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Pointwise multipliers on martingale Campanato spaces

by

EIICHI NAKAI (Mito) and GAKU SADASUE (Kashiwara)

Abstract. We introduce generalized Campanato spaces $\mathcal{L}_{p,\phi}$ on a probability space $(\Omega, \mathcal{F}, P)$, where $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi} = \mathrm{BMO}$. We give a characterization of the set of all pointwise multipliers on $\mathcal{L}_{p,\phi}$.

1. Introduction. We consider a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$, where $\{\mathcal{F}_n\}_{n\ge 0}$ is a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by a countable collection of atoms, where $B \in \mathcal{F}_n$ is called an *atom* (more precisely an $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively.

Let $\mathcal{X}$ be a normed space of $\mathcal{F}$-measurable functions. We say that an $\mathcal{F}$-measurable function g is a *pointwise multiplier* on $\mathcal{X}$ if the pointwise product fg is in $\mathcal{X}$ for any $f \in \mathcal{X}$. We denote by $\mathrm{PWM}(\mathcal{X})$ the set of all pointwise multipliers on $\mathcal{X}$. If $\mathcal{X}$ is a Banach space, then every $g \in \mathrm{PWM}(\mathcal{X})$ is a bounded operator on $\mathcal{X}$ whenever $\mathcal{X}$ has the following property

$$f_n \to f \text{ in } \mathcal{X}\ (n \to \infty) \ \Rightarrow\ \exists\{n(j)\} : f_{n(j)} \to f \text{ a.s. } (j \to \infty). \tag{1.1}$$

Actually, from (1.1) we see that g is a closed operator. Therefore, g is a bounded operator by the closed graph theorem.

It is known that $\mathrm{PWM}(L_p) = L_\infty$ for $p \in (0,\infty]$. More generally, if $\mathcal{X}$ is a (quasi) Banach function space, then $\mathrm{PWM}(\mathcal{X}) = L_\infty$ (see [4, 7]). For Banach function spaces, see Kikuchi [2].

In this paper we consider pointwise multipliers on generalized Campanato spaces which are not Banach function spaces in general. We always assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, that is, the operator E_0 coincides with E. Then we introduce generalized Campanato spaces $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^{\natural}_{p,\phi}$ as follows:

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Definition 1.1. Let $p \in [1,\infty)$ and ϕ be a function from $(0,1]$ to $(0,\infty)$. For $f \in L_1$, let

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{n\ge 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |f-E_nf|^p\,dP\right)^{1/p}, \tag{1.2}$$

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |Ef|. \tag{1.3}$$

Define

$$\mathcal{L}_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\} \quad \text{and} \quad \mathcal{L}^\natural_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}^\natural_{p,\phi}} < \infty\}.$$

If $\phi(r) = r^\lambda$, $\lambda \in (-\infty,\infty)$, we simply denote $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ by $\mathcal{L}_{p,\lambda}$ and $\mathcal{L}^\natural_{p,\lambda}$, respectively; the latter spaces were introduced in [9].

Note that $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ coincide as sets of measurable functions. We see that $\mathcal{L}_{p,\phi} = (\mathcal{L}_{p,\phi}, \|\cdot\|_{\mathcal{L}_{p,\phi}})$ is a seminormed space and $\mathcal{L}^\natural_{p,\phi} = (\mathcal{L}^\natural_{p,\phi}, \|\cdot\|_{\mathcal{L}^\natural_{p,\phi}})$ is a normed space. Moreover, $\mathcal{L}^\natural_{p,\phi}$ is a Banach space, but it is not a Banach function space in general. It is easy to see that $\mathcal{L}^\natural_{p,\phi}$ has the property (1.1), since

$$\|f\|_{L_1} \le E[|f-Ef|] + |Ef| \le \max(1,\phi(1))\|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

For $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$, let

$$\|g\|_{\mathrm{Op}} = \sup_{f\not\equiv 0} \frac{\|fg\|_{\mathcal{L}^\natural_{p,\phi}}}{\|f\|_{\mathcal{L}^\natural_{p,\phi}}}.$$

We also define BMO and Lip_α as follows:

Definition 1.2. For $\phi \equiv 1$, denote $\mathcal{L}_{1,\phi}$ and $\mathcal{L}^\natural_{1,\phi}$ by BMO and $\mathrm{BMO}^\natural$, respectively. For $\phi(r) = r^\alpha$, $\alpha > 0$, denote $\mathcal{L}_{1,\phi}$ and $\mathcal{L}^\natural_{1,\phi}$ by Lip_α and $\mathrm{Lip}^\natural_\alpha$, respectively.

Let

$$L_{1,0} = \{f \in L_1 : Ef = 0\}.$$

Then $\mathrm{BMO} \cap L_{1,0} = \mathrm{BMO}^\natural \cap L_{1,0}$ and $\mathrm{Lip}_\alpha \cap L_{1,0} = \mathrm{Lip}^\natural_\alpha \cap L_{1,0}$. These spaces coincide respectively with BMO and Lip_α defined by Weisz [12, 13], under the assumption that every σ-algebra $\mathcal{F}_n$ is generated by a countable collection of atoms, see [9] for details.

