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On a problem posed by M. M. Popov

by

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Abstract. We show that if X is a non-locally convex quasi-Banach space with a rich dual, there exists a continuous function $f : [0,1] \to X$ failing to have a primitive. This answers a twenty year-old question raised by M. M. Popov in this journal.

1. Introduction. When X is a Banach space, every continuous function $f : [a,b] \to X$ is Riemann-integrable and the corresponding integral function $F(t) = \int_a^t f$ is a primitive of f. The situation changes dramatically when X is a non-locally convex F-space. Indeed, by an old result of Mazur and Orlicz [9] we know that there exist continuous functions on $[a,b]$ mapping into X which fail to be Riemann-integrable and so the usual way of getting primitives for integrable functions may break down in this setting.

It is therefore natural to ask whether every continuous function from a compact interval of the real line into a given F-space will have a primitive. This question was formally raised by M. M. Popov almost twenty years ago in [10]. Two years later, in a paper of classical elegance, N. Kalton gave a positive partial answer by showing that if X is quasi-Banach space with trivial dual (like the spaces $L_p[0,1]$ for $0 < p < 1$) then every continuous function $f : [a,b] \to X$ has a primitive [5].

In this note we analyze what happens on the opposite side of the spectrum. We prove that if X belongs to a wide class of quasi-Banach spaces that includes those with separating dual, then there is a continuous function $f : [0,1] \to X$ failing to have a primitive. This applies in particular to the non-existence of primitives for continuous functions mapping into the spaces ℓ_p for $0 < p < 1$, a case that was recently settled in [2].

We refer the reader to [10, 5, 2] for background and to [7, 11] for the needed terminology and notation on quasi-Banach spaces and F-spaces.

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2. Preliminaries. Unlike for Banach spaces, the fundamental theorem of calculus does not hold for continuous functions $f : [a, b] \to X$ mapping into a non-locally convex quasi-Banach space (see [3]). This "pathology" motivates the study of sufficient conditions on f that guarantee its integrability, either in the sense of Riemann or in the sense of Vogt. The construction of the Riemann integral for F-spaces is classical and appears in [11, 10]. The specifics on Vogt integrability can be found in [18, 8] (see also [3, Section 5] for a fast reminder of the construction of this integral designed for p-normed spaces). Even in the case when f is integrable, differentiating the integral function $F(t) = \int_a^t f$ is not a trivial question. The important point to note here is that the differentiation of functions taking values in a non-locally convex space faces many issues from a very early stage, amongst which we highlight the failure of the mean value property (see [1]). This fact opens the door to the existence of functions with continuous derivative that are not Lipschitz [2] and leads to consider the space $\mathcal{C}^{(1)}([a, b], X)$. Since we make use of this ingredient throughout, with more prominence in Section 4, the first subsection is meant to recall the particularities of this class of functions in the setting of quasi-Banach spaces.

Our answer in the negative to Popov's question relies on the ability to rig the method that we employed in [2] for constructing functions from the unit interval of the real line into quasi-Banach spaces. §2.3 provides an exposition of this machinery, enhanced for better performance. As it turned out, proving the non-existence of primitives for the general case required less effort and ended up being less technical than for functions mapping into ℓ_p for $0 < p < 1$. The simplification is due to an alternative approach that uses the notion of galb of an F-space, a tool introduced by Turpin in 1971. For the convenience of the reader we have gathered the main properties of these objects in §2.2.

Here and subsequently X will be an infinite-dimensional real quasi-Banach space. Recall that a *quasi-norm* on X is a map $\|\cdot\| : X \to [0, \infty)$ satisfying the axioms of a norm except for the triangle law, which is replaced with the condition

$$\|x + y\| \leq \kappa(\|x\| + \|y\|), \qquad \forall x, y \in X,$$

for some constant $\kappa \geq 1$ independent of x and y. In particular, if $\kappa = 1$ we obtain a norm. A quasi-norm $\|\cdot\|$ is called a *p-norm* ($0 < p < 1$) if it is *p-subadditive*, that is,

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \qquad \forall x, y \in X.$$

In this case X is said to be a *p-convex quasi-Banach space* (*p-Banach space* for short). Although what follows applies to functions $f : [a, b] \to X$ defined on any compact interval $[a, b]$ of the real line, without loss of generality and for the sake of simplicity we will work on the unit interval $I = [0, 1]$.

