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Calderón–Zygmund operators acting on generalized Carleson measure spaces

by

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Abstract. We study Calderón–Zygmund operators acting on generalized Carleson measure spaces $\mathrm{CMO}_r^{\alpha,q}$ and show a necessary and sufficient condition for their boundedness. The spaces $\mathrm{CMO}_r^{\alpha,q}$ are a generalization of BMO, and can be regarded as the duals of homogeneous Triebel–Lizorkin spaces as well.

1. Introduction. To generalize the Hilbert transform and Riesz transforms, Calderón and Zygmund [3, 4] developed a class of singular integral operators which are convolution operators. The L^2-boundedness of such operators follows from the Plancherel theorem. It is well known that Calderón–Zygmund convolution operators are bounded on L^p for $1 < p < \infty$, on Hardy spaces H^p for $0 < p \le 1$, and on BMO as well. However, for non-convolution operators such as the Calderón commutators, the Cauchy integral on Lipschitz curves, the double layer potential on Lipschitz surfaces, the multilinear operators of Coifman and Meyer, new methods have to be developed to obtain L^2 estimates. The remarkable $T1$ theorem given by David and Journé [7] provides a general criterion for the L^2-boundedness of these generalized singular integral operators (cf. [1, 2, 5, 6]). In recent years, the boundedness of Calderón–Zygmund operators on other function spaces such as Hardy spaces, Sobolev spaces, Besov spaces, and Triebel–Lizorkin spaces has been obtained by many authors; see [9, 13, 14, 21, 24, 26] for example.

The purpose of this article is to study the theory of Calderón–Zygmund on generalized Carleson measure spaces $\mathrm{CMO}_r^{\alpha,q}$ which are a generalization of BMO. We begin by recalling some basic results about Calderón–Zygmund operator theory. As usual, we denote by $\mathscr{D}$ the set of C^∞ functions with compact support.

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We say that T is a *singular integral operator*, denoted by $T \in \mathrm{SIO}(\varepsilon)$, if T is a continuous linear operator from $\mathscr{D}(\mathbb{R}^n)$ into its dual associated to a kernel $K(x,y)$, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}$, satisfying the following conditions: there exist constants $C>0$ and $0<\varepsilon\le 1$ such that

$$|K(x,y)| \le C|x-y|^{-n} \qquad \text{for all } x \neq y, \tag{1.1}$$

$$|K(x,y)-K(x',y)| \le C|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon} \tag{1.2}$$

for all x, x' and y in $\mathbb{R}^n$ with $|x-x'| \le |x-y|/2$, and

$$|K(x,y)-K(x,y')| \le C|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon} \tag{1.3}$$

for all y, y' and x in $\mathbb{R}^n$ with $|y-y'| \le |x-y|/2$. Moreover, the operator T can be represented by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)\,dy\,dx$$

for all $f, g \in \mathscr{D}(\mathbb{R}^n)$ with $\mathrm{supp}(f)\cap\mathrm{supp}(g)=\emptyset$. We say that T is a *Calderón–Zygmund operator*, denoted by $T \in \mathrm{CZO}(\varepsilon)$, if $T \in \mathrm{SIO}(\varepsilon)$ and is bounded on L^2.

Let C_0^η denote the space of continuous functions f with compact support such that

$$\|f\|_\eta := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\eta} < \infty.$$

Let $T : C_0^\eta \to (C_0^\eta)'$, $\eta > 0$, be a continuous linear operator. We say that T has the *weak boundedness property*, denoted by $T \in \mathrm{WBP}$, if, for each $\eta > 0$, there is a constant $C > 0$ such that, for all cubes Q with diameter at most $t > 0$ and all $f, g \in C_0^\eta$ supported in Q,

