher dimensions has been obtained. In most cases, it is difficult to establish the spectrum-tiling relation. So the spectrality of the self-affine measure $\mu_{M,D}$ becomes of considerable interest (see [2], [3], [4], [15] and the references cited therein).

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Here we consider the following question: Under what conditions is $\mu_{M,D}$ a spectral measure?

The spectrality of $\mu_{M,D}$ is directly connected with the Fourier transform

$$\hat{\mu}_{M,D}(\xi) := \int e^{2\pi i\langle x,\xi\rangle}\, d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi) \qquad (\xi \in \mathbb{R}^n)$$

and its zero set $Z(\hat{\mu}_{M,D}) = \{\xi \in \mathbb{R}^n : \hat{\mu}_{M,D}(\xi) = 0\}$, where

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle} \qquad (x \in \mathbb{R}^n).$$

Let $\Theta_0 = \{x \in \mathbb{R}^n : m_D(x) = 0\}$. Then $Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}\Theta_0$, where M^* denotes the transposed conjugate matrix of M, in fact $M^* = M^t$.

When dealing with the spectrality of a self-affine measure, the notion of compatible pair plays an important role. Dutkay and Jorgensen [2, Conjecture 2.5], [4, Conjecture 1.1] (see also [3, Problem 1]) conjectured that for an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$ with $0 \in D$, if there exists a subset $S \subset \mathbb{Z}^n$ with $0 \in S$ such that $(M^{-1}D, S)$ is a compatible pair (or (M, D, S) is a Hadamard triple), then $\mu_{M,D}$ is a spectral measure. This conjecture is proved in some special cases. It should be pointed out that in higher dimensions ($n \geq 2$), there are many spectral measures that cannot be obtained from a compatible pair. Besides the condition of compatible pair, there are a few other conditions guaranteeing that $\mu_{M,D}$ is a spectral measure. For example, in the special case when $|\det(M)| = |D| = p$ is a prime, the author [15] obtained the following conditions for $\mu_{M,D}$ to be a spectral measure with lattice spectrum.

THEOREM A. *Let $M \in M_n(\mathbb{Z})$ be an expanding matrix such that $|\det(M)| = p$ is a prime and one of the following three conditions holds:*

(a) $p\mathbb{Z}^n \not\subseteq M^2(\mathbb{Z}^n)$;
(b) $p(\mathbb{Z}^n \setminus M(\mathbb{Z}^n)) \subseteq M(\mathbb{Z}^n \setminus M(\mathbb{Z}^n))$;
(c) $p\mathbb{Z}^2 \neq M^2(\mathbb{Z}^2)$ *in the case when* $n = 2$.

Let $D \subset \mathbb{Z}^n$ be a finite digit set of cardinality $|D| = |\det(M)|$ with $0 \in D$. If $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n \neq \emptyset$ or if there are two points $s_1, s_2 \in \mathbb{R}^n$ with $s_1 - s_2 \in \mathbb{Z}^n$ such that the exponential functions $e_{s_1}(x), e_{s_2}(x)$ are orthogonal in $L^2(\mu_{M,D})$, then there exists $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that $D = M^r\tilde{D}$, where $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$, and hence $\mu_{M,D}$ is a spectral measure with lattice spectrum.

This gives several sufficient conditions, different from the condition of compatible pair, for a self-affine measure to be a spectral measure with lattice spectrum. In the present paper we further the above research by providing some new sufficient conditions for $\mu_{M,D}$ to be a spectral measure with lattice spectrum. This constitutes the content of Section 2. As an application, we obtain in Section 3 some properties of the digit set D of an integral self-affine tile $T(M,D)$ with prime determinant $\det(M)$. These properties are closely connected with a conjecture of Lagarias and Wang.

2. Sufficient conditions for spectrality. The first main result on the spectrality of $\mu_{M,D}$ with prime determinant $\det(M)$ is the following.

THEOREM 2.1. *Let $M\in M_n(\mathbb{Z})$ be an expanding matrix such that $|\det(M)| = p$ is a prime and one of the following three conditions holds:*

(d) $p\mathbb{Z}^n \not\subseteq M^{*2}(\mathbb{Z}^n)$;
(e) $p(\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)) \subseteq M^*(\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n))$;
(f) $p\mathbb{Z}^2 \neq M^{*2}(\mathbb{Z}^2)$ *in the case when* $n=2$.

If $D\subset\mathbb{Z}^n$ is a finite digit set of cardinality $|D| = |\det(M)|$ with $0\in D$ such that $Z(\hat{\mu}_{M,D})\cap\mathbb{Z}^n\neq\emptyset$, then there exists $r\in\mathbb{N}_0$ such that $D = M^r\tilde{D}$, where $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$, and hence $\mu_{M,D}$ is a spectral measure with lattice spectrum.