2.1. The space $\mathcal{C}^{(1)}(I, X)$ when X is quasi-Banach. Following Kalton [5], we will denote by $\mathcal{C}^{(1)}(I, X)$ the space of all functions $F : I \to X$ for which there exists $f : I \to X$ such that the function $S(F, f) : I^2 \to X$ defined by

$$S(F,f)(s,t) = \begin{cases} \dfrac{F(s) - F(t)}{s - t} & \text{for } s \neq t, \\ f(t) & \text{for } t = s, \end{cases}$$

is continuous. It is straightforward to verify that $\mathcal{C}^{(1)}(I, X)$ is a quasi-Banach space under the quasi-norm

$$\|F\|_{\mathcal{C}^1} = \|F(0)\| + \sup_{0 \le s < t \le 1} \frac{\|F(t) - F(s)\|}{t - s}.$$

Clearly, if F and f are as above it follows that F is differentiable, f is continuous, and $F'(t) = f(t)$ for all $t \in I$.

We are aware that this notation can be misleading since, in contrast to the case when X is a Banach space, the space of continuously differentiable functions from I into a quasi-Banach space X is contained in but not equal to $\mathcal{C}^{(1)}(I, X)$. Indeed, a function in $\mathcal{C}^{(1)}(I, X)$ in particular is Lipschitz on I, but as the authors showed in [2, Theorem 4.1] there are differentiable functions with continuous derivative mapping from I into a quasi-Banach space which fail to be Lipschitz!

Kalton brought into play the space $\mathcal{C}^{(1)}(I, X)$ to give a positive partial answer to Popov's question by proving that if X is a quasi-Banach space with trivial dual then every continuous function $f : I \to X$ has a primitive that belongs to $\mathcal{C}^{(1)}(I, X)$.

2.2. A brief account on galbs of F-spaces. The notion of galb was introduced and developed by Turpin in a series of papers (cf. [14, 15]) and a monograph ([13]) in the early 1970's. In this subsection we touch only a few aspects of the theory and summarize without proofs the properties that are relevant to our work.

Suppose that X is an F-space. The *galb* of X, here denoted by $\mathcal{G}(X)$, is defined to be the vector space of all sequences $(\lambda_k)_{k=1}^{\infty}$ of real numbers that have the following topological property: for any 0-neighborhood $\mathcal{U}$ there is another 0-neighborhood $\mathcal{V}$ so that $(x_k)_{k=1}^{N} \in \mathcal{V}^N$ implies $\sum_{k=1}^{N} \lambda_k x_k \in \mathcal{U}$. In other words, $(\lambda_k)_{k=1}^{\infty}$ belongs to $\mathcal{G}(X)$ if and only if the linear operator from $c_{00}(X)$ (endowed with the topology of uniform convergence) into X given by $(x_k)_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \lambda_k x_k$ is continuous.

With the help of the closed graph theorem we can identify the members $(\lambda_k)_{k=1}^{\infty} \in \mathcal{G}(X)$ by checking either of the following equivalent statements:

(i) The series $\sum_{k=1}^\infty \lambda_k x_k$ converges in X for any $(x_k)_{k=1}^\infty \in c_0(X)$.
(ii) The linear operator $T_\alpha : c_0(X) \to X$ given by $(x_k)_{k=1}^\infty \mapsto \sum_{k=1}^\infty \lambda_k x_k$ is well-defined and continuous.

Now, thanks to this characterization it is easily seen that if X is a locally bounded F-space (i.e., a quasi-Banach space) then so is $\mathcal{G}(X)$ with the quasi-norm given by

$$\|(\lambda_k)_{k=1}^\infty\|_{\mathcal{G}} = \sup\Big\{\Big\|\sum_{k=1}^N \lambda_k x_k\Big\| : N \in \mathbb{N},\ (x_k)_{k=1}^N \in X \text{ with } \|x_k\| \le 1\Big\}$$
$$= \sup\Big\{\Big\|\sum_{k=1}^\infty \lambda_k x_k\Big\| : (x_k)_{k=1}^\infty \in c_0(X)\Big\} = \|T_\alpha\|.$$

In general, the galb of an F-space X cannot be expected to have a structure of F-space. This might have motivated Turpin to investigate this concept in the more general category of *espaces vectoriels à convergence*, in his own terminology. Nevertheless, thanks to validity of the uniform boundedness principle and the closed graph theorem, if E is a subspace of $\mathcal{G}(X)$ that can be endowed with a structure of F-space, then the mapping

$$E \times c_0(X) \to X, \qquad ((\lambda_k)_{k=1}^\infty, (x_k)_{k=1}^\infty) \mapsto \sum_{k=1}^\infty \lambda_k x_k,$$

is well-defined, bilinear, and continuous.

