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A unified approach to the strong approximation property and the weak bounded approximation property of Banach spaces

by

Aleksei Lissitsin (Tartu)

Abstract. We consider convex versions of the strong approximation property and the weak bounded approximation property and develop a unified approach to their treatment introducing the inner and outer Λ-bounded approximation properties for a pair consisting of an operator ideal and a space ideal. We characterize this type of properties in a general setting and, using the isometric DFJP-factorization of operator ideals, provide a range of examples for this characterization, eventually answering a question due to Lima, Lima, and Oja: Are there larger Banach operator ideals than $\mathcal{W}$ yielding the weak bounded approximation property?

1. Introduction. Let X and Y be Banach spaces (over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X to Y, and by $\mathcal{F}(X, Y)$ and $\mathcal{K}(X, Y)$ its subspaces of finite-rank and compact operators, respectively. If $X = Y$, then we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$, and similarly for other spaces of operators.

A Banach space X is said to have the *approximation property* (AP) if for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists a finite-rank operator $S \in \mathcal{F}(X)$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$. The approximation property is said to be *metric* if, in addition, $S \leq 1$.

In analogy with this basic property, one defines the $\mathcal{A}$-*approximation property*, for which the operator S is allowed to belong to an operator ideal $\mathcal{A}$. If X is a Banach lattice and S can be chosen to be positive then X is said to have the *positive approximation property*. Let us also mention a very recent concept, the *bounded approximation property for pairs of Banach spaces*, due to Figiel, Johnson, and Pełczyński [FJP].

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The variations of the approximation property described above, as well as their metric versions, include *convex approximation properties*. By the latter concept, occasionally introduced in [LMO] and studied in [LisO], we mean the following.

DEFINITION. Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. The space X has the *A-approximation property* if for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists an operator $S \in A$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$.

Observe, for instance, that the positive approximation property is precisely the A-approximation property where A is the convex cone of positive finite-rank operators.

While in general the AP and the metric AP of a Banach space are different properties (see [FJ]), it is an open problem whether the same holds if the space in question is a dual space. The *strong approximation property* introduced by Oja [O2], and the *weak bounded approximation property* introduced by Lima and Oja [LO] are more fine-grained and sit between the AP and the metric AP.

The purpose of the present paper is to approach the strong AP and the weak BAP and their convex versions in a unified way (see Section 5).

In Section 2 we start with the standard descriptions of the notions, postponing the proofs until Sections 3 and 5. In Section 3 we recall necessary tools needed for our techniques and prove Theorem 2.4, which characterizes the convex approximation property and serves as a template for the results in Section 5. In Section 4 we consider a range of examples, when the isometric version of the Davis–Figiel–Johnson–Pełczyński factorization lemma can be applied. These examples are also suitable for the application of general theorems from Section 5. Section 6 presents a convex version for the impact of the Radon–Nikodým property (RNP) on the weak bounded approximation property due to [O1]. Finally, using simple results from Section 5 and examples from Section 4, in Section 6 we are able to answer a question due to [LLO1] (see Remark 6.4 and Problem 6.5).

Our notation is standard. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. The identity operator on X is denoted by I_X. The closed unit ball and the unit sphere of X are denoted B_X and S_X, respectively. The closure of a set $K \subset X$ is denoted by $\overline{K}$. The linear span of K is denoted by $\operatorname{span} K$. For Banach spaces X and Y, the components of an operator ideal $\mathcal{A}$ (see [P]) will be denoted $\mathcal{A}(X, Y)$, with the convention $\mathcal{A}(X) := \mathcal{A}(X, X)$; the topology of uniform convergence on compact sets of X, the strong operator topology, and the weak operator topology on the space $\mathcal{L}(X, Y)$ will be denoted $\tau_c(X, Y)$, $\tau_s(X, Y)$, and $\tau_w(X, Y)$ (or simply τ_c, τ_s, and τ_w), respectively.

2. Convex versions of the strong AP and the weak bounded AP. In the following, let X be a Banach space and let $A \subset \mathcal{L}(X)$ be a convex set containing 0.

DEFINITION 2.1 (see [O2]). A Banach space X is said to have the *strong approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, there exists a bounded net $(T_\alpha) \subset \mathcal{F}(X, Z)$ such that $T_\alpha x \to Tx$ for all $x \in X$.

We are interested in the following characterization of the strong approximation property.

PROPOSITION (see [O2, Proposition 4.6]). *A Banach space X has the strong approximation property if and only if for every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that* $\sup_\alpha \|TS_\alpha\| < \infty$ *and $TS_\alpha x \to Tx$ for all $x \in X$.*

This description allows us to extend the notion to the convex approximation properties (for which $A = \mathcal{F}(X)$ below).

DEFINITION 2.2. We say that X has the *strong A-approximation property* (strong A-AP) if for every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha\| < \infty$ and $TS_\alpha x \to Tx$ for all $x \in X$.

One can describe the strong A-AP more akin to Definition 2.1 as follows (see Proposition 5.2 below for the proof in a more general context).

PROPOSITION 2.3. *The space X has the strong A-AP if and only if for every separable reflexive space Z and for every $T \in \mathcal{K}(X, Z)$ there is a net $(S_\alpha) \subset A$ such that* $\sup_\alpha \|TS_\alpha\| < \infty$ *and $TS_\alpha x \to Tx$ for all $x \in X$.*

Observe that the pointwise convergence (i.e., the convergence in $\tau_s(X, Y)$) in Definition 2.2 and Proposition 2.3 can be replaced with the convergence in $\tau_w(X, Y)$ or $\tau_c(X, Y)$ because bounded convex sets have the same closures in all these three topologies (see, e.g., [Gr, Lemma I.20, p. 178]). Removing the boundedness condition in the formally strongest such form of Definition 2.2 results in the A-AP. We shall present a proof of the following theorem in Section 3.