Note that for a non-singular matrix $M\in M_n(\mathbb{R})$, the condition $p\mathbb{Z}^n\subseteq M^2(\mathbb{Z}^n)$ is equivalent to $pM^{-2}(\mathbb{Z}^n)\subseteq\mathbb{Z}^n$, which implies $A := pM^{-2}\in M_n(\mathbb{Z})$, equivalently, $A^* = pM^{*-2}\in M_n(\mathbb{Z})$. So, $A^*(\mathbb{Z}^n)\subseteq\mathbb{Z}^n$ or $p\mathbb{Z}^n\subseteq M^{*2}(\mathbb{Z}^n)$. This shows that $p\mathbb{Z}^n\subseteq M^2(\mathbb{Z}^n)$ is equivalent to $p\mathbb{Z}^n\subseteq M^{*2}(\mathbb{Z}^n)$, that is, (d) is equivalent to (a). In the same way, $p\mathbb{Z}^n = M^2(\mathbb{Z}^n)$ is equivalent to $pM^{-2}(\mathbb{Z}^n)=\mathbb{Z}^n$, which implies $A := pM^{-2}\in M_n(\mathbb{Z})$ is a unimodular matrix (i.e. $A, A^{-1}\in M_n(\mathbb{Z})$), equivalently, $A^* = pM^{*-2}\in M_n(\mathbb{Z})$ is also a unimodular matrix. So, $A^*(\mathbb{Z}^n)=\mathbb{Z}^n$ or $p\mathbb{Z}^n = M^{*2}(\mathbb{Z}^n)$. This shows that $p\mathbb{Z}^n = M^2(\mathbb{Z}^n)$ and $p\mathbb{Z}^n = M^{*2}(\mathbb{Z}^n)$ are equivalent, that is, (f) and (c) are equivalent.

In general, $M^*(\mathbb{Z}^n)\neq M(\mathbb{Z}^n)$, and we cannot expect that the measures $\mu_{M,D}$ and $\mu_{M^*,D}$ have the same spectrality. There exists an expanding matrix $M\in M_n(\mathbb{Z})$ and a finite digit set $D\subset\mathbb{Z}^n$ such that $\mu_{M,D}$ is a spectral measure but $\mu_{M^*,D}$ is not (see [14, Remark 3.8(ii)]). Even so, we modify the method of [15] to give a complete proof of Theorem 2.1. Moreover, the method below leads to another new sufficient condition for $\mu_{M,D}$ to be a spectral measure with lattice spectrum.

Proof of Theorem 2.1. We first write $D = M^r\tilde{D}$ and $\tilde{D} = \{d_0 = 0, d_1,\dots,d_{p-1}\}\subset\mathbb{Z}^n$, where $\tilde{D}\not\subset M(\mathbb{Z}^n)$ and $r\geq 0$ is an integer (see [15, Lemma 1]). Let $l\in Z(\hat{\mu}_{M,D})\cap\mathbb{Z}^n$. From $\hat{\mu}_{M,D}(0)=1$, we have $l\in\mathbb{Z}^n\setminus\{0\}$. Since $M^*(\mathbb{Z}^n)\subset\mathbb{Z}^n$, we divide the proof into two cases: $l\notin M^*(\mathbb{Z}^n)$ and $l\in M^*(\mathbb{Z}^n)$.

CASE 1: $l \notin M^*(\mathbb{Z}^n)$. It follows from $l \in Z(\hat{\mu}_{M,D})$ that

$$0 = \hat{\mu}_{M,D}(l) = \prod_{j=1}^{\infty} m_D(M^{*-j}l) = \prod_{j=1}^{\infty} m_{\tilde{D}}(M^{*(r-j)}l) = \prod_{j=1}^{\infty} m_{\tilde{D}}(M^{*-j}l). \tag{2.1}$$

So, there exists a positive integer $k := k(l)$ such that $m_{\tilde{D}}(M^{*-k}l) = 0$. Let $M^\dagger = pM^{-1}$. Then $M^\dagger \in M_n(\mathbb{Z})$ and

$$\sum_{j=0}^{p-1} e^{2\pi i \langle (M^\dagger)^k d_j, l\rangle / p^k} = 0, \tag{2.2}$$

which yields the following relation:

$$\{0, \langle (M^\dagger)^k d_1, l\rangle, \ldots, \langle (M^\dagger)^k d_{p-1}, l\rangle\} \equiv \{0, p^{k-1}, 2p^{k-1}, \ldots, (p-1)p^{k-1}\} \pmod{p^k}. \tag{2.3}$$

See [15, Lemma 2]. Since $l \in \mathbb{Z}^n \backslash M^*(\mathbb{Z}^n)$, we have $(M^\dagger)^* l \in (M^\dagger)^*(\mathbb{Z}^n) \backslash p\mathbb{Z}^n$, hence there exists some $w \in \mathbb{Z}^n \setminus \{0\}$ such that

$$p \nmid \langle (M^\dagger)^* l, w\rangle. \tag{2.4}$$

For any integer h with $|h| = 1, \ldots, p-1$, (2.4) gives

$$p \nmid \langle (M^\dagger)^* hl, w\rangle \quad \text{and} \quad p \nmid \langle l, M^\dagger hw\rangle, \tag{2.5}$$

which yields $hl \notin M^*(\mathbb{Z}^n)$ and $hw \notin M(\mathbb{Z}^n)$. This shows that

$$\{0, l, 2l, \ldots, (p-1)l\}$$

is a complete set of coset representatives of $\mathbb{Z}^n / M^*(\mathbb{Z}^n)$, and

$$\{0, w, 2w, \ldots, (p-1)w\}$$

is a complete set of coset representatives of $\mathbb{Z}^n / M(\mathbb{Z}^n)$. So, each $\lambda \in \mathbb{Z}^n$ has a unique representation

$$\lambda = jl + M^*\beta \quad \text{for some } 0 \le j \le p-1 \text{ and } \beta \in \mathbb{Z}^n. \tag{2.6}$$

Also $(M^\dagger)^* l \in \mathbb{Z}^n$ has the form

$$(M^\dagger)^* l = j_0 l + M^*\gamma \quad \text{for some } 0 \le j_0 \le p-1 \text{ and } \gamma \in \mathbb{Z}^n. \tag{2.7}$$

CLAIM 1. *Each of the assumptions* (d)–(f) *guarantees that* $j_0 \neq 0$ *in* (2.7).