$$|\langle Tf, g\rangle| \le Ct^{n+2\eta}\|f\|_\eta\|g\|_\eta.$$

Next we recall the definitions of homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$. We say that a cube $Q \subseteq \mathbb{R}^n$ is *dyadic* if $Q = Q_{j\mathbf{k}}$ is defined by $\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : 2^{-j}k_i \le x_i < 2^{-j}(k_i+1),\ i=1,\dots,n\}$ for some $j\in\mathbb{Z}$ and $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n$. Denote by $\ell(Q)=2^{-j}$ the side length of Q and $x_Q = 2^{-j}\mathbf{k}$ the "lower left corner" of Q when $Q=Q_{j\mathbf{k}}$. We use $\sup_Q$ and $\sum_Q$ for the supremum and summation over all dyadic cubes Q, respectively. Also, we denote the summation over all dyadic cubes Q contained in P by $\sum_{Q\subseteq P}$. For any dyadic cubes P and Q, either P and Q are non-overlapping or one contains the other. For any function f defined on $\mathbb{R}^n$, $j\in\mathbb{Z}$ and dyadic cube $Q = Q_{j\mathbf{k}}$, set

$$\begin{aligned} f_Q(x) &= |Q|^{-1/2} f\left(\frac{x - x_Q}{\ell(Q)}\right) = 2^{jn/2} f(2^j x - \mathbf{k}), \\ f_j(x) &= 2^{jn} f(2^j x), \\ \widetilde{f}(x) &= \overline{f(-x)}. \end{aligned}$$

It is clear that $\widetilde{g}_j * f(x_Q) = |Q|^{-1/2}\langle f, g_Q\rangle$, where $\langle f, g\rangle$ denotes the paring in the usual sense for g in a Fréchet space X and f in the dual of X.

Choose a fixed function φ in the Schwartz class $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$, the collection of rapidly decreasing C^∞ functions on $\mathbb{R}^n$, satisfying

$$\text{(1.4)} \qquad \begin{cases} \operatorname{supp}(\widehat{\varphi}) \subseteq \{\xi : 1/2 \le |\xi| \le 2\}, \\ |\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if } 3/5 \le |\xi| \le 5/3. \end{cases}$$

For $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, we say that f belongs to the *homogeneous Triebel–Lizorkin space* $\dot{F}_p^{\alpha,q}$ if $f \in \mathscr{S}'/\mathscr{P}$, the tempered distributions modulo polynomials, satisfies $\|f\|_{\dot{F}_p^{\alpha,q}} < \infty$, where

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \Big\|\Big\{\sum_{k\in\mathbb{Z}} (2^{k\alpha}|\varphi_k * f|)^q\Big\}^{1/q}\Big\|_{L^p} \quad \text{for } 0 < p < \infty$$

and

$$\|f\|_{\dot{F}_\infty^{\alpha,q}} := \sup_P \Big\{|P|^{-1}\int_P \sum_{k=-\log_2 \ell(P)}^{\infty} (2^{k\alpha}|\varphi_k * f(x)|)^q dx\Big\}^{1/q}.$$

When $0 < p < \infty$ and $q = \infty$, the above ℓ^q-norm is modified to be the supremum norm as usual, and $\dot{F}_\infty^{\alpha,\infty}$ is defined to be $\dot{B}_\infty^{\alpha,\infty}$, that is,

$$\|f\|_{\dot{F}_\infty^{\alpha,\infty}} := \sup_{k\in\mathbb{Z}} \sup_{\substack{\ell(Q)=2^{-k} \\ x\in Q}} 2^{k\alpha}|\varphi_k * f(x)| \approx \sup_Q |Q|^{-\alpha/n-1/2}|\langle f, \varphi_Q\rangle| < \infty.$$

Here we summarize some results on singular integral operators bounded on $\dot{F}_p^{\alpha,q}$.

Proposition 1.1 ([13, 21, 24]). *Given $\varepsilon \in (0,1]$, $|\alpha| < \varepsilon$,*

$$\max\Big\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\Big\} < p < \infty \quad \textit{and} \quad \max\Big\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\Big\} < q \le \infty,$$

if $T \in \mathrm{SIO}(\varepsilon)$, $T \in \mathrm{WBP}$ and $T(1) = T^*(1) = 0$, then T extends to a bounded operator on $\dot{F}_p^{\alpha,q}$.