THEOREM 2.4. *The space X has the A-AP if and only if for every separable reflexive space Z and for every $T \in \mathcal{K}(X, Z)$ there is a net $(S_\alpha) \subset A$ such that $TS_\alpha \to T$ in $\tau_c(X, Z)$.*

In the same paper [O2], it was proved that the strong AP shares similar characterizations with the *weak bounded AP*, which was introduced in [LO] and studied, e.g., in [LLO1], [LLO2], [LLO3], [L], and [O4]. The following definition is based on [LO, Theorem 2.4].

Definition 2.5. Let $\lambda \geq 1$. A Banach space X has the *weak λ-bounded AP* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ and $S_\alpha \to I_X$ in the topology of compact convergence.

We extend this notion as follows.

Definition 2.6. Let $\lambda \geq 1$. We say that X has the *weak λ-bounded A-approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ and $S_\alpha \to I_X$ in the topology of compact convergence.

We say that X has the *weak metric A-AP* if X has the weak 1-bounded A-AP, and that X has the *weak bounded A-AP* if X has the weak μ-bounded A-AP for some $\mu \geq 1$.

Observe that the weak λ-bounded A-AP of X means that for every separable reflexive Banach space Z and for every $T \in \mathcal{K}(X,Z)$ the space X has the A_T^λ-AP, where

$$A_T^\lambda := \{S \in A : \|TS\| \leq \lambda\|T\|\}.$$

The following result hints at the link between Definitions 2.2 and 2.6.

Proposition 2.7. *Let $\lambda \geq 1$. The following statements are equivalent:*

(a) *X has the weak λ-bounded A-AP.*
(b) *For every Banach space Y and for every $T \in \mathcal{K}(X,Y)$ there is a net $(S_\alpha) \subset A_T^\lambda$ such that $S_\alpha \to I_X$ in $\tau_c(X,X)$.*
(c) *For every separable reflexive space Z and for every operator $T \in \mathcal{K}(X,Z)$ there is a net $(S_\alpha) \subset A_T^\lambda$ such that $TS_\alpha \to T$ in $\tau_w(X,Z)$.*

We skip the proof of Proposition 2.7 because it is a consequence of a more general Theorem 5.3 below, but we provide the following overview as its corollary.

Corollary 2.8. *Consider the following conditions:*

(a) *X^* has the A-AP with conjugate operators,*
(b) *X has the bounded A-AP,*
(c_1) *X has the weak metric A-AP,*
(c) *X has the weak bounded A-AP,*
(d) *X has the strong A-AP,*
(e) *X has the A-AP.*

Then (a)$\Rightarrow$(c_1)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$ (e) *and* (b)$\Rightarrow$(c). *If $A = \mathcal{F}(X)$, then the implications* (a)$\Rightarrow$(c_1), (c_1)$\Rightarrow$(c), *and* (d)$\Rightarrow$(e) *are strict.*

Proof. The chain (b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e) and the implication (c_1)$\Rightarrow$(c) are clear from the definitions, Theorem 2.4, and Proposition 2.7. The implication (a)$\Rightarrow$(c_1) follows from (a)$\Rightarrow$(b) of [LMO, Corollary 11] and Proposition 2.7.

For (e)$\not\Rightarrow$(d), see [O2, Theorem 2.1, (a)$\not\Rightarrow$(c*), and Proposition 4.6]. For (c)$\not\Rightarrow$(c_1), see [LO, Proposition 2.3]. For (c_1)$\not\Rightarrow$(a), observe that the weak metric AP follows from the metric AP but there is a Banach space having a monotone basis (hence, the metric AP) such that its dual space fails the AP (hence, also the AP with conjugate operators). ∎

It is an open question whether the implications (c)⇒(d) or (b)⇒(c) of Corollary 2.8 can be reversed (see, e.g., [O2, Conjecture 3.5] and [O1, Conjecture 1]). The latter question has a partial positive answer (see Corollary 6.7 below).

3. Isometric factorization, τ_c-continuous functionals, and a proof of Theorem 2.4. To prove Theorem 2.4, we shall employ the famous Davis–Figiel–Johnson–Pełczyński factorization lemma, more precisely, its isometric version due to Lima, Nygaard, and Oja. Let us recall the relevant construction.

Let a be the unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1, \qquad a > 1.$$

Let X and Y be Banach spaces and let K be a closed absolutely convex subset of B_X. For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$. The gauge of B_n gives an equivalent norm $\|\cdot\|_n$ on X. Set

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2},$$

define $X_K = \{x \in X : \|x\|_K < \infty\}$ and $C_K = \{x \in X : \|x\|_K \le 1\}$, and let $J_K : X_K \to X$ denote the identity embedding.