We say $\{\mathcal{F}_n\}_{n\ge 0}$ is *regular* if there exists $R \ge 2$ such that

$$f_n \le Rf_{n-1} \quad \text{for all nonnegative martingales } f = (f_n)_{n\ge 0}. \tag{1.4}$$

A function $\theta : (0,1] \to (0,\infty)$ is said to satisfy the *doubling condition* if there exists a constant $C > 0$ such that

$$\frac{1}{C} \le \frac{\theta(r)}{\theta(s)} \le C \quad \text{for all } r, s \in (0,1] \text{ with } 1/2 \le r/s \le 2.$$

A function $\theta : (0,1] \to (0,\infty)$ is said to be *almost increasing* (resp. *almost decreasing*) if there exists a constant $C > 0$ such that

$$\theta(r) \le C\theta(s) \quad (\text{resp. } \theta(r) \ge C\theta(s)) \quad \text{for } 0 < r \le s \le 1.$$

Our main result is the following:

THEOREM 1.3. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and that*

$$\int_0^r \phi(t)^p\,dt \le Cr\phi(r)^p \quad \text{for all } r \in (0,1]. \tag{1.5}$$

Let

$$\phi_*(r) = 1 + \int_r^1 \frac{\phi(t)}{t}\,dt. \tag{1.6}$$

Then

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}) = \mathcal{L}_{p,\phi/\phi_*} \cap L_\infty.$$

Moreover, for $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$, $\|g\|_{\mathrm{Op}}$ is equivalent to $\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty}$.

See [1, 6, 10, 11, 14] for pointwise multipliers on BMO and Campanato spaces defined on the Euclidean space. Our basic idea comes from [1, 10].

REMARK 1.4. (i) If ϕ satisfies the doubling condition and (1.5), then $r\phi(r)^p$ is almost increasing.

(ii) If ϕ is almost increasing, then so is ϕ/ϕ_*.

(iii) Let

$$\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}} = \sup_{n\ge 0} \sup_{A\in\mathcal{F}_n} \frac{1}{\phi(P(A))} \left(\frac{1}{P(A)} \int_A |f - E_n f|^p\,dP\right)^{1/p}. \tag{1.7}$$

Then $\|f\|_{\mathcal{L}_{p,\phi}} \le \|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ by the definitions. If ϕ is almost increasing, then $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ are equivalent. Actually, for any $A \in \mathcal{F}_n$, there exists a sequence of atoms $B_\ell \in A(\mathcal{F}_n)$, $\ell = 1, 2, \ldots$, such that $A = \bigcup_\ell B_\ell$ and $P(A) = \sum_\ell P(B_\ell)$. Then

$$\begin{aligned}\int_A |f - E_n f|^p\,dP &= \sum_\ell \int_{B_\ell} |f - E_n f|^p\,dP \le \sum_\ell \phi(P(B_\ell))^p P(B_\ell) \|f\|^p_{\mathcal{L}_{p,\phi}} \\ &\le C^p \phi(P(A))^p P(A) \|f\|^p_{\mathcal{L}_{p,\phi}}.\end{aligned}$$

This shows $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}} \le C\|f\|_{\mathcal{L}_{p,\phi}}$. If ϕ is not almost increasing, then $\|f\|_{\mathcal{L}_{p,\phi}}$ is not equivalent to $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ in general (see [9]). The norm (1.7) was introduced in [5] for general $\{\mathcal{F}_n\}_{n\ge 0}$.

Theorem 1.3 has the following two immediate corollaries:

Corollary 1.5. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular and $\mathcal{F}_0=\{\emptyset,\Omega\}$. Then*

$$\mathrm{PWM}(\mathrm{BMO}^\natural)=\mathcal{L}_{1,\psi}\cap L_\infty,$$

where $\psi(r)=1/\log(e/r)$. Moreover, for $g\in\mathrm{PWM}(\mathrm{BMO}^\natural)$, $\|g\|_{\mathrm{Op}}$ is equivalent to $\|g\|_{\mathcal{L}_{1,\psi}}+\|g\|_{L_\infty}$.

Corollary 1.6. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0=\{\emptyset,\Omega\}$ and $\alpha>0$. Then*

$$\mathrm{PWM}(\mathrm{Lip}^\natural_\alpha)=\mathrm{Lip}_\alpha\cap L_\infty.$$

Moreover, for $g\in\mathrm{PWM}(\mathrm{Lip}^\natural_\alpha)$, $\|g\|_{\mathrm{Op}}$ is equivalent to $\|g\|_{\mathrm{Lip}_\alpha}+\|g\|_{L_\infty}$.