Examples and elementary properties

(a) Suppose that X and Y are F-spaces. If X is a subspace of Y then $\mathcal{G}(Y) \subseteq \mathcal{G}(X)$.
(b) Suppose $0 < p \le 1$. Then X is p-convex if and only if $\ell_p \subseteq \mathcal{G}(X)$. In particular, if $\mathcal{G}(X) = \ell_p$ then X cannot be q-convex for any $p < q$.
(c) If X is locally convex then $\mathcal{G}(X) = \ell_1$. In particular $\mathcal{G}(\mathbb{R}) = \ell_1$.
(d) $\mathcal{G}(X) \subseteq \ell_1$ for any F-space X.
(e) Let $0 < p \le 1$. Suppose X is a p-convex quasi-Banach space such that ℓ_p embeds into X. Then $\mathcal{G}(X) = \ell_p$. For instance, $\mathcal{G}(\ell_p) = \mathcal{G}(L_p[0,1]) = G(H_p) = \ell_p$.
(f) If X is a quasi-Banach space, there exists $0 < p \le 1$ such that $\ell_p \subseteq \mathcal{G}(X)$ (and so X is p-convex). This is essentially a restatement of the Aoki–Rolewicz theorem [4, 12].
(g) Suppose $|\lambda_k| \le |\mu_k|$ for all k and $(\mu_k)_{k=1}^\infty \in \mathcal{G}(X)$. Then $(\lambda_k)_{k=1}^\infty \in \mathcal{G}(X)$.
(h) If $(\lambda_k)_{k=1}^\infty$ is a rearrangement of $(\mu_k)_{k=1}^\infty \in \mathcal{G}(X)$ then $(\lambda_k)_{k=1}^\infty \in \mathcal{G}(X)$.
(i) There are F-spaces X with essentially trivial galb, i.e., $\mathcal{G}(X) = c_{00}$. For instance, $\mathcal{G}(L_0[0,1]) = c_{00}$.

(j) For any quasi-Banach space X, $\mathcal{G}(\mathcal{G}(X)) = \mathcal{G}(X)$.
(k) Let X be a quasi-Banach space. If $(\lambda_k)_{k=1}^\infty \in \mathcal{G}(X)$ and $(x_k)_{k=1}^\infty$ in X are such that $\sum_{k=1}^\infty \lambda_k x_k$ converges to some $x \in X$ then $\|x\| \le \|(\lambda_k)\|_{\mathcal{G}} \sup_{k\in\mathbb{N}} \|x_k\|$.

Roughly speaking, in view of (b), it could be said that the galb of an F-space is a measure of its convexity, in the sense that it tells us how far the space is from being locally convex.

2.3. Tailoring functions from $I = [0, 1]$ into a quasi-Banach space. Let $\tau = (t_k)_{k=1}^\infty$ be an increasing sequence of scalars contained in $(0, 1)$ tending to 1. With $t_0 = 0$, let us denote the interval $[t_{k-1}, t_k)$ by I_k and its length by λ_k, i.e., $\lambda_k = |I_k| = t_k - t_{k-1}$. This way we can write $[0, 1) = \bigsqcup_{k=1}^\infty I_k$ (disjoint union). For each $k \in \mathbb{N}$ let $f_{I_k} : I \to \mathbb{R}$ be the non-negative piecewise linear function supported on I_k having a node at the midpoint of I_k, $c_k = (t_k + t_{k-1})/2$, with $f_{I_k}(c_k) = 2$ and $f_{I_k}(t_{k-1}) = f_{I_k}(t_k) = 0$,

$$f_{I_k}(t) = \begin{cases} \dfrac{4}{t_k - t_{k-1}}(t - t_{k-1}) & \text{if } t \in [t_{k-1}, c_k), \\ \dfrac{4}{t_k - t_{k-1}}(t - t_k) & \text{if } t \in [c_k, t_k), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{x} = (x_k)_{k=1}^\infty$ be a sequence of vectors in a quasi-Banach space X. We define the function $f = f(\tau, \mathbf{x}) : I \to X$ as

$$f(t) = \begin{cases} f_{I_k}(t) x_k & \text{if } t \in I_k, \\ 0 & \text{if } t = 1. \end{cases} \tag{2.1}$$