Proof of Claim 1. If $j_0 = 0$ in (2.7), then

$$(M^\dagger)^* l = M^*\gamma \quad \text{for some } \gamma \in \mathbb{Z}^n. \tag{2.8}$$

It follows from (2.6) and (2.8) that for each $\lambda \in \mathbb{Z}^n$,

$$(M^\dagger)^*\lambda = j(M^\dagger)^* l + p\beta = M^*(j\gamma + (M^\dagger)^*\beta) \in M^*(\mathbb{Z}^n).$$

This shows

$$(M^\dagger)^*(\mathbb{Z}^n) \subseteq M^*(\mathbb{Z}^n) \quad \text{or} \quad p\mathbb{Z}^n \subseteq M^{*2}(\mathbb{Z}^n), \tag{2.9}$$

so (d) does not hold.

The condition (e) is equivalent to

$$(M^\dagger)^*(\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)) \subseteq \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n).$$

From $l \in \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)$, we have $(M^\dagger)^* l \in \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)$. Since $\{0, l, 2l, \ldots, (p-1)l\}$ is a complete set of coset representatives of $\mathbb{Z}^n/M^*(\mathbb{Z}^n)$, we know that the p sets

$$M^*(\mathbb{Z}^n),\ l + M^*(\mathbb{Z}^n),\ 2l + M^*(\mathbb{Z}^n),\ \ldots,\ (p-1)l + M^*(\mathbb{Z}^n)$$

are mutually disjoint and

$$\mathbb{Z}^n = M^*(\mathbb{Z}^n) \cup (l + M^*(\mathbb{Z}^n)) \cup (2l + M^*(\mathbb{Z}^n)) \cup \cdots \cup ((p-1)l + M^*(\mathbb{Z}^n)).$$

Then

$$(M^\dagger)^* l \in \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n) = (l + M^*(\mathbb{Z}^n)) \cup (2l + M^*(\mathbb{Z}^n)) \cup \cdots \cup ((p-1)l + M^*(\mathbb{Z}^n)).$$

This guarantees that $j_0 \neq 0$ in (2.7).

For the condition (f), we know, from (2.4) and (2.8), that

$$p \nmid \langle M^*\gamma, w\rangle \quad \text{and} \quad p \nmid \langle M^* h\gamma, w\rangle, \tag{2.10}$$

which yields $h\gamma \notin (M^\dagger)^*(\mathbb{Z}^2)$ for any integer h with $|h| = 1, \ldots, p-1$. In the case when $n = 2$, this shows that $\{0, \gamma, 2\gamma, \ldots, (p-1)\gamma\}$ is a complete set of coset representatives of $\mathbb{Z}^2/(M^\dagger)^*(\mathbb{Z}^2)$. So, each $\tilde{\lambda} \in \mathbb{Z}^2$ has a unique representation

$$\tilde{\lambda} = \tilde{j}\gamma + (M^\dagger)^*\tilde{\beta} \quad \text{for some } 0 \leq \tilde{j} \leq p-1 \text{ and } \tilde{\beta} \in \mathbb{Z}^2. \tag{2.11}$$

Also $M^*\gamma \in \mathbb{Z}^2$ has the form

$$M^*\gamma = \tilde{j}_0\gamma + (M^\dagger)^*\tilde{\eta} \quad \text{for some } 0 \leq \tilde{j}_0 \leq p-1 \text{ and } \tilde{\eta} \in \mathbb{Z}^2. \tag{2.12}$$

(i) If $\tilde{j}_0 = 0$ in (2.12), we see from (2.11) and (2.12) that for each $\tilde{\lambda} \in \mathbb{Z}^2$,

$$M^*\tilde{\lambda} = \tilde{j}M^*\gamma + p\tilde{\beta} = (M^\dagger)^*(\tilde{j}\tilde{\eta} + M^*\tilde{\beta}) \in (M^\dagger)^*(\mathbb{Z}^2). \tag{2.13}$$

This shows

$$M^*(\mathbb{Z}^2) \subseteq (M^\dagger)^*(\mathbb{Z}^2) \quad \text{or} \quad M^{*2}(\mathbb{Z}^2) \subseteq p\mathbb{Z}^2. \tag{2.14}$$

From (2.9) and (2.14), we get $p\mathbb{Z}^2 = M^{*2}(\mathbb{Z}^2)$, a contradiction of (f).

(ii) If $\tilde{j}_0 \neq 0$ in (2.12), we see from (2.8) and (2.12) that

$$(M^\dagger)^* l = \tilde{j}_0\gamma + (M^\dagger)^*\tilde{\eta}. \tag{2.15}$$

If we multiply both sides of (2.15) by M^*, we obtain

$$pl = \tilde{j}_0 M^*\gamma + p\tilde{\eta}, \quad \text{where} \quad 0 < \tilde{j}_0 \leq p-1, \tag{2.16}$$

so, $p \mid M^*\gamma$, a contradiction of (2.10). This completes the proof of Claim 1. ■

From [15, Claim 2], we also have the following.