Example 1.7. Let $\{\mathcal{F}_n\}_{n\ge 0}$, p and ϕ satisfy the assumptions of Theorem 1.3. For a sequence

$$B_0\supset B_1\supset\cdots,\qquad B_n\in A(\mathcal{F}_n),$$

let

$$g=\sin h,\quad\text{where}\quad h=\sum_{n=1}^\infty\frac{\phi(P(B_n))}{\phi_*(P(B_n))}\left(\frac{P(B_{n-1})}{P(B_n)}\chi_{B_n}-\chi_{B_{n-1}}\right). \tag{1.8}$$

Then h is in $\mathcal{L}_{p,\phi/\phi_*}$ (see Lemma 2.4 and Remark 2.5). Hence $g\in\mathcal{L}_{p,\phi/\phi_*}\cap L_\infty$, since $\sin\theta$ is Lipschitz continuous (see Remark 2.7). That is, $g\in\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. If $\phi\equiv 1$, then $\phi(r)/\phi_*(r)=1/\log(e/r)$ and $g\in\mathrm{PWM}(\mathrm{BMO}^\natural)$.

Example 1.8. The following function satisfies the doubling condition and the property (1.5):

$$\phi(r)=r^\alpha(\log(e/r))^{-\beta}\quad(\alpha\in(-1/p,\infty),\ \beta\in(-\infty,\infty)).$$

If $\alpha\in(-1/p,0)$ and $\beta\in(-\infty,\infty)$, then $\phi_*\sim\phi$, that is, there exists a positive constant C such that $C^{-1}\phi(r)\le\phi_*(r)\le C\phi(r)$ for all $r\in(0,1]$. In general, under the assumptions of Theorem 1.3, if $\phi_*\sim\phi$, then $\mathcal{L}_{1,\phi/\phi_*}=\mathrm{BMO}$ and so

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})=\mathrm{BMO}\cap L_\infty=L_\infty.$$

If $\alpha\in[0,\infty)$ and $\beta\in(-\infty,\infty)$, then $\phi_*\not\sim\phi$ and $\phi(r)/\phi_*(r)\to 0$ as $r\to 0$. In this case $\mathcal{L}_{1,\phi/\phi_*}\cap L_\infty\ne L_\infty$ in general (see also Remark 2.6). In particular, if $\alpha=0$ and $\beta\in(1,\infty)$, or if $\alpha\in(0,\infty)$ and $\beta\in(-\infty,\infty)$, then $\phi_*\sim 1$. In general, under the assumption of Theorem 1.3, if $\phi_*\sim 1$, then $\mathcal{L}_{1,\phi/\phi_*}=\mathcal{L}_{1,\phi}\subset L_\infty$ by Lemma 2.2 below, and so

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})=\mathcal{L}_{1,\phi}\cap L_\infty=\mathcal{L}^\natural_{1,\phi}.$$

Moreover, if ϕ is almost increasing, then we can use the John–Nirenberg type inequality of [5, Theorem 2.9], that is,

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})=\mathcal{L}^\natural_{p,\phi}.$$

We can also take the function

$$\phi(r) = r^{\alpha}(\log(e/r))^{-\beta}(\log\log(e/r))^{-\gamma} \quad (\alpha \in (-1/p,\infty),\ \beta,\gamma \in (-\infty,\infty)),$$

and so on.

Next, a martingale $(f_n)_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$, is said to be $\mathcal{L}_{p,\lambda}$-*bounded* if $f_n \in \mathcal{L}_{p,\lambda}$ $(n \geq 0)$ and $\sup_{n\geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}} < \infty$. Similarly, $(f_n)_{n\geq 0}$ is said to be $\mathcal{L}^{\natural}_{p,\lambda}$-*bounded* if $f_n \in \mathcal{L}^{\natural}_{p,\lambda}$ $(n \geq 0)$ and $\sup_{n\geq 0} \|f_n\|_{\mathcal{L}^{\natural}_{p,\lambda}} < \infty$. For martingale theory, see [3] for example.

Let

$$\mathcal{L}_{p,\phi}(\mathcal{F}_n) = \{f \in L_1 : f \text{ is } \mathcal{F}_n\text{-measurable and } \|f\|_{\mathcal{L}_{p,\phi}} < \infty\},$$
$$\mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n) = \{f \in L_1 : f \text{ is } \mathcal{F}_n\text{-measurable and } \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} < \infty\}.$$

Then we have the following:

THEOREM 1.9. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and* (1.5). *Let $g \in L_1$ and $(g_n)_{n\geq 0}$ be its corresponding martingale with $g_n = E_n g$ $(n \geq 0)$. If $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$, then $g_n \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n))$. Conversely, if $g_n \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n))$ and $\sup_{n\geq 0} \|g_n\|_{\mathrm{Op}} < \infty$, then $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$.*

In Section 2 we establish several lemmas in order to prove Theorem 1.3 in Section 3. We prove Theorem 1.9 in Section 4.

To end this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, depend on the subscripts. If $f \leq Cg$, we write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \sim g$.

2. Lemmas. To prove Theorem 1.3 we need several lemmas. The first was proved in [9].

LEMMA 2.1 ([9, Lemma 3.3]). *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots, \qquad B_n \in A(\mathcal{F}_n),$$

has the following property: for each $n \geq 1$,

$$B_n = B_{n-1} \quad \text{or} \quad (1 + 1/R)P(B_n) \leq P(B_{n-1}) \leq RP(B_n),$$

where R is the constant in (1.4).