Lemma 3.1 (see [DFJP] and [LNO]). *With the notation as above, the following holds:*

(i) *$X_K = (X_K, \|\cdot\|_K)$ is a Banach space and $\|J_K\| \le 1$.*
(ii) *$K \subset C_K \subset B_X$.*
(iii) *$C_K \subset B_n$ for all $n \in \mathbb{N}$.*
(iv) *$J_K^*(X^*)$ is norm dense in X_K^*.*
(v) *C_K as a subset of X is compact, separable, or weakly compact if and only if K has the same property.*
(vi) *The weak topologies defined by X and X_K coincide on C_K. Hence, X_K is separable or reflexive if and only if K is separable or weakly compact, respectively.*

Suppose $T \in \mathcal{L}(Y,X)$ with $\|T\| = 1$, let $K = \overline{T(B_X)}$, and let $T_K : Y \to X_K$ be defined by $T_K y = Ty$ for $y \in Y$. Then

(vii) $T = J_K \circ T_K$ *with* $\|T\| = \|J_K\| = 1$ *and both* T_K *and* J_K *are separably valued, weakly compact, or compact if and only if* T *has the same property.*

Another ingredient in proving Theorem 2.4 is Grothendieck's description of τ_c-continuous linear functionals on the space $\mathcal{L}(X, Y)$. Recall (see, e.g., [R, pp. 21–22]) that for any element u of the projective tensor product $X \hat{\otimes} Y$ and for every $\varepsilon > 0$, there exists a representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

with $\sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| < \|u\|_\pi + \varepsilon$. Note that one can actually choose a representation, where $\sum_{n=1}^{\infty} \|x_n\| < \|u\|_\pi + \varepsilon$, $\sup_n \|y_n\| \le 1$, and $y_n \to 0$, or vice versa. The *trace functional* on $X^* \hat{\otimes} X$ is defined as

$$\operatorname{trace}(u) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

for $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^* \hat{\otimes} X$ with $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| < \infty$. It is well defined and does not depend on the representation of u.

Lemma 3.2 (see, e.g., [LT, Proposition 1.e.3]). *Let* X *and* Y *be Banach spaces. There is a surjective linear operator* V *from* $Y^* \hat{\otimes} X$ *to the space* $(\mathcal{L}(X, Y), \tau_c)^*$ *of* τ_c*-continuous linear functionals on* $\mathcal{L}(X, Y)$ *defined by*

$$(Vu)(T) = \operatorname{trace}(Tu)$$

for $u \in Y^* \hat{\otimes} X$ *and* $T \in \mathcal{L}(X, Y)$.

Combining Lemma 3.2 with the Hahn–Banach theorem one gets a "sequential" description of the A-AP.

Lemma 3.3 (see [LMO, Lemma 3]). *Let* X *be Banach space and let* $A \subset \mathcal{L}(X)$ *be a convex set. The space* X *has the* A*-AP if and only if for all sequences* $(x_n^*) \subset X^*$ *and* $(x_n) \subset X$ *such that* $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| < \infty$ *one has*

$$\inf_{S \in A} \Big| \sum_{n=1}^{\infty} x_n^*(S x_n - x_n) \Big| = 0.$$

Proof of Theorem 2.4. Necessity follows from the fact that multiplication with a bounded linear operator preserves τ_c-convergence.

Sufficiency. We shall use Lemma 3.3 to show that X has the A-AP. Take a tensor

$$\sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^* \hat{\otimes} X$$

such that $\sum_{n=1}^{\infty} \|x_n\| \le 1$, $x_n^* \to 0$, and $\|x_n^*\| \le 1$ for all $n \in \mathbb{N}$. Let

$$K := \overline{\operatorname{absconv}}(\{x_1^*, x_2^*, \dots\}) \subset B_{X^*}.$$

Since K is compact, by Lemma 3.1, there is a separable reflexive space $Z := (X^*)_K$ and a compact operator $J := J_K : Z \to X^*$ such that $K \subset J(B_Z)$ and $\|J\| = 1$. For every $n \in \mathbb{N}$ there is $z_n \in B_Z$ such that $Jz_n = x_n^*$.

Since $J^* j_X \in \mathcal{K}(X, Z^*)$ and Z^* is separable and reflexive, the assumption gives us a net $(S_\alpha) \subset A$ such that

$$J^* j_X S_\alpha \to J^* j_X \quad \text{in } \tau_c(X, Z^*). \tag{3.1}$$

Observe that

$$\sum_{n=1}^{\infty} \|z_n\| \, \|x_n\| < \infty$$

and consider the tensor $\sum_{n=1}^{\infty} z_n \otimes x_n$ as a τ_c-continuous linear functional f on $\mathcal{L}(X, Z^*)$. Then (3.1) yields

$$\begin{aligned} \inf_{S \in A} \Big| \sum_{n=1}^{\infty} x_n^*(Sx_n - x_n) \Big| &\le \inf_{\alpha} \Big| \sum_{n=1}^{\infty} (J^* j_X (S_\alpha x_n - x_n))(z_n) \Big| \\ &\le \lim_{\alpha} |f(J^* j_X S_\alpha - J^* j_X)| = 0, \end{aligned}$$

as needed. ∎

4. Factorization of operator ideals.

Let $\mathcal{A}$ be an operator ideal and let A be a space ideal.

Definition 4.1. We say that $\mathcal{A}$ is *DFJP-factorizable through* A if $X_K \in \mathsf{A}$ whenever $T \in \mathcal{A}$ in Lemma 3.1 above, or simply *DFJP-factorizable*, when $\mathsf{A} = \mathrm{Space}(\mathcal{A})$.

For examples of DFJP-factorizable operator ideals we refer to [H]. Recall (see, e.g., [S, p. 13] and [H, pp. 398, 404–405]) that $\mathcal{A}$ is *surjective* if, given Banach spaces X_1, X_2, Y and operators $T \in \mathcal{A}(X_1, Y)$ and $T \in \mathcal{L}(X_2, Y)$, the inclusion $S(B_{X_1}) \subset T(B_{X_2})$ implies $S \in \mathcal{A}$; $\mathcal{A}$ is *injective* if, given Banach spaces X, Y_1, Y_2, an operator $T \in \mathcal{L}(X, Y_1)$ and an injection $J \in \mathcal{L}(Y_1, Y_2)$ the inclusion $JT \in \mathcal{A}$ implies $T \in \mathcal{A}$; and $\mathcal{A}$ has *the* $\sum_2$*-property* if, given sequences (X_n) and (Y_n) of Banach spaces and an operator $T \in \mathcal{L}(\ell_2(X_n), \ell_2(Y_n))$ one has $T \in \mathcal{A}$ whenever $i_n T p_m \in \mathcal{A}(X_n, Y_m)$ for all m and n (here $i_m : X_m \to \ell_2(X_n)$ and $p_m : \ell_2(Y_n) \to Y_m$ denote the respective natural injection and natural projection).