Claim 2. *There exists* $d_{q0} \in \tilde{D}$ $(1 \le q_0 \le p-1)$ *such that*

$$d_{q0} = j_{q0} w + M\beta_{q0} \quad \text{for some } 1 \le j_{q0} \le p-1 \text{ and } \beta_{q0} \in \mathbb{Z}^n. \tag{2.17}$$

Secondly, it follows from (2.7) and Claim 1 that for any positive integer $\sigma \in \mathbb{N}$,

$$(M^\dagger)^{*\sigma} l = (j_0)^\sigma l + M^* \gamma_\sigma \tag{2.18}$$

for some $0 < j_0 \le p-1$ and $\gamma_\sigma \in \mathbb{Z}^n$. Combining this with (2.17), we see that for any positive integer $\sigma \in \mathbb{N}$,

$$\begin{aligned} \langle (M^\dagger)^\sigma d_{q0}, l\rangle &= \langle d_{q0}, (M^\dagger)^{*\sigma} l\rangle \\ &= j_{q0} j_0^{\sigma-1} \langle w, (M^\dagger)^* l\rangle + j_0^{\sigma-1} p \langle \beta_{q0}, l\rangle + p\langle d_{q0}, \gamma_{\sigma-1}\rangle \\ &\equiv j_{q0} j_0^{\sigma-1} \langle (M^\dagger)^* l, w\rangle \pmod p \\ &\not\equiv 0 \pmod p \quad \text{(by (2.4))}, \end{aligned} \tag{2.19}$$

where $\gamma_0 = 0$.

Next, comparing (2.3) and (2.19), we find that $k = k(l) = 1$ and (2.3) becomes

$$\{0, \langle M^\dagger d_1, l\rangle, \ldots, \langle M^\dagger d_{p-1}, l\rangle\} \equiv \{0, 1, 2, \ldots, (p-1)\} \pmod p. \tag{2.20}$$

In this case, if $d_{i_1} - d_{i_2} = M\lambda$ for some $\lambda \in \mathbb{Z}^n$ and $d_{i_1}, d_{i_2} \in \tilde{D}$, then $\langle M^\dagger(d_{i_1} - d_{i_2}), l\rangle = p\langle \lambda, l\rangle \in p\mathbb{Z}$ contradicts (2.20). Thus, $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

Case 2: $l \in M^*(\mathbb{Z}^n)$. From $(M^*)^\sigma(\mathbb{Z}^n) \subseteq (M^*)^{\sigma-1}(\mathbb{Z}^n)(\sigma = 1, 2, \ldots)$ and $\bigcap_{\sigma=1}^\infty (M^*)^\sigma(\mathbb{Z}^n) = \{0\}$, we know that there exists a non-negative integer $\tilde{\gamma}$ and $\tilde{l} \in \mathbb{Z}^n \setminus \{0\}$ such that $l = (M^*)^{\tilde{\gamma}} \tilde{l}$ and $\tilde{l} \notin M^*(\mathbb{Z}^n)$. Then (2.1) can be written as $0 = \prod_{j=1}^\infty m_{\tilde{D}}(M^{*-j}\tilde{l})$. So, applying Case 1 to $\tilde{l}$ in place of l, we conclude that $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

Thus, we have proved that $\tilde{D}$ is always a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$. This implies that $\mu_{M,\tilde{D}}$ is a spectral measure with some lattice spectrum Γ^*, and therefore $\mu_{M,D}$ is a spectral measure with lattice spectrum $(M^*)^{-r}\Gamma^*$. The proof of Theorem 2.1 is complete. ∎

Remark 2.2. (i) It should be pointed out that for an expanding matrix $M \in M_n(\mathbb{Z})$ such that $|\det(M)| = p$ is a prime, none of the conditions (a)–(f) in Theorem A and in Theorem 2.1 can be omitted. For example, consider the expanding integer matrix $M \in M_2(\mathbb{Z})$ and the digit set D given by

$$M = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{2.21}$$

Then $p\mathbb{Z}^2 = M^{*2}(\mathbb{Z}^2)$ and the conditions (a)–(f) are not satisfied. For this pair (M,D), $\Theta_0=\{x\in\mathbb{R}^n : m_D(x)=0\}$ equals

$$\left\{\begin{pmatrix}1/3+k_1\\2/3+k_2\end{pmatrix} : k_1,k_2\in\mathbb{Z}\right\}\cup\left\{\begin{pmatrix}2/3+\tilde{k}_1\\1/3+\tilde{k}_2\end{pmatrix} : \tilde{k}_1,\tilde{k}_2\in\mathbb{Z}\right\},$$

and $Z(\hat{\mu}_{M,D})=\bigcup_{j=1}^{\infty}M^{*j}\Theta_0=(\bigcup_{j=1}^{\infty}M^{*2j}\Theta_0)\cup(\bigcup_{j=0}^{\infty}M^{*(2j+1)}\Theta_0)$ equals

$$Z(\hat{\mu}_{M,D})=\Big(\bigcup_{j=1}^{\infty}3^j\Theta_0\Big)\cup\Big(\bigcup_{j=0}^{\infty}\begin{bmatrix}0&3\\1&0\end{bmatrix}(3^j\Theta_0)\Big). \tag{2.22}$$

Hence $Z(\hat{\mu}_{M,D})\cap\mathbb{Z}^n\neq\emptyset$. But there is no $r\in\mathbb{N}_0$ such that $D=M^r\tilde{D}$ and $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

(ii) In the plane $\mathbb{R}^2$, let $M\in M_2(\mathbb{Z})$ be an expanding matrix such that $|\det(M)|=p\geq 3$ is a prime. If $\mathrm{Trace}(M)=0$, then $p\mathbb{Z}^2=M^{*2}(\mathbb{Z}^2)$. Conversely, if $p\mathbb{Z}^2=M^{*2}(\mathbb{Z}^2)$, we first have $\mathrm{Trace}(M)=p\rho$ for some $\rho\in\mathbb{Z}$; then, the expansivity of $M\in M_2(\mathbb{Z})$ yields the conclusion that (a) if $\det(M)=-p$, then $\mathrm{Trace}(M)=0$; (b) if $\det(M)=p$, then $\mathrm{Trace}(M)=0$ or $\mathrm{Trace}(M)=\pm p$. This gives a relation between the condition $p\mathbb{Z}^2=M^{*2}(\mathbb{Z}^2)$ and $\mathrm{Trace}(M)$. To prove these assertions, we let

$$M=\begin{bmatrix}a&b\\c&d\end{bmatrix},\quad M^2=\begin{bmatrix}a^2+bc&b(a+d)\\c(a+d)&d^2+bc\end{bmatrix},\quad e_1=\begin{pmatrix}1\\0\end{pmatrix},\quad e_2=\begin{pmatrix}0\\1\end{pmatrix}.$$