For a function $f \in L_1$ and an atom $B \in A(\mathcal{F}_n)$, let

$$f_B = \frac{1}{P(B)} \int_B f\, dP.$$

For a function $\phi : (0,1] \to (0,\infty)$, let ϕ_* be defined by (1.6). If ϕ satisfies the doubling condition, then $\phi(r) \leq C\phi_*(r)$ for all $r \in (0,1]$.

LEMMA 2.2. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition. For $f \in \mathcal{L}^{\natural}_{p,\phi}$ and $B \in \bigcup_{n\ge 0} A(\mathcal{F}_n)$,*

$$|f_B| \le C\phi_*(P(B))\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}. \tag{2.1}$$

Proof. By Lemma 2.1, we can choose $B_{k_j} \in A(\mathcal{F}_{k_j})$ with $0 = k_0 < k_1 < \cdots < k_m \le n$ such that $B_{k_0} \supset B_{k_1} \supset \cdots \supset B_{k_m} = B$ and $(1+1/R)P(B_{k_j}) \le P(B_{k_{j-1}}) \le RP(B_{k_j})$. Then

$$\begin{aligned}
|f_{B_{k_j}} - f_{B_{k_{j-1}}}| &= \left|\frac{1}{P(B_{k_j})}\int\limits_{B_{k_j}} f(\omega)\,dP - \frac{1}{P(B_{k_{j-1}})}\int\limits_{B_{k_{j-1}}} f(\omega)\,dP\right| \\
&= \left|\frac{1}{P(B_{k_j})}\int\limits_{B_{k_j}} [f - E_{k_{j-1}}f](\omega)\,dP\right| \\
&\le \left(\frac{1}{P(B_{k_j})}\int\limits_{B_{k_j}} |f - E_{k_{j-1}}f|^p\,dP\right)^{1/p} \\
&\lesssim \left(\frac{1}{P(B_{k_{j-1}})}\int\limits_{B_{k_{j-1}}} |f - E_{k_{j-1}}f|^p\,dP\right)^{1/p} \\
&\le \phi(P(B_{k_{j-1}}))\,\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.
\end{aligned}$$

Since ϕ satisfies the doubling condition,

$$\begin{aligned}
|f_B - f_{B_0}| &\le \sum_{j=1}^{m} |f_{B_{k_j}} - f_{B_{k_{j-1}}}| \lesssim \sum_{j=1}^{m} \phi(P(B_{k_{j-1}}))\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \\
&\lesssim \sum_{j=1}^{m} \int\limits_{P(B_{k_j})}^{P(B_{k_{j-1}})} \frac{\phi(t)}{t}\,dt\,\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} = \int\limits_{P(B)}^{1} \frac{\phi(t)}{t}\,dt\,\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \\
&= \{\phi_*(P(B)) - 1\}\,\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.
\end{aligned}$$

On the other hand,

$$|f_{B_0}| = |Ef| \le \|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Therefore, we have (2.1). ∎

LEMMA 2.3. *Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that $r\phi(r)^p$ is almost increasing. For any atom $B \in \bigcup_{n\ge 0} A(\mathcal{F}_n)$, the characteristic function χ_B is in $\mathcal{L}^{\natural}_{p,\phi}$ and there exists a positive constant C, independent of B, such that*

$$\|\chi_B\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C/\phi(P(B)). \tag{2.2}$$

Proof. Let $B \in A(\mathcal{F}_n)$ and $B' \in A(\mathcal{F}_k)$. Let $B_j \in A(\mathcal{F}_j)$, $0 \le j \le n$, be such that $B_0 \supset B_1 \supset \cdots \supset B_n = B$.

If $k \ge n$, then $\chi_B - E_k\chi_B = 0$ and

$$\int_{B'} |\chi_B - E_k\chi_B|^p \, dP = 0.$$

If $k < n$ and $B' \ne B_k$, then $B' \cap B_k = \emptyset$ and

$$\int_{B'} |\chi_B - E_k\chi_B|^p \, dP = 0.$$

Hence, we have

$$\|\chi_B\|_{\mathcal{L}_{p,\phi}} = \sup_{k<n} \frac{1}{\phi(P(B_k))} \left(\frac{1}{P(B_k)} \int_{B_k} |\chi_B - E_k\chi_B|^p \, dP \right)^{1/p}.$$

For $k < n$, since $r\phi(r)^p$ is almost increasing,

$$\begin{aligned}
&\frac{1}{\phi(P(B_k))^p} \frac{1}{P(B_k)} \int_{B_k} |\chi_B - E_k\chi_B|^p \, dP \\
&\quad = \frac{1}{\phi(P(B_k))^p P(B_k)} \\
&\qquad \times \left\{ P(B_n) \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + (P(B_k) - P(B_n)) \left(\frac{P(B_n)}{P(B_k)}\right)^p \right\} \\
&\quad \lesssim \frac{1}{\phi(P(B_n))^p P(B_n)} \\
&\qquad \times \left\{ P(B_n) \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + (P(B_k) - P(B_n)) \left(\frac{P(B_n)}{P(B_k)}\right)^p \right\} \\
&\quad = \frac{1}{\phi(P(B_n))^p} \left\{ \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + \left(1 - \frac{P(B_n)}{P(B_k)}\right) \left(\frac{P(B_n)}{P(B_k)}\right)^{p-1} \right\} \\
&\quad \lesssim \frac{1}{\phi(P(B_n))^p} = \frac{1}{\phi(P(B))^p}.
\end{aligned}$$