The following result is essentially well known (see [H] and [G]). For completeness, we provide a proof for the isometric case.

Proposition 4.2. *Let $\mathcal{A}$ be an operator ideal and let $T \in \mathcal{A}(Y, X)$ be as in Lemma* 3.1 *above.*

(i) *If $\mathcal{A}$ is closed and injective, then $T_K \in \mathcal{A}$.*
(ii) *If $\mathcal{A}$ is closed and surjective, then $J_K \in \mathcal{A}$.*

(iii) *If $\mathcal{A}$ is injective, surjective, and has the $\sum_2$-property, then $X_K \in$* Space($\mathcal{A}$) (*i.e., $\mathcal{A}$ is DFJP-factorizable*).

Proof. (i) Since for $x \in K$ and an integer $N \geq 1$ one has

$$\|x\|_K^2 \leq \sum_{n \leq N} \|x\|_n^2 + \sum_{n > N} \frac{a^n}{(a^n+1)^2},$$

it follows that for any $\varepsilon > 0$ one can find $C > 0$ such that for all $x \in K$ one has

$$\|x\|_K \leq C\|x\| + \varepsilon,$$

which together with [J1, Theorem 20.7.3] implies the claim.

For (ii), combine (iii) of Lemma 3.1 with [J2, Proposition 2.9] (see also [GG, Lemma 2]).

(iii) If $\mathcal{A}$ has the $\sum_2$-property, then it is closed (see [H, p. 405]). Therefore $J_K \in \mathcal{A}$ by (ii). Put $X_n = X_K$ and $Y_n = (X, \|\cdot\|_n)$ for all $n \in \mathbb{N}$, and let $q_1 : \ell_2(X_n) \to X_K$ and $j : X_K \to \ell_2(Y_n)$ denote the respective natural projection and inclusion. Since $q_1 i_n = 0$ if $n \neq 1$, $q_1 i_1 = I_{X_K}$, and $p_n j = J_K \in \mathcal{A}(Y_K, (Y, \|\cdot\|_n))$, the $\sum_2$-property of $\mathcal{A}$ gives $jq_1 \in \mathcal{A}$. Then $I_{X_K} \in \mathcal{A}$ because $\mathcal{A}$ is injective and surjective. ∎

Definition 4.3. We say that an operator ideal $\mathcal{A}$ is *DFJP-surjective* if $J_K \in \mathcal{A}$ whenever $T \in \mathcal{A}$ in Lemma 3.1 above.

Clearly, every closed and surjective operator ideal is DFJP-surjective and every DFJP-surjective operator ideal is surjective.

Whenever $\mathcal{A}$ is DFJP-factorizable through A, one has $\mathcal{A} \subset \mathrm{Op}(\mathsf{A})$. If, in addition, $\mathcal{A}$ is DFJP-surjective, then $\mathcal{A} = \mathcal{A} \circ \mathrm{Op}(\mathsf{A})$; or if $\mathcal{A}$ is closed and injective, then $\mathcal{A} = \mathrm{Op}(\mathsf{A}) \circ \mathcal{A}$. The latter property means that given Banach spaces X, Y and an operator $T \in \mathcal{A}(X, Y)$ there are a Banach space $Z \in \mathsf{A}$ and operators $T_1 \in \mathcal{L}(X, Z)$ and $T_2 \in \mathcal{A}(Z, Y)$ such that $T = T_2 T_1$. Let us also note that if $\mathcal{A}$ is closed and surjective (injective), then $\mathcal{A}^{\mathrm{dual}}$ is closed and injective (surjective).

Example 4.4. See [GG, p. 471] for extra examples and references.

- The following operator ideals are closed and surjective: compact operators $\mathcal{K}$, Grothendieck operators, limited operators, strictly cosingular operators.
- The following operator ideals are closed and injective: compact operators $\mathcal{K}$, completely continuous operators $\mathcal{V}$, weakly Banach–Saks operators, strictly singular operators, Radon–Nikodým operators $\mathcal{RN}$, absolutely continuous operators.
- The following operator ideals are DFJP-factorizable (see [LNO], [H, Theorem 2.3] and Proposition 4.2): finite-rank operators $\mathcal{F}$, separable operators $\mathcal{X}$, weakly compact operators $\mathcal{W}$, Banach–Saks opera-

tors $\mathcal{BS}$, Asplund operators $\mathcal{RN}^{\mathrm{dual}}$, Rosenthal operators $\mathcal{V}^{-1}\circ\mathcal{K}$, and also their intersections. They factor through the space ideals of finite-dimensional spaces $\mathsf{F} = \mathrm{Space}(\mathcal{F})$, separable spaces $\mathsf{X} = \mathrm{Space}(\mathcal{X})$, reflexive spaces $\mathsf{W} = \mathrm{Space}(\mathcal{W})$, Banach–Saks spaces $\mathsf{BS} = \mathrm{Space}(\mathcal{BS})$, Asplund spaces $\mathsf{RN}^{\mathrm{dual}} = \mathrm{Space}(\mathcal{RN})^{\mathrm{dual}}$, and $\mathrm{Space}(\mathcal{V}^{-1}\circ\mathcal{K})$, respectively.
- Other pairs $(\mathcal{A},\mathsf{A})$, where $\mathcal{A}$ is DFJP-factorizable through A, include, for instance, $(\mathcal{K},\mathsf{X}\cap\mathsf{BS})$ and $(\mathcal{K},\mathsf{X}\cap\mathsf{W})$, that is, compact operators are DFJP-factorizable through separable reflexive spaces and through separable Banach–Saks spaces. (Note the strict inclusions $\mathcal{K}\subset\mathcal{X}\cap\mathcal{BS}\subset\mathcal{X}\cap\mathcal{W}$.)