Then $p\mathbb{Z}^2=M^{*2}(\mathbb{Z}^2)$ implies that $M^{*2}e_1,M^{*2}e_2\in p\mathbb{Z}^2$, that is, there exist $l_1,l_2,l_3,l_4\in\mathbb{Z}$ such that

$$a^2+bc=l_1p,\quad b(a+d)=l_2p,\quad c(a+d)=l_3p,\quad d^2+bc=l_4p.$$

This shows $p\,|\,(a+d)$ (otherwise, $p\,|\,b$ and $p\,|\,c$, which yields $p\,|\,a$ and $p\,|\,d$, so $p\,|\,(a+d)$, a contradiction). Hence $\mathrm{Trace}(M)=a+d=p\rho$ for some $\rho\in\mathbb{Z}$. Next, $\det(\lambda I-M)=\lambda^2-\mathrm{Trace}(M)\lambda+\det(M)=\lambda^2-p\rho\lambda\pm p$. The expansivity of $M\in M_2(\mathbb{Z})$ yields

$$|p\rho+\sqrt{p^2\rho^2\mp 4p}|>2\quad\text{and}\quad|p\rho-\sqrt{p^2\rho^2\mp 4p}|>2,$$

which shows that (a) if $\det(M)=-p$ ($p\geq 3$ is a prime), then $\rho=0$; (b) if $\det(M)=p$ ($p\geq 3$ is a prime), then $\rho=0$ or $\rho=\pm 1$.

(iii) For any expanding matrix $M\in M_n(\mathbb{Z})$ with $|\det(M)|=2$ and for any two-element digit set $D\subset\mathbb{Z}^n$, $\mu_{M,D}$ is always a spectral measure with lattice spectrum. So, when $|\det(M)|=|D|=p$, we may always assume that $p\geq 3$. Also, the conditions (a), (b), (d), (e) are always satisfied in the one-dimensional case ($n=1$).

In the following discussion, we may assume that $|\det(M)|=|D|=p\geq 3$ and $n\geq 2$. It is well-known that if $x\in\mathbb{R}$ is a root of an equation $x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n=0$ with integral coefficients of which the first is unity, then x is either integral or irrational. So, if $M\in M_n(\mathbb{Z})$

is an expanding matrix such that $|\det(M)| = p$ is a prime, then for each $j_0 \in \{1, \dots, p-1\}$, the matrix $pI_n - j_0 M^*$ is invertible. Based on these simple facts, the above method leads to the following more general result.

THEOREM 2.3. *Let $M \in M_n(\mathbb{Z})$ be an expanding matrix such that $|\det(M)| = p \geq 3$ is a prime and $n \geq 2$. If $D \subset \mathbb{Z}^n$ is a finite digit set of cardinality $|D| = |\det(M)|$ with $0 \in D$ such that*

$$\Big(\bigcup_{j_0=1}^{p-1} (pI_n - j_0 M^*)^{-1} M^{*(k+2)}(\mathbb{Z}^n)\Big) \cap Z(\hat{\mu}_{M,D}) \cap (M^{*k}(\mathbb{Z}^n) \setminus M^{*(k+1)}(\mathbb{Z}^n)) \neq \emptyset \tag{2.23}$$

for some $k \in \mathbb{N}_0$, then there exists $r \in \mathbb{N}_0$ such that $D = M^r \tilde{D}$, where $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$, and hence $\mu_{M,D}$ is a spectral measure with lattice spectrum.

Proof. From (2.23), we can choose a non-zero integer l such that

$$l \in Z(\hat{\mu}_{M,D}) \cap (M^{*k}(\mathbb{Z}^n) \setminus M^{*(k+1)}(\mathbb{Z}^n)) \tag{2.24}$$

and

$$l \in \bigcup_{j_0=1}^{p-1} (pI_n - j_0 M^*)^{-1} M^{*(k+2)}(\mathbb{Z}^n). \tag{2.25}$$

(i) When $k = 0$, we see from (2.24) that $0 = \hat{\mu}_{M,D}(l)$ and $l \in \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)$. Thus, (2.1)–(2.7) hold as in Case 1 above. At the same time, by (2.25), there exists $\hat{j}_0 \in \{1, \dots, p-1\}$ such that $l \in (pI_n - \hat{j}_0 M^*)^{-1} M^{*2}(\mathbb{Z}^n)$. Furthermore, there exists $\hat{\gamma} \in \mathbb{Z}^n$ such that $l = (pI_n - \hat{j}_0 M^*)^{-1} M^{*2} \hat{\gamma}$, equivalently,

$$(M^\dagger)^* l = \hat{j}_0 l + M^* \hat{\gamma} \quad \text{for some } 1 \leq \hat{j}_0 \leq p-1 \text{ and } \hat{\gamma} \in \mathbb{Z}^n. \tag{2.26}$$

By considering (2.26) instead of (2.7), we obtain the desired conclusion from the above proof of Theorem 2.1.