Therefore, we have

$$\|\chi_B\|_{\mathcal{L}_{p,\phi}} \lesssim 1/\phi(P(B)). \tag{2.3}$$

On the other hand, since $r\phi(r)^p$ is almost increasing,

$$|E\chi_B| = P(B) \le P(B)^{1/p} \lesssim 1/\phi(P(B)). \tag{2.4}$$

Combining (2.3) and (2.4), we have (2.2). ∎

Lemma 2.4. *Let $\{\mathcal{F}_n\}_{n\ge0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and* (1.5). *For a sequence*

$$B_0 \supset B_1 \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

let

$$f_0 = \chi_{B_0}, \quad u_k = \phi(P(B_k))\left(\frac{P(B_{k-1})}{P(B_k)}\chi_{B_k} - \chi_{B_{k-1}}\right),$$

and let

$$f_n = f_0 + \sum_{k=1}^{n} u_k. \tag{2.5}$$

Then $(f_n)_{n\ge 0}$ *is a martingale and is* $\mathcal{L}^{\natural}_{p,\phi}$*-bounded. The sum* $f \equiv f_0 + \sum_{k=1}^{\infty} u_k$ *converges a.s. and in* L_p*, and* $E_n f = f_n$ *for* $n \ge 0$*. Moreover, there exist positive constants* C_1 *and* C_2*, independent of the sequence of atoms, such that*

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C_1 \quad \text{and} \quad |f_{B_n}| \ge C_2\phi_*(P(B_n)), \quad n \ge 0. \tag{2.6}$$

Proof. Since $E_n[u_k] = 0$ for $k > n$, $(f_n)_{n\ge 0}$ is a martingale. We show that the sum $f_0 + \sum_{k=1}^{\infty} u_k$ converges in L_p. If $\lim_{k\to\infty} P(B_k) > 0$ then the convergence is clear because there exists m such that $B_m = B_n$ for all $n \ge m$. So assume that $\lim_{k\to\infty} P(B_k) = 0$. By Lemma 2.1, we can take a sequence of integers $0 = k_0 < k_1 < \cdots$ that satisfies

$$(1 + 1/R)P(B_{k_j}) \le P(B_{k_{j-1}}) \le RP(B_{k_j}), \tag{2.7}$$

and $B_{k_{j-1}} = B_k$ if $k_{j-1} \le k < k_j$. In this case we can write

$$f_n = \chi_{B_0} + \sum_{1\le k_j\le n} \phi(P(B_{k_j}))\left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})}\chi_{B_{k_j}} - \chi_{B_{k_{j-1}}}\right).$$

Note that, by Remark 1.4 and [8, Lemma 7.1], the doubling condition and (1.5) imply

$$\int_0^r \phi(t)t^{1/p-1}\,dt \le C_p\phi(r)r^{1/p} \quad \text{for all } r \in (0,1]. \tag{2.8}$$

Using the doubling condition and (2.8), we have

$$\begin{aligned}
&\sum_{k_j>n} \phi(P(B_{k_j}))\left\|\frac{P(B_{k_{j-1}})}{P(B_{k_j})}\chi_{B_{k_j}} - \chi_{B_{k_{j-1}}}\right\|_{L_p} \\
&\quad\le \sum_{k_j>n} \phi(P(B_{k_j}))(R\|\chi_{B_{k_j}}\|_{L_p} + \|\chi_{B_{k_{j-1}}}\|_{L_p}) \\
&\quad\le 2R\sum_{k_j>n} \phi(P(B_{k_j}))P(B_{k_j})^{1/p} \le C\sum_{k_j>n} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \phi(t)t^{1/p-1}\,dt \\
&\quad\le C\int_0^{P(B_n)} \phi(t)t^{1/p-1}\,dt \le CC_p\phi(P(B_n))P(B_n)^{1/p}.
\end{aligned} \tag{2.9}$$

We can deduce from (2.9) that $f \equiv f_0 + \sum_{k=1}^{\infty} u_k$ converges in L_p. By the martingale convergence theorem, $f_0 + \sum_{k=1}^{\infty} u_k$ also converges almost surely. Moreover, $E_n f = f_n$ and

$$\left(\frac{1}{P(B_n)} \int_{B_n} |f - E_n f|^p \, dP\right)^{1/p} \le C C_p \phi(P(B_n)). \tag{2.10}$$

For $B' \in A(\mathcal{F}_n)$, we have

$$(f - E_n f)\chi_{B'} = \begin{cases} f - E_n f & (B' = B_n), \\ 0 & (B' \neq B_n). \end{cases} \tag{2.11}$$

Combining (2.10) and (2.11), we have $\|f\|_{\mathcal{L}_{p,\phi}} \le C$ where C is a positive constant independent of the sequence of atoms. Moreover, since $B_0 = \Omega$,

$$|Ef| = |f_0| = 1.$$

Therefore, $\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C_1$ where C_1 is a positive constant independent of the sequence of atoms.