Note that f is continuous and Riemann-integrable on $[0, 1)$ since for each $s < 1$ the set $f([0, s])$ is a finite-dimensional subspace of X. Let $F = F(\tau, \mathbf{x})$ be the corresponding integral function on $[0, 1)$,

$$F(t) = \int_0^t f(u)\, du. \tag{2.2}$$

The following lemma is a version of [2, Proposition 2.1] that has been customized to fit our current needs.

LEMMA 2.1. *For a given pair $(\tau, \mathbf{x})$ we have the following:*

(i) *The function $f = f(\tau, \mathbf{x}) : I \to X$ is continuous at 1, hence continuous on I, if and only if $x_k \to 0$.*
(ii) *$F = F(\tau, \mathbf{x})$ can be extended continuously to I if and only if the series $\sum_{k=1}^\infty \lambda_k x_k$ converges in X.*
(iii) *$F \in \mathcal{C}^{(1)}(I, X)$ if and only if*

$$(2.3)\qquad \lim_{m,n\to\infty}\frac{\|\sum_{m+1\le k\le n}\lambda_k x_k\|}{\sum_{m+1\le k\le n}\lambda_k}=\lim_{m,n\to\infty}\frac{\|\sum_{m+1\le k\le n}\lambda_k x_k\|}{t_n-t_m}=0.$$

Proof. We assume, by Aoki–Rolewicz's theorem, that X is p-convex for some $0<p\le 1$. Statements (i) and (ii) were proved in [2], so only (iii) needs a proof.

Suppose that $F\in\mathcal{C}^{(1)}(I,X)$. By construction, it has to be the case that $F'(t)=f(t)$ for all $t\in I$. Then

$$\lim_{m,n\to\infty}\frac{\sum_{k=m+1}^n\lambda_k x_k}{\sum_{k=m+1}^n\lambda_k}=\lim_{m,n\to\infty}\frac{F(t_n)-F(t_m)}{t_n-t_m}=\lim_{m,n\to\infty}S(F,f)(t_n,t_m)$$
$$=S(F,f)(1,1)=f(1)=0.$$

Conversely, suppose that (2.3) holds. Since $\sum_k\lambda_k$ is a Cauchy series, so is $\sum_k\lambda_k x_k$, hence it converges. Defining $F(1)=\sum_k\lambda_k x_k$, we see that F is continuous on I. To prove that $S(F,f)$ is continuous it suffices to show that

$$\lim_{\substack{t,s\to 1^-\\ s>t}}\frac{F(s)-F(t)}{s-t}=0.$$

For any $\epsilon>0$ there exists $N\in\mathbb{N}$ such that for all $n>m\ge N$,

$$\frac{\|\sum_{m+1\le k\le n}\lambda_k x_k\|}{\sum_{m+1\le k\le n}\lambda_k}=\frac{\|\sum_{m+1\le k\le n}\lambda_k x_k\|}{t_n-t_m}\le\frac{\epsilon}{3^{1/p-1}}\quad\text{and}\quad\|x_n\|\le\frac{\epsilon}{3^{1/p-1}}.$$

Given $s>t\ge t_{N-1}$ we have $s\in I_n$, $t\in I_m$ for some $n\ge m\ge N$. Then

$$\begin{aligned}&\frac{\|F(s)-F(t)\|^p}{(s-t)^p}\\ &\quad\le\frac{\|F(s)-F(t_n)\|^p+\|F(t_n)-F(t_m)\|^p+\|F(t_m)-F(t)\|^p}{(s-t)^p}\\ &\quad=\frac{\|x_n\|^p(\int_{t_n}^s f_{I_n}(u)\,du)^p+\|\sum_{k=m+1}^n\lambda_k x_k\|^p+\|x_m\|^p(\int_t^{t_m}f_{I_m}(u)\,du)^p}{(s-t)^p}\\ &\quad\le\frac{3^{p-1}\epsilon^p(t_m-t)^p+3^{p-1}\epsilon^p(t_n-t_m)^p+3^{p-1}\epsilon^p(s-t_n)^p}{(s-t)^p}\le\epsilon^p.\ \blacksquare\end{aligned}$$