(ii) When $k \neq 0$, we let $l = M^{*k} \hat{l}$; it follows from (2.24) that $0 = \hat{\mu}_{M,D}(l) = \hat{\mu}_{M,D}(M^{*k} \hat{l}) = \hat{\mu}_{M,D}(\hat{l})$ and $\hat{l} \in \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)$. Then l satisfies (2.25) with $k \neq 0$ if and only if $\hat{l}$ satisfies (2.25) with $k = 0$. So, applying case (i) to $\hat{l}$ in place of l, we get the desired conclusion. ■

REMARK 2.4. (i) Observe that

$$(pI_n - j_0 M^*)^{-1} M^{*(k+2)} = M^{*(k+2)} (pI_n - j_0 M^*)^{-1},$$

and let $q := |\det(pI_n - j_0 M^*)|$. Then

$$\begin{aligned}(pI_n - j_0 M^*)^{-1} M^{*(k+2)}(q\mathbb{Z}^n) &= M^{*(k+2)} (pI_n - j_0 M^*)^{-1}(q\mathbb{Z}^n) \\ &= M^{*(k+2)} (pI_n - j_0 M^*)^{\dagger}(\mathbb{Z}^n) \subseteq M^{*(k+2)}(\mathbb{Z}^n) \subseteq M^{*k}(\mathbb{Z}^n),\end{aligned} \tag{2.27}$$

and for each $k \in \mathbb{N}_0$, $M^{*k}(\mathbb{Z}^n) \cap (pI_n - j_0M^*)^{-1}M^{*(k+2)}(\mathbb{Z}^n) \neq \emptyset$. The above condition (2.23) cannot be replaced by

$$\Big(\bigcup_{j_0=1}^{p-1}(pI_n - j_0M^*)^{-1}M^{*(k+2)}(\mathbb{Z}^n)\Big) \cap Z(\hat{\mu}_{M,D}) \cap M^{*k}(\mathbb{Z}^n) \neq \emptyset \tag{2.28}$$

for some $k \in \mathbb{N}_0$. For example, consider the pair (M, D) given by (2.21); then we have

$$\bigcup_{j_0=1}^{p-1}(pI_n - j_0M^*)^{-1}M^{*2}(\mathbb{Z}^n) = \begin{bmatrix} 3/2 & 3/2 \\ 1/2 & 3/2 \end{bmatrix}(\mathbb{Z}^2) \cup \begin{bmatrix} -3 & -6 \\ -2 & -3 \end{bmatrix}(\mathbb{Z}^2). \tag{2.29}$$

Combining this with (2.22), we see that

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \Big(\bigcup_{j_0=1}^{p-1}(pI_n - j_0M^*)^{-1}M^{*2}(\mathbb{Z}^n)\Big) \cap Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n, \tag{2.30}$$

and (2.28) holds with $k = 0$, but there is no $r \in \mathbb{N}_0$ such that $D = M^r\tilde{D}$ and $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

(ii) (2.23) implies the condition $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n \neq \emptyset$, which plays an important role in this section. For this single condition, we have the following conclusion.

PROPOSITION 2.5. *For an expanding matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n \neq \emptyset$ or if there are two points $s_1, s_2 \in \mathbb{R}^n$ with $s_1 - s_2 \in \mathbb{Z}^n$ such that the exponential functions $e_{s_1}(x), e_{s_2}(x)$ are orthogonal in $L^2(\mu_{M,D})$, then there are infinite families of orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,D})$ with $\Lambda \subseteq \mathbb{Z}^n$.*

In fact, let $l = s_1 - s_2 \in Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n$. Then there exists a positive integer $k := k(l)$ such that $m_D(M^{*-k}l) = 0$. From the tiling property $\mathbb{R}^n = [0,1)^n + \mathbb{Z}^n$, we write $M^{*-k}l = \hat{\alpha} + \tilde{\alpha}$, where $\hat{\alpha} \in [0,1)^n$ and $\tilde{\alpha} \in \mathbb{Z}^n$. Then $m_D(\hat{\alpha} + \tilde{\alpha}) = m_D(\hat{\alpha}) = 0$, i.e., $\hat{\alpha} \in Z := \{x \in [0,1)^n : m_D(x) = 0\}$. For each integer $\sigma \geq k = k(l)$, we have

$$M^{*\sigma}\hat{\alpha} = M^{*\sigma}(M^{*-k}l - \tilde{\alpha}) = M^{*(\sigma-k)}l - M^{*\sigma}\tilde{\alpha} \in \mathbb{Z}^n,$$

which yields the desired result from [13, Theorem 2].

Furthermore, for an expanding matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, let $|\det(M)| = m = p_1^{b_1} \cdots p_r^{b_r}$ be the standard prime factorization, where $p_1 < \cdots < p_r$ are prime numbers and $b_j > 0$. Denote by $W(m)$ the set of non-negative integer combinations of $p_1, \ldots, p_r$. If $|D| \notin W(m)$, then (i) $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n = \emptyset$; (ii) there is no finite subset $S \subset \mathbb{Z}^n$ such that $(M^{-1}D, S)$ is a compatible pair; (iii) there are no points $s_1, s_2 \in \mathbb{R}^n$ with $s_1 - s_2 \in \mathbb{Z}^n$ such that the exponential functions $e_{s_1}(x), e_{s_2}(x)$ are orthogonal in $L^2(\mu_{M,D})$ (see [12, Section 3]).

3. Application to tile digit sets. Let $M \in M_n(\mathbb{Z})$ be an expanding matrix, and $D \subset \mathbb{Z}^n$ be a finite digit set of cardinality $|D| = |\det(M)|$ with $0 \in D$. For most pairs (M, D), the set $T(M, D)$ has Lebesgue measure $\mu_L(T(M, D)) = 0$.