In the following it will be more convenient to use a version of Lemma 3.1 with a restriction on the set K and not on the operator T. To this end, consider the following notion due to [S].

Let $\mathcal{A}$ be an operator ideal. The corresponding *ideal system of sets* $b_{\mathcal{A}}$ is defined as follows: given a Banach space X, a set $K\subset X$ is in $b_{\mathcal{A}}(X)$ if there is a Banach space Y and $T\in\mathcal{A}(Y,X)$ such that $K\subset T(B_Y)$.

The following is an easy observation. Recall that a *bornology* on a Banach space X is a covering of X which respects inclusions and finite unions.

PROPOSITION 4.5. *Let $\mathcal{A}$ be an operator ideal, let A be a space ideal, and let X and Y be Banach spaces. Then $b_{\mathcal{A}}(X)$ is a bornology on X, which respects set sums, multiplication by a scalar, and absolutely convex hulls.*

If $\mathcal{A}$ is surjective, then $T\in\mathcal{A}(X,Y)$ if and only if $T\in\mathcal{L}(X,Y)$ and $T(B_X)\in b_{\mathcal{A}}(Y)$.

If $\mathcal{A}$ is DFJP-surjective, then $b_{\mathcal{A}}$ also respects set closures.

The operator ideal $\mathcal{A}$ is DFJP-surjective (*respectively, DFJP-factorizable through* A) *if and only if $K\in b_{\mathcal{A}}(X)$ implies $J_K\in\mathcal{A}$* (*respectively,* $X_K\in\mathsf{A}$) *in Lemma* 3.1 *above.*

5. Unified approach. In the following, let X be a Banach space, and let $A\subset\mathcal{L}(X)$ be convex and contain 0. Let $\mathcal{A}$ be an operator ideal and let A be a space ideal. Let $\Lambda\subset[1,\infty)$ be non-empty and let τ be one of the topologies τ_c, τ_s, τ_w, or the norm topology $\tau_{\|\cdot\|}$.

For the unified investigation of the strong approximation property and weak bounded approximation property and their flavors consider the following general definition.

DEFINITION 5.1.

- We say that X has the *τ-inner* (*or τ-outer*) *Λ-bounded A-AP for the pair* $(\mathcal{A},\mathsf{A})$ if for every space $Y\in\mathsf{A}$ and for every operator $T\in\mathcal{A}(X,Y)$

there is $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $S_\alpha \to I_X$ (or $TS_\alpha \to T$) in the topology τ.
- If there is no restriction on Banach spaces, i.e., if $\mathsf{A} = \mathsf{L}$ above, we say that X has the corresponding property *for* $\mathcal{A}$.
- If $\tau = \tau_c$ we omit "τ_c-inner" in the definition above. Similarly we just say "outer" in place of "τ_c-outer".
- We say that X has the *Λ-bounded A-AP* if it has the λ-bounded A-AP for some $\lambda \in \Lambda$.
- If $\Lambda = \{\lambda\}$, we replace Λ with λ in the above notions.

The definition above is modelled after the definition of "λ-bounded AP for $\mathcal{B}$" in [LLO1] (see also [O3]), where $\mathcal{B}$ is a Banach operator ideal. Our definition is not consistent with the original Lima–Lima–Oja definition where the approximating sets are given using the operator ideal norm. For simplicity, we use instead the usual operator norm. However, for closed Banach operator ideals these two norms coincide. Therefore, in that case, the definitions are still consistent.

Note also that our "inner" and "outer" terminology is different from those used in [T] or in [O4].

Let us point out some observations relating to Definition 5.1.

- Let $\mathcal{B}$ be an operator ideal such that $\mathcal{B} \subset \mathcal{A}$ and let B be a space ideal such that $\mathsf{B} \subset \mathsf{A}$. Then the τ-inner (or τ-outer) Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ implies the corresponding property for $(\mathcal{B}, \mathsf{B})$
- The weak λ-bounded A-AP is the λ-bounded A-AP for $(\mathcal{K}, \mathsf{X} \cap \mathsf{W})$.
- The τ-outer Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ is simply "outer" if τ is any of the topologies τ_c, τ_s, or τ_w (see the note after Proposition 2.3).
- The strong A-AP is the outer $[1, \infty)$-bounded A-AP for $\mathcal{K}$.
- The Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ implies the A-AP. Indeed, $0 \in \mathcal{A}(X, Y)$ for every space $Y \in \mathsf{A}$, so for some $\lambda \in \Lambda$ the space X has the A_0^λ-AP. But $A_0^\lambda = A$.
- The Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ implies the respective "τ_s-inner" property, which, in turn, implies the respective "outer" property.
- The outer Λ-bounded A-AP for $(\mathcal{K}, \mathsf{X} \cap \mathsf{W})$ implies the A-AP by Theorem 2.4.
- The $\tau_{\|\cdot\|}$-outer (1-bounded) A-AP for $(\mathcal{K}, \mathsf{X} \cap \mathsf{W})$ of X coincides with the A-AP with conjugate operators of X^* by [LMO, Corollary 11].
- The outer Λ-bounded A-AP for $\mathcal{L}$ coincides with the Λ-bounded A-AP. Indeed, take $T = I_X \in \mathcal{L}(X)$ in Definition 5.1; then for any $\lambda \geq 1$ one has
$$A_T^\lambda = \lambda B_{\mathcal{L}(X)} \cap A.$$

Consider the following simple factorization result (see Example 4.4 for its possible applications). Among other things, it yields Proposition 2.3.