This article is motivated by [16, 17]. To show the $\mathrm{CMO}_r^{\alpha,q}$-boundedness of Calderón–Zygmund operators, we cannot apply the duality argument directly. In [16], the authors introduced a weak density property to overcome this difficulty. In this article, we adopt the same method to deal with the boundedness of Calderón–Zygmund operators acting on $\mathrm{CMO}_r^{\alpha,q}$. We now recall the space $\mathrm{CMO}_r^{\alpha,q}$ introduced in [22].

Definition 1.2. Let $\varphi \in \mathscr{S}$ satisfy (1.4). For $\alpha, r \in \mathbb{R}$ and $0 < q \le \infty$, the *generalized Carleson measure space* $\mathrm{CMO}_r^{\alpha,q}$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}$ satisfying $\|f\|_{\mathrm{CMO}_r^{\alpha,q}} < \infty$, where

$$\|f\|_{\mathrm{CMO}_r^{\alpha,q}} := \sup_P \Big\{ |P|^{-r} \int_P \sum_{Q \subseteq P} \big(|Q|^{-\alpha/n-1/2} |\langle f, \varphi_Q\rangle| \chi_Q(x)\big)^q \, dx \Big\}^{1/q}$$

for $0 < q < \infty$ and

$$\|f\|_{\mathrm{CMO}_r^{\alpha,\infty}} := \sup_P \sup_{Q \subseteq P} |Q|^{-\alpha/n-1/2} |\langle f, \varphi_Q\rangle| = \sup_Q |Q|^{-\alpha/n-1/2} |\langle f, \varphi_Q\rangle|.$$

As usual, χ_Q denotes the characteristic function of Q.

Remark 1.3. It follows from [23, p. 154, Theorem 4] that $\mathrm{CMO}_1^{0,2} = \mathrm{BMO}$, and hence $\mathrm{CMO}_r^{\alpha,q}$ is a generalization of BMO. For $\alpha = 0$, $q = 2$ and $r = 2/p - 1$, the space $\mathrm{CMO}_{2/p-1}^{0,2}$ reduces to the Carleson measure space CMO^p which was first introduced in [18] for the multiparameter Hardy spaces associated with flag singular integrals, and in [20] for the wavelet characterization of the weighted Carleson measure space CMO_w^p with constant weight function.

We will prove the following main result of this article.

Theorem 1.4. *Let T be a Calderón–Zygmund operator with regularity exponent ε and $T^*(1) = 0$. Suppose $-\varepsilon < \alpha < \varepsilon$ and*

$$\max\Big\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\Big\} < p \le 1 < q < \infty.$$

Set $r = q/p - q/q'$. If $T(1) = 0$, then T can be extended to a bounded operator from $\mathrm{CMO}_r^{\alpha,q}$ to itself, and moreover there exists a constant $C > 0$ independent of f such that

$$\|T(f)\|_{\mathrm{CMO}_r^{\alpha,q}} \le C \|f\|_{\mathrm{CMO}_r^{\alpha,q}} \quad \text{for all } f \in \mathrm{CMO}_r^{\alpha,q}.$$

Conversely, for (i) $-\varepsilon < \alpha < 0$ *and* $1 < q < \infty$ *or* (ii) $\alpha = 0$ *and* $2 \le q < \infty$, *if T admits a bounded extension from $\mathrm{CMO}_r^{\alpha,q}$ to itself, then $T(1) = 0$.*

Remark 1.5. Recently Lee [19] obtained the boundedness of Riesz transforms acting on CMO_w^p. The key idea of his proof is that an almost orthogonality estimate can be applied to convolution operators, and hence the assumption $T(1) = T^*(1) = 0$ is not used explicitly. In practice $T(1) = T^*(1) = 0$ holds for any convolution operator T. We do not use the almost orthogonality estimate and consider more general non-convolution operators here. Lee's result is essentially a special case of Theorem 1.4 whenever the weight function w is constant, and our approach is easier.

The article is organized as follows. In Section 2, we state two key lemmas and use them to show our main result. Then we give the proofs of the key lemmas in Section 3. Throughout, we use C to denote a universal constant which does not depend on the main variables but may differ from line to line. Also, Q and P always mean dyadic cubes in $\mathbb{R}^n$.