For example, consider the pair (M, D) given by (2.21). Then the attractor $T = T(M, D)$ satisfies $M(T) = T + D$, which yields $M^2(T) = T + D + M(D)$. From (2.21), the set $D + M(D)$ contains eight elements. By taking the Lebesgue measure, we have

$$9\mu_L(T) = \mu_L(M^2(T)) = \mu_L(T + D + M(D)) \leq 8\mu_L(T),$$

and hence $\mu_L(T) = 0$.

If $\mu_L(T(M, D)) > 0$, we call $T(M, D)$ an *integral self-affine tile* and the corresponding D a *tile digit set* (with respect to M). Associated with the pair (M, D) is the smallest M-invariant sublattice of $\mathbb{Z}^n$ containing D, which is denoted by $\mathbb{Z}[M, D]$. If $\mathbb{Z}[M, D] = \mathbb{Z}^n$, we call the digit set D *primitive* (with respect to M). It should be pointed out that $\mathbb{Z}[M, D] \not\subseteq M(\mathbb{Z}^n)$ is equivalent to $D \not\subseteq M(\mathbb{Z}^n)$.

It is known that most of the measure and tiling questions on $T(M, D)$ can be reduced to the case of primitive tiles. More precisely, Lagarias and Wang provide the following useful fact (see [9], [10, Theorem 1.2]).

Proposition 3.1. *If the columns of a matrix $B \in M_n(\mathbb{Z})$ form a basis of $\mathbb{Z}[M, D]$, that is, $\mathbb{Z}[M, D] = B(\mathbb{Z}^n)$, then there exists a matrix*

$$\tilde{M} := B^{-1}MB \in M_n(\mathbb{Z})$$

and a digit set

$$\tilde{D} := B^{-1}D \subset \mathbb{Z}^n$$

such that $\mathbb{Z}[\tilde{M}, \tilde{D}] = \mathbb{Z}^n$, $0 \in \tilde{D}$, and $T(M, D) = B(T(\tilde{M}, \tilde{D}))$.

With the same notation as in Proposition 3.1, we follow the terminology of [9], and say that D is a *standard digit set* (with respect to M) if $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/\tilde{M}(\mathbb{Z}^n)$. Here we note that the technique from Proposition 3.1 has its limitations. It is unsuitable for those pairs (M, D) where $\mathbb{Z}[M, D]$ is not a full rank lattice. On the other hand, for a given expanding matrix $\tilde{M} \in M_n(\mathbb{Z})$, it is not always possible to find a digit set $\tilde{D}$ primitive with respect to $\tilde{M}$ (see example in [11, pp. 192–193]). Hence the above technique cannot be applied to such expanding matrices. However, for an expanding matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, there always exists $r \in \mathbb{N}_0$ and a finite subset $\tilde{D} \subset \mathbb{Z}^n$ such that

$$D = M^r\tilde{D} \quad \text{and} \quad \tilde{D} \not\subseteq M(\mathbb{Z}^n). \tag{3.1}$$

See [15, Lemma 1], [11, p. 189]. Also, for the digit sets in (3.1), we have

$T(M,D) = M^r(T(M,\tilde{D}))$ and

$$\mu_L(T(M,D)) = (|\det(M)|)^r \mu_L(T(M,\tilde{D})). \tag{3.2}$$

This shows that D is a tile digit set if and only if $\tilde{D}$ is. Therefore, we may always assume that the digit set D in the IFS $\{\phi_d(x)\}_{d\in D}$ satisfies the condition $D \not\subseteq M(\mathbb{Z}^n)$.

For the digit sets of integral self-affine tiles with prime determinant, Kenyon [7] proved the following.

THEOREM 3.2. *Let p be a prime and $D \subset \mathbb{Z}$ be a primitive digit set with $|D| = |p|$. Then $T(p,D)$ is an integral self-affine tile if and only if D is a complete set of residues modulo p.*

This result has been generalized by Lagarias and Wang [9, Theorem 4.1], [16, Theorem 4.2] to show that nonstandard digit sets do not exist for many M such that $|\det(M)| = p$ is a prime. In fact, they stated the following.

THEOREM 3.3 ([9, Theorem 4.1]). *Let $M \in M_n(\mathbb{Z})$ be expanding such that $|\det(M)| = p$ is a prime and $p\mathbb{Z}^n \not\subseteq M^2(\mathbb{Z}^n)$. If $D \subset \mathbb{Z}^n$ is a digit set with $|D| = p$, then $\mu_L(T(M,D)) > 0$ if and only if D is a standard digit set.*

Lagarias and Wang also formulated the following conjecture in [9]:

CONJECTURE 1. The condition $p\mathbb{Z}^n \not\subseteq M^2(\mathbb{Z}^n)$ in Theorem 3.3 is redundant.

Some partial results concerning this conjecture can be found in [8] and [6]. Since $p\mathbb{Z}^n \not\subseteq M^2(\mathbb{Z}^n)$ is equivalent to $pB^{-1}(\mathbb{Z}^n) \not\subseteq \tilde{M}^2(B^{-1}(\mathbb{Z}^n))$, which is not of the form $p\mathbb{Z}^n \not\subseteq \tilde{M}^2(\mathbb{Z}^n)$, the author [11] observed that there is a gap in the proof of Theorem 3.3 in [9]. The proof there essentially yields the following.