We now show $|f_{B_n}| \ge C_2 \phi_*(P(B_n))$. On the atom B_n, we have

$$f_n = 1 + \sum_{1 \le k_j \le n} \phi(P(B_{k_j}))\left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} - 1\right) \ge 1 + \frac{1}{R} \sum_{1 \le k_j \le n} \phi(P(B_{k_j})).$$

Therefore,

$$\begin{aligned} |f_{B_n}| &= \left|\frac{1}{P(B_n)} \int_{B_n} f_n \, dP\right| \\ &\ge 1 + \frac{1}{R} \sum_{1 \le k_j \le n} \phi(P(B_{k_j})) \sim 1 + \sum_{1 \le k_j \le n} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \frac{\phi(t)}{t} \, dt \\ &= 1 + \int_{P(B_n)}^{1} \frac{\phi(t)}{t} \, dt = \phi_*(P(B_n)). \end{aligned}$$

That is, $|f_{B_n}| \ge C_2 \phi_*(P(B_n))$ where C_2 is a positive constant independent of the sequence of atoms. ■

Remark 2.5. From the proof of Lemma 2.4 we see that, for

$$h = \sum_{k=1}^{\infty} u_k, \qquad h_0 = 0, \qquad h_n = \sum_{k=1}^{n} u_k \quad (n \ge 1), \tag{2.12}$$

h is in $\mathcal{L}_{p,\phi}$ and $(h_n)_{n \ge 0}$ is its corresponding martingale with $h_n = E_n h$ $(n \ge 0)$.

REMARK 2.6. Let $(\Omega, \mathcal{F}, P)$ be as follows:

$$\Omega = [0,1), \quad A(\mathcal{F}_n) = \{I_{n,j} = [j2^{-n}, (j+1)2^{-n}) : j = 0,1,\dots,2^n - 1\},$$
$$\mathcal{F}_n = \sigma(A(\mathcal{F}_n)), \quad \mathcal{F} = \sigma\Big(\bigcup_n \mathcal{F}_n\Big), \quad P = \text{the Lebesgue measure}.$$

If $\phi(r) = 1/\log(e/r)$, then h in (2.12) is unbounded. Actually,

$$u_k = \frac{1}{1 + k\log 2}(2\chi_{B_k} - \chi_{B_{k-1}}),$$

and

$$h = \sum_{k=1}^{n} \frac{1}{1 + k\log 2} - \frac{1}{1 + (n+1)\log 2} \quad \text{on } B_n \setminus B_{n+1}.$$

REMARK 2.7. If $F : \mathbb{C} \to \mathbb{C}$ is Lipschitz continuous, that is,

$$|F(z_1) - F(z_2)| \le C|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C},$$

then, for $B \in \mathcal{F}_n$,

$$\int_B |F(f) - E_n[F(f)]|\,dP \le 2C \int_B |f - E_n f|\,dP.$$

Actually,

$$\begin{aligned}
&\int_B |F(f) - E_n[F(f)]|\,dP \\
&\qquad \le \int_B |F(f) - F(E_n f)|\,dP + \int_B |F(E_n f) - E_n[F(f)]|\,dP \\
&\qquad = \int_B |F(f) - F(E_n f)|\,dP + \int_B |E_n[F(E_n f) - F(f)]|\,dP \\
&\qquad \le 2\int_B |F(f) - F(E_n f)|\,dP \le 2C\int_B |f - E_n f|\,dP.
\end{aligned}$$

LEMMA 2.8. *Let $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Suppose $f \in \mathcal{L}_{p,\phi}$ and $g \in L_\infty$. Then $fg \in \mathcal{L}_{p,\phi}$ if and only if*

$$F(f,g) := \sup_{n\ge 0} \sup_{B\in A(\mathcal{F}_n)} \frac{|f_B|}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |g - E_n g|^p\,dP\right)^{1/p} < \infty. \tag{2.13}$$

In this case,

$$\big|F(f,g) - \|fg\|_{\mathcal{L}_{p,\phi}}\big| \le 2\|f\|_{\mathcal{L}_{p,\phi}}\|g\|_{L_\infty}. \tag{2.14}$$

Proof. Let $f \in \mathcal{L}_{p,\phi}$ and $g \in L_\infty$. Let $B \in A(\mathcal{F}_n)$. Since $E_n f = f_B$ on B, we can use the same method as in [6, Lemma 3.5] to obtain

$$(2.15)\qquad \left|\left(\frac{1}{P(B)}\int_B |fg-E_n[fg]|^p\,dP\right)^{1/p} - |f_B|\left(\frac{1}{P(B)}\int_B |g-E_ng|^p\,dP\right)^{1/p}\right|$$
$$\le 2\left(\frac{1}{P(B)}\int_B |(f-E_nf)g|^p\,dP\right)^{1/p} \le 2\phi(P(B))\|f\|_{\mathcal{L}_{p,\phi}}\|g\|_{L_\infty}.$$

Therefore, $fg\in\mathcal{L}_{p,\phi}$ if and only if $F(f,g)<\infty$. In this case, we can deduce (2.14) from (2.15). ∎