Moreover, it says that we can replace "separable reflexive" with "separable Banach–Saks" in Proposition 2.3, Theorem 2.4, and Proposition 2.7.

PROPOSITION 5.2. *Let $\mathcal{A} = \mathrm{Op}(\mathsf{A}) \circ \mathcal{A}$. Then X has the τ-inner (or τ-outer) Λ-bounded A-AP for $\mathcal{A}$ if and only if X has the corresponding property for $(\mathcal{A}, \mathsf{A})$.*

Proof. We only need to prove sufficiency. Let $T \in \mathcal{A}(X,Y)$. By assumption there are $Z \in \mathsf{A}$, $T_1 \in \mathcal{A}(X,Z)$, and $T_2 \in \mathcal{L}(Z,Y)$ such that $T = T_2T_1$. We may assume that $\|T\| = \|T_2\|\|T_1\|$ (see [H, Lemma 1.2]). Clearly, if $(S_\alpha) \subset \mathcal{L}(X)$, then the τ-convergence $T_1S_\alpha \to T_1$ implies the τ-convergence $TS_\alpha \to T$. It remains to observe that $A^\lambda_{T_1} \subset A^\lambda_T$ for any $\lambda \ge 1$. Indeed, if $S \in A$ and $\|T_1S\| \le \lambda\|T_1\|$, then

$$\|TS\| = \|T_2T_1S\| \le \|T_2\| \cdot \lambda\|T_1\| = \lambda\|T\|. \ \blacksquare$$

Next we would like to establish some sufficient conditions, when the Λ-bounded A-AP for $\mathcal{A}$ is actually "outer". Our method (which is an enhanced version of the proof for Theorem 2.4) seems to work only in the case when $\Lambda = \{\lambda\}$. In the following, let $\lambda \ge 1$.

THEOREM 5.3. *Let $\mathcal{A}^{\mathrm{dual}}$ be DFJP-surjective and DFJP-factorizable through $\mathsf{A}^{\mathrm{dual}}$, and let $\mathcal{K} \subset \mathcal{A}^{\mathrm{dual}}$. If X has the outer λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$, then X has the λ-bounded A-AP for $(\mathcal{A}^{\mathrm{dual}})^{\mathrm{dual}}$.*

Proof. Let Y be a Banach space and let $T \in (\mathcal{A}^{\mathrm{dual}})^{\mathrm{dual}}(X,Y)$. We may assume that $\|T\| = 1$. We shall use Lemma 3.3 to show that X has the A^λ_T-AP.

Take a tensor $\sum_{n=1}^\infty x_n^* \otimes x_n \in X^* \hat{\otimes} X$ such that $\sum_{n=1}^\infty \|x_n\| \le 1$, $x_n^* \to 0$, and $\|x_n^*\| \le 1$ for all $n \in \mathbb{N}$. Let

$$K := \overline{\mathrm{absconv}}(\{x_1^*, x_2^*, \dots\} \cup T^*(B_{Y^*})) \subset B_{X^*}.$$

Since K belongs to $b_{\mathcal{A}^{\mathrm{dual}}}(X^*)$ (see Proposition 4.5), by assumption, there is a space $Z := (X^*)_K \in \mathsf{A}^{\mathrm{dual}}$ and an operator $J := J_K : Z \to X^*$ such that $J \in \mathcal{A}^{\mathrm{dual}}$, $K \subset J(B_Z)$, and $\|J\| = 1$. For every $n \in \mathbb{N}$ there is $z_n \in B_Z$ such that $J_K z_n = x_n^*$. Moreover, for the astriction $T_K \in \mathcal{L}(Y^*, Z)$ of T^* to Z, we have $T^* = JT_K$ and $\|T_K\| = 1$.

Since $J^*j_X \in \mathcal{A}(X, Z^*)$ and $Z^* \in \mathsf{A}$, the assumption gives us a net $(S_\alpha) \subset A^\lambda_{J^*j_X}$ such that

$$J^*j_XS_\alpha \to J^*j_X \quad \text{in } \tau_c(X, Z^*). \tag{5.1}$$

Fix α. Note that

$$(J^*j_XS_\alpha)^*j_Z = S_\alpha^*j_X^*J^{**}j_Z = S_\alpha^*j_X^*j_{X^*}J = S_\alpha^*J,$$

so that the inclusion $S_\alpha \in A^\lambda_{J^*j_X}$ implies

$$\|S_\alpha^*J\| = \|(J^*j_XS_\alpha)^*j_Z\| \le \|J^*j_XS_\alpha\| \le \lambda.$$

Therefore,

$$\|TS_\alpha\| = \|S_\alpha^* T^*\| = \|S_\alpha^* J T_K\| \le \lambda.$$

That is, $(S_\alpha) \subset A_T^\lambda$. Observe that

$$\sum_{n=1}^{\infty} \|z_n\| \, \|x_n\| < \infty$$

and consider the tensor $\sum_{n=1}^{\infty} z_n \otimes x_n$ as a τ_c-continuous linear functional f on $\mathcal{L}(X, Z^*)$. Then (5.1) gives

$$\begin{aligned}\inf_{S \in A_T^\lambda} \Big| \sum_{n=1}^{\infty} x_n^*(Sx_n - x_n) \Big| &\le \lim_\alpha \Big| \sum_{n=1}^{\infty} (J^* j_X(S_\alpha x_n - x_n))(z_n) \Big| \\ &= \lim_\alpha |f(J^* j_X S_\alpha - J^* j_X)| = 0,\end{aligned}$$

as needed. ■

Theorem 5.3 can be applied when $\mathcal{A}$ is any closed and injective operator ideal and $\mathsf{A} = \mathsf{L}$, as well as when $\mathcal{A} = \mathcal{RN}$ and $\mathsf{A} = \mathsf{RN}$ because $\mathcal{RN}^{\mathrm{dual}}$ is DFJP-factorizable.