THEOREM 3.4 ([16, Theorem 4.2]). *Let $M \in M_n(\mathbb{Z})$ be expanding such that $|\det(M)| = p$ is a prime and $p\mathbb{Z}^n \not\subseteq M^2(\mathbb{Z}^n)$. Let $D \subset \mathbb{Z}^n$ with $|D| = |\det(M)|$ be primitive. Then $\mu_L(T(M,D)) > 0$ if and only if D is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.*

Since the sufficiency of the theorems above was proved by Bandt [1] under a much weaker condition, one often concentrates on the necessity of the theorems in higher dimensions (see [9, p. 174]), especially on the above Conjecture 1. Based on previous research, the author [11] extended Theorems 3.2, 3.3 and 3.4, giving in particular a complete proof of Theorem 3.3. As an application of Section 2, we present the following more general result on the digit sets of integral self-affine tiles with prime determinant.

THEOREM 3.5. *Let $M \in M_n(\mathbb{Z})$ be expanding such that $|\det(M)| = p$ is a prime and one of the conditions* (a)–(f) *holds. Suppose that $D \subset \mathbb{Z}^n$ is a tile digit set with respect to M, and $0 \in D$. Then:*

(i) *If* $\mathbb{Z}[M,D] \not\subseteq M(\mathbb{Z}^n)$, *then* D *is a complete set of coset representatives of* $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

(ii) *If* $\mathbb{Z}[M,D] \subseteq M(\mathbb{Z}^n)$, *then there exists a positive integer* $r \in \mathbb{N}$ *such that* $D = M^r\tilde{D}$ *and* $\tilde{D}$ *is a complete set of coset representatives of* $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

Proof. We first write $D = M^{\tilde{r}}\tilde{D}$, where $\tilde{D} \subset \mathbb{Z}^n$, $\tilde{D} \not\subset M(\mathbb{Z}^n)$ and $\tilde{r} \geq 0$ is an integer. The property that $\tilde{D}$ is a tile digit set implies

$$\mathbb{Z}^n \setminus \{0\} \subseteq Z(\hat{\mu}_{M,\tilde{D}}). \tag{3.3}$$

See [9, Theorem 2.1], [12, p. 636]. This gives

$$Z(\hat{\mu}_{M,\tilde{D}}) \cap (\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)) = \mathbb{Z}^n \setminus M^*(\mathbb{Z}^n) \neq \emptyset. \tag{3.4}$$

Since the cases $p = 2$ and $n = 1$ are trivial, we may assume that $p \geq 3$ and $n \geq 2$ in the following discussion.

If $|\det(M)| = p$ is a prime and one of the conditions (d)–(f) holds, then the method in Section 2 yields

$$\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n) \subseteq \bigcup_{j_0=1}^{p-1} (pI_n - j_0M^*)^{-1}M^{*2}(\mathbb{Z}^n). \tag{3.5}$$

It follows from (3.4) and (3.5) that (2.23) holds for $k = 0$. Hence, from Theorem 2.3, there exists $r \in \mathbb{N}_0$ such that $D = M^r\tilde{D}$ and $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

If $|\det(M)| = p$ is a prime and one of the conditions (a)–(c) holds, it follows from (3.4) and the method of [15] that one can take any $l \in Z(\hat{\mu}_{M,\tilde{D}}) \cap (\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n))$ to conclude that there exists $r \in \mathbb{N}_0$ such that $D = M^r\tilde{D}$ and $\tilde{D}$ is a complete set of coset representatives of $\mathbb{Z}^n/M(\mathbb{Z}^n)$.

The case $r = 0$ in $D = M^r\tilde{D}$ corresponds to the conclusion (i), while the case $r \geq 1$ in $D = M^r\tilde{D}$ corresponds to (ii). ∎

To end the paper, we point out that: (i) in order to prove Conjecture 1, one only needs to consider the case where all the following conditions:

(a′) $p\mathbb{Z}^n \subseteq M^2(\mathbb{Z}^n)$;
(b′) $p(\mathbb{Z}^n \setminus M(\mathbb{Z}^n)) \not\subseteq M(\mathbb{Z}^n \setminus M(\mathbb{Z}^n))$;
(c′) $p\mathbb{Z}^2 = M^2(\mathbb{Z}^2)$ in the case when $n = 2$;
(d′) $p\mathbb{Z}^n \subseteq M^{*2}(\mathbb{Z}^n)$;
(e′) $p(\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n)) \not\subseteq M^*(\mathbb{Z}^n \setminus M^*(\mathbb{Z}^n))$;
(f′) $p\mathbb{Z}^2 = M^{*2}(\mathbb{Z}^2)$ in the case when $n = 2$

are satisfied for an expanding matrix $M \in M_n(\mathbb{Z})$ with prime determinant $|\det(M)| = p$; (ii) the conclusion of Theorem 3.5 also implies that $T(M,D)$ is a spectral set with lattice spectrum. This gives some sufficient conditions for an integral self-affine tile $T(M,D)$ to be a spectral set (see the open problem of [12, p. 636]); (iii) for the integer case: $M \in M_n(\mathbb{Z})$ and $D \subset \mathbb{Z}^n$, we know that $\mu_L(T(M,D)) > 0$ if and only if $\mathbb{Z}^n \setminus \{0\} \subseteq Z(\hat{\mu}_{M,D})$ (see [9, Theorem 2.1]). The differences between the question considered in Section 2 and the question considered in Section 3 lie mainly in the differences between the condition $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n \neq \emptyset$ and $\mathbb{Z}^n \setminus \{0\} \subseteq Z(\hat{\mu}_{M,D})$. Since the latter condition is much stronger, the method here shows that the results obtained under the condition $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^n \neq \emptyset$ can be applied to characterize tile digit sets.

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