LEMMA 2.9. *Let $\{\mathcal{F}_n\}_{n\ge0}$ be regular, $\mathcal{F}_0=\{\emptyset,\Omega\}$, $p\in[1,\infty)$ and $\phi:(0,1]\to(0,\infty)$. Assume that $r\phi(r)^p$ is almost increasing and that ϕ satisfies the doubling condition. If $g\in\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$, then $g\in L_\infty$ and $\|g\|_{L_\infty}\le C\|g\|_{\mathrm{Op}}$ for some positive constant C independent of g.*

Proof. Let $g\in\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. Since the constant function 1 is in $\mathcal{L}^\natural_{p,\phi}$, the pointwise product $g=g\cdot1$ is in $\mathcal{L}^\natural_{p,\phi}$, which implies $g\in L_1$. Then

$$E[|g|]\le E[|g-Eg|]+|Eg|\le\max(1,\phi(1))\|g\|_{\mathcal{L}^\natural_{p,\phi}}\lesssim\|g\|_{\mathrm{Op}}\|1\|_{\mathcal{L}^\natural_{p,\phi}}=\|g\|_{\mathrm{Op}}.$$

Since $\{\mathcal{F}_n\}_{n\ge0}$ is regular, we also have $E_ng\in L_\infty$:

$$E_n[|g|]\le RE_{n-1}[|g|]\le\cdots\le R^nE_0[|g|]=R^nE[|g|].$$

Next we shall show that there exists a positive constant C such that $\|g\|_{L_\infty}\le C\|g\|_{\mathrm{Op}}$. Then we will have the conclusion. Let $B\in A(\mathcal{F}_n)$ such that $|g_B|\ge\|E_ng\|_{L_\infty}/2$. By Lemma 2.1 there exists $B'\in A(\mathcal{F}_{n'})$ with $B\subset B'$ such that $(1+1/R)P(B)\le P(B')\le RP(B)$. Then

$$\|g\chi_B\|_{\mathcal{L}^\natural_{p,\phi}}\ge\frac{1}{\phi(P(B'))}\left(\frac{1}{P(B')}\int_{B'}|g\chi_B-E_{n'}[g\chi_B]|^p\,dP\right)^{1/p}$$
$$\ge\frac{1}{\phi(P(B'))}\left(\frac{1}{P(B')}\int_{B'\setminus B}|g\chi_B-E_{n'}[g\chi_B]|^p\,dP\right)^{1/p}$$
$$=\frac{1}{\phi(P(B'))}\left(\frac{1}{P(B')}\int_{B'\setminus B}\big|E_{n'}[[E_ng]\chi_B]\big|^p\,dP\right)^{1/p}.$$

Since $|[E_ng]\chi_B|=|g_B\chi_B|\ge\|E_ng\|_{L_\infty}\chi_B/2$, we have

$$\int_{B'\setminus B}\big|E_{n'}[[E_ng]\chi_B]\big|^p\,dP\ge\left(\frac{\|E_ng\|_{L_\infty}}{2}\right)^p\left(\frac{P(B)}{P(B')}\right)^pP(B'\setminus B).$$

Hence,

$$(2.16)\qquad \|g\chi_B\|_{\mathcal{L}^\natural_{p,\phi}}\ge\frac{\|E_ng\|_{L_\infty}}{2R(R+1)^{1/p}\phi(P(B'))}.$$

Using (2.16), Lemma 2.3 and the doubling condition on ϕ, we have

$$\|E_n g\|_{L_\infty} \le 2R(R+1)^{1/p}\phi(P(B'))\|g\chi_B\|_{\mathcal{L}^\natural_{p,\phi}} \lesssim \|g\|_{\mathrm{Op}}\frac{\phi(P(B'))}{\phi(P(B))} \lesssim \|g\|_{\mathrm{Op}}.$$

Therefore,

$$\|g\|_{L_\infty} = \sup_{n\ge 0}\|E_n g\|_{L_\infty} \le C\|g\|_{\mathrm{Op}}. \blacksquare$$

3. Proof of Theorem 1.3. We first show that

$$\mathcal{L}_{p,\phi/\phi_*}\cap L_\infty \subset \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}) \quad \text{and} \quad \|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty}). \tag{3.1}$$

Let $g \in \mathcal{L}_{p,\phi/\phi_*}\cap L_\infty$ and $f \in \mathcal{L}^\natural_{p,\phi}$. Let $F(f,g)$ be as in Lemma 2.8. Then, by the definition of $F(f,g)$ and Lemma 2.2,

$$F(f,g) \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} < \infty.$$

Therefore, by Lemma 2.8, we have $fg \in \mathcal{L}_{p,\phi}$ and

$$\|fg\|_{\mathcal{L}_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + 2\|f\|_{\mathcal{L}_{p,\phi}}\|g\|_{L_\infty}. \tag{3.2}$$

On the other hand,

$$|E[fg]| \le \|g\|_{L_\infty}E[|f|] \le \|g\|_{L_\infty}\max(1,\phi(1))\|f\|_{\mathcal{L}^\natural_{p,\phi}}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain (3.1).