3. An answer in the negative to Popov's question. The question of Popov whether every continuous function $f:I\to X$ mapping into a non-locally convex F-space has a primitive ([10, p. 206]) was tackled by Kalton in [5] in terms of the surjectivity of the operator

$$\mathcal{D}:\mathcal{C}^{(1)}(I,X)\to\mathcal{C}(I,X),\qquad F\mapsto\mathcal{D}(f)=F',$$

and was given a definite form as an intrinsic property of the space X by the authors in [2]. They defined a quasi-Banach space X to be a P-*space* (or to have *property* (P)) if every continuous function $f:I\to X$ has a primitive, and proved that no ℓ_p-space has property (P) when $0 < p < 1$. In this section we extend this result by showing that, in fact, no quasi-Banach space with separating dual does. The strategy is different. It depends on the next proposition, which will also lead us to the classical Mazur–Orlicz theorem [9] via an alternative path.

PROPOSITION 3.1. *Let X be a non-locally convex quasi-Banach space. Then there exist continuous functions $f : I \to X$ and $F : [0,1) \to X$ such that*

(a) *F is differentiable at every $t \in [0,1)$ with $F'(t) = f(t)$, but*
(b) *F does not admit a continuous extension to I.*

Proof. Since X is not locally convex, property (b) in §2.2 yields the existence of a sequence $(\lambda_k)_{k=1}^\infty \in \ell_1 \setminus \mathcal{G}(X)$, so that the series $\sum_{k=1}^\infty \lambda_k x_k$ does not converge in X for some sequence of vectors $\mathbf{x} = (x_k)_{k=1}^\infty$ in $c_0(X)$. Without loss of generality we assume $\lambda_k > 0$ for all k and $\sum_{k=1}^\infty \lambda_k = 1$.

Define $\tau = (t_n)_{n=1}^\infty$ by $t_n = \sum_{k=1}^n \lambda_k$, so that $t_n - t_{n-1} = \lambda_n$. Now, we feed the function-tailoring machine described in §2.3 with the pair $(\tau, \mathbf{x})$ to manufacture $f = f(\tau, \mathbf{x})$ and $F = F(\tau, \mathbf{x})$ with the desired properties, namely, f is continuous on $[0,1]$ and F is continuous on $[0,1)$ but cannot be continuously extended to I. Note that F is differentiable on $[0,1)$ with derivative $F'(t) = f(t)$ at every $t \in [0,1)$. ∎

COROLLARY 3.2 (Mazur–Orlicz). *Let X be a non-locally convex quasi-Banach space. Then there exists a continuous function $f : I \to X$ which is Riemann-integrable on $[0, 1-\epsilon]$ for every $\epsilon > 0$ but fails to be Riemann-integrable on I.*

Proof. Let f and F be the functions in Proposition 3.1. Since, by the construction of f, the set $f([0,s])$ is a finite-dimensional subspace of X for every $s < 1$, it follows that f is Riemann-integrable on $[0,s]$. Suppose the function f were Riemann-integrable on I. Then an appeal to [10, Proposition 1.2] would show that $G(t) = \int_0^t f(s)\,ds$ is (uniformly) continuous on $[0,1]$. But $G([0,s])$ maps into a finite-dimensional subspace of X for each $s < 1$, and so it is differentiable on $[0,1)$ with $G' = F'$ on $[0,1)$. This implies $G(t) = F(t)$ for all $t \in [0,1)$ so that F would admit a continuous extension to I, a contradiction. ∎

REMARK 3.3. The same argument shows that f fails to be integrable in the sense of Vogt on I.

In [6], Kalton introduced the notion of *core* of a quasi-Banach space X as the largest subspace of X with trivial dual. Note that if X^* separates the points of X then $\operatorname{core}(X) = \{0\}$, so the following theorem applies to quasi-Banach spaces with separating dual.