2. Proof of the main result. Define a linear map S_φ from $\mathscr{S}'/\mathscr{P}$ into the family of complex sequences by

$$S_\varphi(f) = \{\langle f, \varphi_Q\rangle\}_Q.$$

Let

$$\mathscr{S}_\infty = \Big\{ f \in \mathscr{S} : \int x^{\mathbf{k}} f(x)\,dx = 0 \text{ for all } \mathbf{k} \in (\mathbb{N}\cup\{0\})^n \Big\}.$$

For $g \in \mathrm{CMO}_p^{-\alpha,q'}$, define a linear functional L_g by

$$L_g(f) = \langle S_\psi(g), S_\varphi(f)\rangle = \sum_Q \langle g, \psi_Q\rangle \langle f, \varphi_Q\rangle \quad \text{for } f \in \mathscr{S}_\infty. \tag{2.1}$$

The following duality result can be found in [22].

PROPOSITION 2.1. *Suppose $\alpha \in \mathbb{R}$, $0 < p \le 1$ and $1 < q < \infty$. Then the dual of $\dot F_p^{\alpha,q}$ is $\mathrm{CMO}_{q'/p-q'/q}^{-\alpha,q'}$ in the following sense.*

- (i) *For $g \in \mathrm{CMO}_{q'/p-q'/q}^{-\alpha,q'}$, the linear functional L_g given by (2.1), defined initially on $\mathscr{S}_\infty$, extends to a continuous linear functional on $\dot F_p^{\alpha,q}$ with $\|L_g\| \le C\|g\|_{\mathrm{CMO}_{q'/p-q'/q}^{-\alpha,q'}}$.*
- (ii) *Conversely, every continuous linear functional L on $\dot F_p^{\alpha,q}$ satisfies $L = L_g$ for some $g \in \mathrm{CMO}_{q'/p-q'/q}^{-\alpha,q'}$ with $\|g\|_{\mathrm{CMO}_{q'/p-q'/q}^{-\alpha,q'}} \le C\|L\|$.*

In order to prove the main theorem, we need the following two lemmas.

LEMMA 2.2. *Let $\varepsilon \in (0,1]$. For $-\varepsilon < \alpha < \varepsilon$ and $\max\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\} < p \le 1 < q < \infty$, let $r = q/p - q/q'$. Then the space $\mathrm{CMO}_r^{\alpha,q} \cap L^2$ is dense in $\mathrm{CMO}_r^{\alpha,q}$ in the weak topology $(\dot F_p^{-\alpha,q'}, \mathrm{CMO}_r^{\alpha,q})$. More precisely, for $f \in \mathrm{CMO}_r^{\alpha,q}$, there exists a sequence $\{f_m\} \subseteq \mathrm{CMO}_r^{\alpha,q} \cap L^2$ such that $\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}$ and for $g \in \dot F_p^{-\alpha,q'} \cap L^2$, $\lim_{m\to\infty}\langle f_m, g\rangle = \langle f, g\rangle$, where the constant C is independent of m and f.*

LEMMA 2.3. *Let T be a Calderón–Zygmund operator with regularity exponent ε and $T(1) = T^*(1) = 0$. For $-\varepsilon < \alpha < \varepsilon$ and $\max\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\} < p \le 1 < q < \infty$, let $r = q/p - q/q'$. Then for $f \in \mathrm{CMO}_r^{\alpha,q} \cap L^2$, there exists a constant $C > 0$ independent of f such that*

$$\|T(f)\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}.$$

Assuming these two lemmas for the moment, let T be a Calderón–Zygmund operator with regularity exponent ε and $T(1) = 0$. (Note that we have the assumption $T^*(1) = 0$.) We first define $T(f)$ for $f \in \mathrm{CMO}_r^{\alpha,q}$ as follows. Given $f \in \mathrm{CMO}_r^{\alpha,q}$, by Lemma 2.2 there exists a sequence $\{f_m\}$ in $\mathrm{CMO}_r^{\alpha,q} \cap L^2$ such that $\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}$ and, for $g \in \dot{F}_p^{-\alpha,q'} \cap L^2$, $\langle f_m, g\rangle \to \langle f, g\rangle$ as $m \to \infty$. Thus, for $f \in \mathrm{CMO}_r^{\alpha,q}$, define

$$\langle T(f), g\rangle = \lim_{m\to\infty} \langle T(f_m), g\rangle \quad \text{for } g \in \mathrm{CMO}_r^{\alpha,q} \cap L^2.$$