We now show the converse, that is,

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}) \subset \mathcal{L}_{p,\phi/\phi_*}\cap L_\infty \quad \text{and} \quad \|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty} \le C\|g\|_{\mathrm{Op}}. \tag{3.4}$$

Let $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. By Lemma 2.9, we have $g \in L_\infty$ and $\|g\|_{L_\infty} \le C\|g\|_{\mathrm{Op}}$. Let $B \in A(\mathcal{F}_n)$. We take $B_j \in A(\mathcal{F}_j)$ with $B_n = B$ such that $B_0 \supset B_1 \supset \cdots$. Let f be the function described in Lemma 2.4. Then, combining Lemmas 2.4 and 2.8, we obtain

$$\begin{aligned}
&\frac{C_2\phi_*(P(B))}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |g-E_n g|^p\,dP\right)^{1/p} \\
&\quad \le \frac{|f_B|}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |g-E_n g|^p\,dP\right)^{1/p} \le F(f,g) \\
&\quad \le \|fg\|_{\mathcal{L}_{p,\phi}} + 2\|g\|_{L_\infty}\|f\|_{\mathcal{L}_{p,\phi}} \le \|g\|_{\mathrm{Op}}\|f\|_{\mathcal{L}^\natural_{p,\phi}} + 2C\|g\|_{\mathrm{Op}}\|f\|_{\mathcal{L}_{p,\phi}} \\
&\quad \lesssim \|g\|_{\mathrm{Op}}\|f\|_{\mathcal{L}^\natural_{p,\phi}} \le C_1\|g\|_{\mathrm{Op}}.
\end{aligned}$$

Therefore, we have (3.4).

4. Proof of Theorem 1.9. To prove Theorem 1.9 we use the following proposition. It can be shown in the same way as [9, Proposition 2.2] which deals with the case $\phi(r) = r^\lambda$, $\lambda \in (-\infty, \infty)$.

PROPOSITION 4.1. *Let* $1 \le p < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Let* $f \in L_1$ *and* $(f_n)_{n\ge 0}$ *be its corresponding martingale with* $f_n = E_n f$ $(n \ge 0)$.

(i) *If* $f \in \mathcal{L}_{p,\phi}$, *then* $(f_n)_{n\ge 0}$ *is* $\mathcal{L}_{p,\phi}$*-bounded and*

$$\|f\|_{\mathcal{L}_{p,\phi}} \ge \sup_{n\ge 0} \|f_n\|_{\mathcal{L}_{p,\phi}}.$$

Conversely, if $(f_n)_{n\ge 0}$ *is* $\mathcal{L}_{p,\phi}$*-bounded, then* $f \in \mathcal{L}_{p,\phi}$ *and*

$$\|f\|_{\mathcal{L}_{p,\phi}} \le \sup_{n\ge 0} \|f_n\|_{\mathcal{L}_{p,\phi}}.$$

(ii) *If* $f \in \mathcal{L}^\natural_{p,\phi}$, *then* $(f_n)_{n\ge 0}$ *is* $\mathcal{L}^\natural_{p,\phi}$*-bounded and*

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} \ge \sup_{n\ge 0} \|f_n\|_{\mathcal{L}^\natural_{p,\phi}}.$$

Conversely, if $(f_n)_{n\ge 0}$ *is* $\mathcal{L}^\natural_{p,\phi}$*-bounded, then* $f \in \mathcal{L}^\natural_{p,\phi}$ *and*

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} \le \sup_{n\ge 0} \|f_n\|_{\mathcal{L}^\natural_{p,\phi}}.$$

REMARK 4.2. In general, for $f \in \mathcal{L}_{p,\phi} \cap L_{1,0}$ (resp. $f \in \mathcal{L}^\natural_{p,\phi}$), its corresponding martingale $(f_n)_{n\ge 0}$ with $f_n = E_n f$ does not always converge to f in $\mathcal{L}_{p,\phi}$ (resp. $\mathcal{L}^\natural_{p,\phi}$). See [9, Remark 3.7] for the case $\phi(r) = r^\lambda$.

Proof of Theorem 1.9. Let $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$ and $f \in \mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n)$. Then, using Proposition 4.1, we have

$$\|E_n[g]f\|_{\mathcal{L}^\natural_{p,\phi}} = \|E_n[gf]\|_{\mathcal{L}^\natural_{p,\phi}} \le \|gf\|_{\mathcal{L}^\natural_{p,\phi}} \le \|g\|_{\mathrm{Op}} \|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

Therefore, $E_n g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n))$.

Conversely, assume $E_n g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n))$ and $\sup_{n\ge 0} \|E_n g\|_{\mathrm{Op}} < \infty$. Then, using Proposition 4.1 and Theorem 1.3, we have

$$\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty} \le \sup_{n\ge 0} \|E_n g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \sup_{n\ge 0} \|E_n g\|_{L_\infty} \lesssim \sup_{n\ge 0} \|E_n g\|_{\mathrm{Op}} < \infty.$$

Using Theorem 1.3 again, we obtain $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. ∎

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Eiichi Nakai
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310-8512, Japan
E-mail: enakai@mx.ibaraki.ac.jp

Gaku Sadasue
Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: sadasue@cc.osaka-kyoiku.ac.jp

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