THEOREM 3.4. *Let X be a non-locally convex quasi-Banach space with null core. Then there exists a continuous function $f : [0,1] \to X$ failing to have a primitive.*

Proof. Pick out functions f and F as in Proposition 3.1. Suppose there exists a differentiable function $G : I \to X$ such that $G'(t) = f(t)$ for all $t \in I$. Then $(F - G)'(t) = 0$ for all $t \in [0,1)$. Now, the fact that $\operatorname{core}(X) = \{0\}$ yields $F(t) = G(t) + C$ for all $t \in [0,1)$, where C is some constant (see [2, Lemma 3.1]). Hence, F admits a continuous extension to I, a contradiction. ∎

4. An application to integration theory in quasi-Banach spaces. In [18] Vogt introduced a concept of integrability quite different from that of Riemann. Let (Ω, Σ, μ) be a measure space and X be a p-Banach space. A function $f : \Omega \to X$ is said to be *integrable in the sense of Vogt*, and we write $f \in L^1_V(\mu, X)$ (also, $f \in L^1_V(I, X)$ when μ is the Lebesgue measure on a subset $I \subseteq \mathbb{R}^d$) if f admits an expression of the following guise:

$$f(t) = \sum_{n=1}^{\infty} x_n f_n(t) \qquad \text{a.e. } t \in I, \tag{4.1}$$

where $\mathbf{x} = (x_n)_{n=1}^{\infty}$ in X and $\mathbf{f} = (f_n)_{n=1}^{\infty}$ in $L_1(\mu, \mathbb{R})$ satisfy the condition

$$N(\mathbf{x}, \mathbf{f}) = \sum_{n=1}^{\infty} \|x_n\|^p \|f_n\|_1^p < \infty. \tag{4.2}$$

The space $L^1_V(\mu, X)$ equipped with the gauge

$$\|f\|_{1,V} = \inf\{N(\mathbf{x}, \mathbf{f})^{1/p} : (4.1) \text{ and } (4.2) \text{ hold}\}$$

is a p-Banach space. Moreover, for $E \in \Sigma$ the expression

$$\sum_{n=1}^{\infty} x_n \int_E f_n \, d\mu$$

does not depend on the decomposition (4.1) chosen for f, and so it is consistent to define the *Vogt integral* of f on E as

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} x_n \int_E f_n \, d\mu.$$

The crucial fact in the work of Vogt is the possibility to identify isometrically $L^1_V(\mu, X)$ with the completion of the tensor product $X \otimes L_1(\mu, \mathbb{R})$ endowed with the quasi-norm

$$\|\Phi\| = \inf\Big\{\Big(\sum_{n=1}^{N} \|x_n\|^p \|f_n\|_1^p\Big)^{1/p} : \Phi = \sum_{n=1}^{N} x_n \otimes f_n,\ N \in \mathbb{N}\Big\}.$$

In their search for sufficient conditions of integrability in the sense of Vogt for functions taking values in F-spaces, Turpin and Waelbroeck [16, 17] dealt with spaces of functions of the form $\mathcal{C}^{(r)}(U, X)$, where U is an open set of the Euclidean space $\mathbb{R}^n$ and $0 < r < \infty$. Although we will not entertain a discussion concerning these spaces, it is perhaps worth mentioning that the authors defined the corresponding class $\mathcal{C}^{(r)}(I, X)$ in their study of sufficient conditions for Riemann integrability ([3]), and that the space $\mathcal{C}^{(1)}(I, X)$ introduced in Section 2.1 happens to be a member under disguise of that class. For our purposes, in this section it suffices to recall that if $f \in \mathcal{C}^{(1)}(I, X)$, where X is a quasi-Banach space with separating dual, and f' is Riemann-integrable on I then Barrow's rule holds, i.e.,

$$\int_a^b f'(u)\, du = f(b) - f(a).$$

The hypothesis that f' is Riemann-integrable on I is not redundant, as the following theorem shows. However, increasing the smoothness degree of f by imposing that f belongs to $\mathcal{C}^{(r)}(I, X)$ with $r > (p+1)/p$ does the trick (see [3] for more details).