Corollary 5.4. *Let $\mathcal{A}$ be completely symmetric and DFJP-surjective, and let $\mathcal{K} \subset \mathcal{A}$. Assume that*

(i) *$\mathcal{A}$ is DFJP-factorizable through $\mathsf{A}^{\mathrm{dual}}$,*

or

(ii) $\mathcal{A} = \mathrm{Op}(\mathsf{A}) \circ \mathcal{A}$.

Then X has the λ-bounded A-AP for $\mathcal{A}$ if and only if X has the outer λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$.

Corollary 5.4 can be applied to pairs $(\mathcal{A}, \mathsf{A})$ such as $(\mathcal{L}, \mathsf{L})$, $(\mathcal{W}, \mathsf{W})$, $(\mathcal{X} \cap \mathcal{W}, \mathsf{X} \cap \mathsf{W})$, and $(\mathcal{K}, \mathsf{X} \cap \mathsf{W})$; in particular it implies Proposition 2.7.

Corollary 5.5. *A Banach space X has the τ_s-inner λ-bounded A-AP for $\mathcal{F}$ if and only if X has the outer λ-bounded A-AP for $(\mathcal{F}, \mathsf{F})$.*

Proof. In the proof of Theorem 5.3 consider $T \in \mathcal{F}(X, Y)$ and τ_w-continuous functionals (i.e., tensors of the form $\sum_{n=1}^{k} x_n^* \otimes x_n$ for $k \in \mathbb{N}$). Then $K \in b_{\mathcal{F}}(X^*)$, and the rest follows in the same way. ■

Remark 5.6. Every Banach space has the τ_s-inner 1-bounded $\mathcal{F}(X)$-AP for $\mathcal{F}$. Indeed, it easily follows from [O2, Corollary 4.4] that for any Banach spaces X, Y and an operator $T \in \mathcal{F}(X, Y)$ there is a sequence $(S_n) \subset \mathcal{F}(X)$ such that $TS_n \to T$. That is, every Banach space has the $\tau_{\|\cdot\|}$-outer (1-bounded) $\mathcal{F}(X)$-AP for $\mathcal{F}$, and Corollary 5.5 implies the claim.

6. The weak bounded AP and the RNP impact. The impact of the Radon–Nikodým property on the weak bounded AP was discovered by Oja in [O1]. The prototype of the following result is [O1, Theorem 2].

LEMMA 6.1. *Let X and Y be Banach spaces, let $A \subset \mathcal{L}(X)$ be convex and contain 0, and let $\lambda \geq 1$. Let X have the weak λ-bounded A-AP. Let $T \in \mathcal{L}(X, Y)$ be such that $\{TS : S \in A\} \subset \mathcal{K}(X, Y)$. If X^{**} or Y^* has the RNP, then X has the A_T^λ-AP.*

Proof. We may assume that $\|T\| = 1$. We show that X has the $A_T^{\lambda+\delta}$-AP for every $\delta > 0$. The claim would then follow because A is convex and contains 0.

Fix $\delta > 0$, a compact set $C \subset X$, and $\varepsilon > 0$. Define

$$\mathcal{C} = \{TS : S \in A,\ \|Sa - a\| < \varepsilon\ \forall a \in C\} \subset \mathcal{K}(X, Y).$$

We need to show that $\mathcal{C} \cap (\lambda + \delta)B_{\mathcal{K}(X,Y)}$ is not empty. Observe that $\mathcal{C}$ is convex and not empty because X has the A-AP, while $(\lambda + \delta)B_{\mathcal{K}(X,Y)}$ is convex with non-empty interior. Therefore by the Hahn–Banach separation theorem, it remains to show that

$$\inf_{TS \in \mathcal{C}} \Re\varphi(TS) < \sup\{\Re\varphi(R) : R \in \mathcal{K}(X, Y),\ \|R\| \leq \lambda + \delta\} = \lambda + \delta$$

for every $\varphi \in (\mathcal{K}(X, Y))^*$ with $\|\varphi\| = 1$.

Let $\varphi \in \mathcal{K}(X, Y)^*$ with $\|\varphi\| = 1$. Since X^{**} or Y^* has the RNP, from the theorem of Feder and Saphar (see [FS, Theorem 1]), there is $u \in Y^* \hat{\otimes} X^{**}$ such that $\|u\|_\pi = 1$ and

$$\varphi(R) = \operatorname{trace}(R^{**}u)$$

for all $R \in \mathcal{K}(X, Y)$. Pick a representation

$$u = \sum_{n=1}^{\infty} y_n^* \otimes x_n^{**} \in Y^* \hat{\otimes} X^{**}$$

such that $1 \geq \|y_n^*\| \to 0$ and $\sum_{n=1}^\infty \|x_n^{**}\| < 1 + \delta/\lambda$.