To see the existence of this limit, we write $\langle T(f_j - f_k), g\rangle = \langle f_j - f_k, T^*(g)\rangle$ since $f_j - f_k$ and g both belong to L^2, and both T, T^* are bounded on L^2. By Proposition 1.1, T^* is bounded on $\dot{F}_p^{-\alpha,q'}$ since $(T^*)^* = T$ and hence $(T^*)^*(1) = 0$. Therefore $T^*(g) \in \dot{F}_p^{-\alpha,q'} \cap L^2$. Consequently, by Lemma 2.2 again, $\langle f_j - f_k, T^*(g)\rangle$ tends to zero as $j, k \to \infty$. It is also easy to see that the above definition of $T(f)$ is independent of the choice of the sequence $\{f_m\}$ satisfying the condition in Lemma 2.2.

We now conclude the proof of Theorem 1.4 as follows. We first prove the "if" part. Given $f \in \mathrm{CMO}_r^{\alpha,q}$, by Lemma 2.2 we choose a sequence $\{f_m\}$ in $\mathrm{CMO}_r^{\alpha,q} \cap L^2$ such that $\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}$ and $\langle T(f), g\rangle = \lim_{m\to\infty}\langle T(f_m), g\rangle$ for $g \in \dot{F}_p^{-\alpha,q'} \cap L^2$. In particular, taking for g the function φ_Q as in Definition 1.2, we get $\langle T(f), \varphi_Q\rangle = \lim_{m\to\infty}\langle T(f_m), \varphi_Q\rangle$, and hence Fatou's lemma implies, for each dyadic cube P in $\mathbb{R}^n$,

$$\begin{aligned}\Big\{|P|^{-r} \int_P \sum_{Q\subseteq P} \big(|Q|^{-\alpha/n-1/2}|\langle T(f), \varphi_Q\rangle|\chi_Q(x)\big)^q\,dx\Big\}^{1/q} \\ \le \liminf_{m\to\infty} \Big\{|P|^{-r} \int_P \sum_{Q\subseteq P} \big(|Q|^{-\alpha/n-1/2}|\langle T(f_m), \varphi_Q\rangle|\chi_Q(x)\big)^q\,dx\Big\}^{1/q}.\end{aligned}$$

This yields

$$\begin{aligned}\|T(f)\|_{\mathrm{CMO}_r^{\alpha,q}} &\le \liminf_{m\to\infty} \|T(f_m)\|_{\mathrm{CMO}_r^{\alpha,q}} \le C \liminf_{m\to\infty} \|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \\ &\le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}},\end{aligned}$$

where the second and the last inequalities follow from Lemmas 2.3 and 2.2, respectively.

To show the "only if" part, we assume that $T \in \mathrm{CZO}(\varepsilon)$ is bounded on $\mathrm{CMO}_r^{\alpha,q}$ for $-\varepsilon < \alpha < 0$ and $1 < q < \infty$. Then T^* is bounded on L^2 and $\dot{F}_p^{-\alpha,q'}$, the predual of $\mathrm{CMO}_r^{\alpha,q}$. Now for $g \in \mathscr{S}_\infty$, we have $g \in L^2 \cap \dot{F}_p^{-\alpha,q'}$ and $T^*(g) \in L^2 \cap \dot{F}_p^{-\alpha,q'}$ since T is bounded on L^2 and on $\mathrm{CMO}_r^{\alpha,q}$. By the embedding theorem (cf. [25, p. 129] or [26, Theorem 2.4]), $\dot{F}_p^{-\alpha,q'} \hookrightarrow \dot{F}_{p_1}^{0,2} = H^{p_1}$, where $-\varepsilon < \alpha < 0$ and $\alpha + n/p = n/p_1$. Thus, $T^*(g) \in L^2 \cap H^{p_1}$ and hence (cf. [15, Corollary 1.4])

$$0 = \int_{\mathbb{R}^n} T^*(g)(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(y)K(y,x)\,dy\,dx = \langle T(1), g\rangle \tag{2.2}$$

for all $g \in \mathscr{S}_\infty$, which implies $T(1) = 0$. For the case $\alpha = 0$ and $2 \le q < \infty$, we use the embedding $\dot F_p^{0,q'} \hookrightarrow \dot F_p^{0,2} = H^p$ to obtain (2.2) again. This completes the proof of Theorem 1.4.