THEOREM 4.1. *Let X be a non-locally convex quasi-Banach space. Then there exists $G \in \mathcal{C}^{(1)}(I, X)$ so that G' is neither Riemann-integrable nor integrable in the sense of Vogt on I.*

Proof. Assume that X is p-convex for some $0 < p < 1$. Consider, as in the proof of Proposition 3.1, $(\lambda_k)_{k=1}^\infty$, and $\mathbf{x} = (x_k)_{k=1}^\infty \in c_0(X)$ such that $\lambda_k > 0$ for all k, $\sum_{k=1}^\infty \lambda_k = 1$, and the series $\sum_{k=1}^\infty \lambda_k x_k$ does not converge.

With $t_0 = s_0 = 0$, define $\tau = (t_n)_{n=1}^\infty$, $\sigma = (s_n)_{n=1}^\infty$ and $\mathbf{y} = (y_k)_{k=1}^\infty$ by

$$t_n = \sum_{k=1}^n \lambda_k,$$

$$s_n = \begin{cases} \frac{1}{2}(t_{(n-1)/2} + t_{(n+1)/2}) & \text{if } n \text{ is odd,} \\ t_{n/2} & \text{if } n \text{ is even,} \end{cases}$$

$$y_n = \begin{cases} x_{(n+1)/2} & \text{if } n \text{ is odd,} \\ -x_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Define $\mu_n = s_n - s_{n-1}$. We have $t_n - t_{n-1} = \lambda_n$ and $\mu_n = \frac{1}{2}\lambda_{\lfloor (n+1)/2 \rfloor}$.

We feed again the function-tailoring machine described in §2.3 with the pair $(\sigma, \mathbf{y})$. Let $g = f(\sigma, \mathbf{y})$ and $G = F(\sigma, \mathbf{y})$. We have

$$\sum_{m+1 \le k \le n} \mu_k y_k = \begin{cases} 0 & \text{if } m \text{ and } n \text{ are both even,} \\ \mu_n x_n & \text{if } n \text{ is odd and } m \text{ is even,} \\ -\mu_{m+1} x_{m+1} & \text{if } n \text{ is even and } m \text{ is odd,} \\ \mu_n x_n - \mu_{m+1} x_{m+1} & \text{if } m \text{ and } n \text{ are both odd.} \end{cases}$$

In any case,

$$\frac{\|\sum_{m+1\le k\le n}\mu_k y_k\|}{\sum_{m+1\le k\le n}\mu_k} \le 2^{1/p-1}\frac{\mu_n\|x_n\|+\mu_{m+1}\|x_{m+1}\|}{\sum_{m+1\le k\le n}\mu_k}$$
$$\le 2^{1/p-1}\max\{\|x_{m+1}\|,\|x_n\|\} \xrightarrow{m,n\to\infty} 0.$$

Appealing to Lemma 2.1(iii) we get $G \in \mathcal{C}^{(1)}(I,X)$.

Suppose that $G' = g$ is integrable in the sense of Vogt on I. Consider the set

$$\Omega = \bigcup_{n=1}^{\infty}[s_{2n-2}, s_{2n-1})$$

equipped with the measure μ given by $\mu(A) = 2|A|$. Let $h = g|_\Omega$. Obviously, h is integrable in the sense of Vogt on the measure space (Ω,μ). Notice that the bijective mapping $\phi : [0,1) \to \Omega$ given by

$$\phi(t) = \frac{t+t_{n-1}}{2}, \qquad t \in [t_{n-1}, t_n),$$

is measure-preserving. Hence, $h \circ \phi$ is integrable in the sense of Vogt on $[0,1)$. Moreover, it is easy to check that $h \circ \phi = f(\tau, \mathbf{x})$. But, in view of Remark 3.3, $f(\tau,\mathbf{x})$ is not integrable in the sense of Vogt, and we reach a contradiction.