Let $K := \overline{\{T^*y_1^*, T^*y_2^*, \dots\}} \subset B_{X^*}$. Since K is compact, by Lemma 3.1, we can construct a separable reflexive Banach space Z, sitting inside X^*, such that the embedding operator $J \in \mathcal{K}(Z, X^*)$ has norm 1, and $K \subset J(B_Z)$. For all $n \in \mathbb{N}$ let $z_n \in B_Z$ be such that $Jz_n = T^*y_n^*$. We have $J^*j_X \in \mathcal{K}(X, Z^*)$. By assumption we can find $S \in A$ such that $\|J^*j_XS\| \leq \lambda$ and $\|Sa - a\| < \varepsilon$ for all $a \in C$. Since Z^* is reflexive, we get

$$\|J^*S^{**}\| = \|J^{***}j_X^{**}S^{**}\| \leq \lambda$$

and

$$|\varphi(TS)| = \Big|\sum_{n=1}^{\infty}(S^{**}x_n^{**})(T^*y_n^*)\Big| = \Big|\sum_{n=1}^{\infty}(J^*S^{**}x_n^{**})(z_n)\Big|$$
$$\le \lambda\sum_{n=1}^{\infty}\|x_n^{**}\| < \lambda\Big(1+\frac{\delta}{\lambda}\Big) = \lambda+\delta,$$

as required. ■

As an immediate consequence of Lemma 6.1 we obtain the next theorem. Recall that $(\mathcal{A}^{-1}\circ\mathcal{K})(X)$ consists of all operators $S\in\mathcal{L}(X)$ such that for every Banach space Y and for every $T\in\mathcal{A}(X,Y)$ one has $TS\in\mathcal{K}(X,Y)$.

THEOREM 6.2. *Let X be a Banach space, let $\mathcal{A}$ be an operator ideal, let A be a convex subset of $(\mathcal{A}^{-1}\circ\mathcal{K})(X)$ containing 0, and let $\lambda\ge 1$. Let X have the weak λ-bounded A-AP. Then:*

(i) *X has the λ-bounded A-AP for $(\mathcal{A},\mathsf{RN}^{\mathrm{dual}})$;*
(ii) *if X^{**} has the RNP, then X has the λ-bounded A-AP for $\mathcal{A}$.*

Apart from the case when A consists of compact operators (see below), Theorem 6.2 can be applied, for instance, when $\mathcal{A}=\mathcal{V}$. Observe that $\mathcal{W}\subset\mathcal{V}^{-1}\circ\mathcal{K}$ (see [P, p. 61]).

In the case when $A\subset\mathcal{K}(X)$, Theorem 6.2 allows us to nicely describe the weak λ-bounded A-AP.

COROLLARY 6.3. *Let X be a Banach space, let A be a convex subset of $\mathcal{K}(X)$ containing 0, and let $\lambda\ge 1$. The following properties are equivalent for X:*

(a) *weak λ-bounded A-AP,*
(b) *outer λ-bounded A-AP for $(\mathcal{K},\mathsf{X}\cap\mathsf{BS})$,*
(c) *λ-bounded A-AP for $\mathcal{K}$.*
(d) *λ-bounded A-AP for $\mathcal{W}$,*
(e) *λ-bounded A-AP for $\mathcal{RN}^{\mathrm{dual}}$.*

Proof. Implication (a)⇒(e) follows from (i) of Theorem 6.2, the fact that $\mathcal{RN}^{\mathrm{dual}}$ factors through $\mathsf{RN}^{\mathrm{dual}}$, and Proposition 5.2; (b)⇔(c) follows from Corollary 5.4, while (e)⇒(d)⇒(c)⇒(a) are obvious. ■

REMARK 6.4. In the case when $A=\mathcal{F}(X)$, the equivalences (a)⇔(c)⇔(d) ⇔ "outer λ-bounded A-AP for $(\mathcal{K},\mathsf{X}\cap\mathsf{W})$" have been established in [LO, Theorem 2.4]. Note that the equivalence (a)⇔(e) answers the following question.

PROBLEM 6.5 (see [LLO1, Problems 5.1 and 5.2]). *Are there larger Banach operator ideals than $\mathcal{W}$ yielding the weak bounded approximation property? Does $\mathcal{RN}^{\mathrm{dual}}$ yield the weak bounded approximation property?*

We would like to point out a related open problem.

PROBLEM 6.6 (see [O3, Problem 5.5]). *Describe the λ-bounded $\mathcal{F}(X)$-AP for $\mathcal{RN}$, $\mathcal{V}$, or $\mathcal{U}$ ($\mathcal{U}$ denotes the operator ideal of unconditionally summing operators).*

The fact that the impact of the Radon–Nikodým property enables one to pass from the weak λ-bounded A-AP to the λ-bounded A-AP was first established in [O1, Corollary 1] for $A = \mathcal{F}(X)$. In [LisO, Theorem 5.1] it was noticed that the same proof actually holds in the case when $A \subset \mathcal{K}(X)$. The following corollary is essentially the latter result.

COROLLARY 6.7. *Let X be a Banach space and let A be a convex subset of $\mathcal{K}(X)$ containing* 0. *If X^* or X^{**} has the RNP, then the weak λ-bounded A-AP and the λ-bounded A-AP are equivalent for X.*

Proof. The case when X^{**} has the RNP follows from Theorem 6.2(ii). The case when X^* has the RNP follows from Lemma 6.1 applied to $T = I_X$. ∎

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Aleksei Lissitsin
Faculty of Mathematics and Computer Science
Tartu University
J. Liivi 2
EE-50409 Tartu, Estonia
E-mail: aleksei.lissitsin@ut.ee

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