3. Proofs of the key lemmas. Let us recall a result on the φ-transform introduced by Frazier and Jawerth in [10, 11, 12].

LEMMA 3.1. *Suppose $\varphi \in \mathscr{S}(\mathbb{R}^n)$ satisfies* (1.4). *Then there exists a function $\psi \in \mathscr{S}(\mathbb{R}^n)$ satisfying* (1.4) *such that*

$$f = \sum_Q \langle f, \varphi_Q\rangle \psi_Q,$$

where the series converges in $L^2(\mathbb{R}^n)$, $\mathscr{S}_\infty(\mathbb{R}^n)$, and $(\mathscr{S}_\infty)'(\mathbb{R}^n)$.

For $m \in \mathbb{N}$, let

$$\mathscr{Q}_m = \{Q \text{ dyadic cube in } \mathbb{R}^n : 2^{-m} \le \ell(Q) \le 2^m \text{ and } Q \subseteq [-2^m, 2^m]^n\}.$$

For $f \in \mathrm{CMO}_r^{\alpha,q}$, set

$$f_m = \sum_{Q\in\mathscr{Q}_m} \langle f, \varphi_Q\rangle \psi_Q. \tag{3.1}$$

Note that the cardinality of $\mathscr{Q}_m$ is finite and hence $f_m \in L^2$.

LEMMA 3.2. *Let $f \in \mathrm{CMO}_r^{\alpha,q}$ and f_m be given by* (3.1). *Then $f_m \in \mathrm{CMO}_r^{\alpha,q}$ and*

$$\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}},$$

where the constant C is independent of m and f.

Proof. It follows from [22, Lemma 3.2] that any almost diagonal operator is bounded on $\mathrm{CMO}_r^{\alpha,q}$. Hence

$$\begin{aligned}\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} &= \sup_P \Big\{|P|^{-r} \int_P \sum_{Q\subseteq P} \big(|Q|^{-\alpha/n-1/2}|\langle f_m, \varphi_Q\rangle|\chi_Q(x)\big)^q\,dx\Big\}^{1/q}\\ &= \|A_m(f)\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}},\end{aligned}$$

where A_m is the almost diagonal operator defined by

$$\begin{aligned}A_m(g) &= \sum_{Q\in\mathscr{Q}_m} \Big\langle \sum_R \langle g, \varphi_R\rangle\psi_R, \varphi_Q\Big\rangle \psi_Q\\ &= \sum_{Q\in\mathscr{Q}_m} \sum_R \langle \psi_R, \varphi_Q\rangle\langle g, \varphi_R\rangle\psi_Q.\end{aligned}$$

Hence A_m is bounded on $\mathrm{CMO}_r^{\alpha,q}$ and its norm is dominated by the norm of A defined by

$$A(g) = \sum_Q \sum_R \langle \psi_R, \varphi_Q \rangle \langle g, \varphi_R \rangle \psi_Q.$$

This shows that the constant C is independent of m and f. ∎

We now apply Lemmas 3.1 and 3.2, and Proposition 2.1, to prove Lemma 2.2.

Proof of Lemma 2.2. For each $h \in \mathscr{S}$, by (3.1),

$$\langle f - f_m, h \rangle = \Big\langle \sum_{Q \notin \mathscr{Q}_m} \langle f, \varphi_Q \rangle \psi_Q, h \Big\rangle = \Big\langle f, \sum_{Q \notin \mathscr{Q}_m} \langle h, \psi_Q \rangle \varphi_Q \Big\rangle.$$