Let us assume now that g is Riemann-integrable on I. Then $\int_0^1 g = \sum_{k=1}^{\infty}\mu_k y_k = 0$. Fix any $\epsilon > 0$ (for instance, $\epsilon = 1$). There exists $\delta > 0$ such that

$$\|\sigma(g,\pi)\| < (2^p-1)^{1/p}\epsilon$$

for all Riemann sums $\sigma(f,\pi)$ of f associated with a partition π of I with diameter at most δ. Let $N \in \mathbb{N}$ be such that $1 - t_N < \delta$. Since $\sum_{k=1}^{\infty}\lambda_k x_k$ is not a Cauchy series, there exist $N \le m < n$ such that $\|\sum_{k=m+1}^{n}\lambda_k x_k\| \ge \epsilon$. Now, since $\int_0^{t_m} g = \sum_{k=0}^{2m}\mu_k y_k = 0$, we can pick a Riemann sum $\sigma(g,\pi_1)$ associated with a partition of $[0,t_m]$,

$$\pi_1 = \{0 = a_0 < \cdots < a_{l-1} < a_l < \cdots < a_L = t_m\},$$

of diameter at most δ, such that $\|\sigma(g,\pi_1)\| \le \epsilon$. Consider the partition

$$\pi = \{a_0 < \cdots < a_l < \cdots < a_L = t_m < \cdots < t_k < \cdots < t_n < 1\},$$

and the Riemann sum

$$\sigma(g,\pi) = \sigma(g,\pi_1) + \left(\sum_{k=m+1}^{n} g\left(\frac{3}{4}t_{k-1} + \frac{1}{4}t_k\right)(t_k - t_{k-1})\right) + g(t_{n+1})(1-t_n).$$

We observe that the diameter of π is at most δ and that

$$\sigma(g,\pi) = \sigma(g,\pi_1) + 2\sum_{k=m+1}^{n}\lambda_k x_k.$$

But

$$\left\|\sigma(g,\pi_1)+2\sum_{k=m+1}^{n}\lambda_k x_k\right\|^p \geq 2^p\epsilon^p-\epsilon^p,$$

a contradiction. ■

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References

[1] F. Albiac, *The role of local convexity in Lipschitz maps*, J. Convex Anal. 18 (2011), 983–997.
[2] F. Albiac and J. L. Ansorena, *Lipschitz maps and primitives for continuous functions in quasi-Banach spaces*, Nonlinear Anal. 75 (2012), 6108–6119.
[3] F. Albiac and J. L. Ansorena, *Integration and the fundamental theorem of calculus in quasi-Banach spaces*, submitted.
[4] T. Aoki, *Locally bounded linear topological spaces*, Proc. Imp. Acad. Tokyo 18 (1942), 588–594.
[5] N. J. Kalton, *The existence of primitives for continuous functions in a quasi-Banach space*, Atti Sem. Mat. Fis. Univ. Modena 44 (1996), 113–117.
[6] N. J. Kalton, *Curves with zero derivative in F-spaces*, Glasgow Math. J. 22 (1981), 19–29.
[7] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Lecture Note Ser. 89, Cambridge Univ. Press, Cambridge, 1984.
[8] B. Maurey, *Intégration dans les espaces p-normés*, Ann. Scuola Norm. Sup. Pisa (3) 26 (1972), 911–931.
[9] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires I*, Studia Math. 10 (1948), 184–208.
[10] M. M. Popov, *On integrability in F-spaces*, Studia Math. 110 (1994), 205–220.
[11] S. Rolewicz, *Metric Linear Spaces*, Math. Appl. (East Eur. Ser.) 20, Reidel, Dordrecht, 1985.
[12] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), 471–473.
[13] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Dissertationes Math. (Rozprawy Mat.) 131 (1976).
[14] P. Turpin, *Opérateurs linéaires entre espaces d'Orlicz non localement convexes*, Studia Math. 46 (1973), 153–165.

[15] P. Turpin, *Espaces et intersections d'espaces d'Orlicz non localement convexes*, Studia Math. 46 (1973), 167–195.
[16] P. Turpin et L. Waelbroeck, *Sur l'approximation des fonctions différentiables à valeurs dans les espaces vectoriels topologiques*, C. R. Acad. Sci. Paris Sér. A-B 267 (1968), A94–A97.
[17] P. Turpin et L. Waelbroeck, *Intégration et fonctions holomorphes dans les espaces localement pseudo-convexes*, C. R. Acad. Sci. Paris Sér. A-B 267 (1968), A160–A162.
[18] D. Vogt, *Integrationstheorie in p-normierten Räumen*, Math. Ann. 173 (1967), 219–232.

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