Lemma 3.1 implies that

$$\sum_{Q \notin \mathscr{Q}_m} \langle h, \psi_Q \rangle \varphi_Q$$

tends to zero in $\mathscr{S}_\infty$ as $m \to \infty$. Since $\mathscr{S}_\infty$ is dense in $\dot{F}_p^{-\alpha,q'}$, this implies that for each $g \in \dot{F}_p^{-\alpha,q'}$, $\langle f - f_m, g \rangle$ tends to 0 as $m \to \infty$. Indeed, for any given $\varepsilon > 0$, there exists $h \in \mathscr{S}_\infty$ such that $\|g - h\|_{\dot{F}_p^{-\alpha,q'}} < \varepsilon$. By Proposition 2.1 first and then Lemma 3.2 (i.e. $\|f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \le C \|f\|_{\mathrm{CMO}_r^{\alpha,q}}$),

$$\begin{aligned} |\langle f - f_m, g \rangle| &\le |\langle f - f_m, g - h \rangle| + |\langle f - f_m, h \rangle| \\ &\le C \|f - f_m\|_{\mathrm{CMO}_r^{\alpha,q}} \|g - h\|_{\dot{F}_p^{-\alpha,q'}} + |\langle f - f_m, h \rangle| \\ &\le C\varepsilon \|f\|_{\mathrm{CMO}_r^{\alpha,q}} + |\langle f - f_m, h \rangle|, \end{aligned}$$

which implies $\langle f - f_m, g \rangle \to 0$ as $m \to \infty$. ∎

Lemma 2.3 follows from Propositions 1.1 and 2.1:

Proof of Lemma 2.3. Let $f \in \mathrm{CMO}_r^{\alpha,q} \cap L^2$ and $g \in \dot{F}_p^{-\alpha,q'} \cap L^2$. By Propositions 2.1 and 1.1 (i.e. T^* is bounded on $\dot{F}_p^{-\alpha,q'}$ for $\max\{\frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha}\} < p \le 1$),

$$\begin{aligned} |\langle T(f), g \rangle| = |\langle f, T^*(g) \rangle| &\le \|f\|_{\mathrm{CMO}_r^{\alpha,q}} \|T^*(g)\|_{\dot{F}_p^{-\alpha,q'}} \\ &\le C \|f\|_{\mathrm{CMO}_r^{\alpha,q}} \|g\|_{\dot{F}_p^{-\alpha,q'}}. \end{aligned}$$

This implies that for each $f \in \mathrm{CMO}_r^{\alpha,q} \cap L^2$, $\ell_f(g) = \langle T(f), g \rangle$ is a continuous linear functional on $\dot{F}_p^{-\alpha,q'} \cap L^2$. Note that $\dot{F}_p^{-\alpha,q'} \cap L^2$ is dense in $\dot{F}_p^{-\alpha,q'}$. Thus, $\ell_f(g) = \langle T(f), g \rangle$ belongs to the dual of $\dot{F}_p^{-\alpha,q'}$, and the norm of this linear functional is dominated by $C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}$. By Proposition 2.1 again, there exists $h \in \mathrm{CMO}_r^{\alpha,q}$ such that $\langle T(f), g \rangle = \langle h, g \rangle$ for $g \in \mathscr{S}_\infty$ and $\|h\|_{\mathrm{CMO}_r^{\alpha,q}} \le \|\ell_f\| \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}$. In particular, taking for g the function φ_Q as in Definition 1.2, we conclude that $\langle T(f), \varphi_Q \rangle = \langle h, \varphi_Q \rangle$ for each Q.

Therefore,

$$\begin{aligned}\|T(f)\|_{\mathrm{CMO}_r^{\alpha,q}} &:= \sup_P \Big\{|P|^{-r}\int_P \sum_{Q\subseteq P} \big(|Q|^{-\alpha/n-1/2}|\langle T(f),\varphi_Q\rangle|\chi_Q(x)\big)^q dx\Big\}^{1/q}\\ &= \sup_P \Big\{|P|^{-r}\int_P \sum_{Q\subseteq P} \big(|Q|^{-\alpha/n-1/2}|\langle h,\varphi_Q\rangle|\chi_Q(x)\big)^q dx\Big\}^{1/q}\\ &= \|h\|_{\mathrm{CMO}_r^{\alpha,q}} \le C\|f\|_{\mathrm{CMO}_r^{\alpha,q}}. \ \blacksquare\end{aligned}$$

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