A spectral mapping theorem for Banach modules

by

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Abstract. Let $G$ be a locally compact abelian group, $M(G)$ the convolution measure algebra, and $X$ a Banach $M(G)$-module under the module multiplication $\mu \circ x, \mu \in M(G), x \in X$. We show that if $X$ is an essential $L^1(G)$-module, then $\sigma(T_\mu) = \overline{\widehat{\mu} \cdot \text{sp}(X)}$ for each measure $\mu$ in $\text{reg}(M(G))$, where $T_\mu$ denotes the operator in $B(X)$ defined by $T_\mu x = \mu \circ x$, $\sigma(\cdot)$ the usual spectrum in $B(X)$, $\text{sp}(X)$ the hull in $L^1(G)$ of the ideal $I_X = \{f \in L^1(G) \mid T_f = 0\}$, $\widehat{\mu}$ the Fourier–Stieltjes transform of $\mu$, and $\text{reg}(M(G))$ the largest closed regular subalgebra of $M(G)$; $\text{reg}(M(G))$ contains all the absolutely continuous measures and discrete measures.

1. Introduction. Let $G$ be a locally compact abelian group, $\widehat{G}$ its dual group, $L^1(G)$ the group algebra of $G$, and $M(G)$ the Banach algebra of all bounded regular complex Borel measures on $G$. It is well known that $M(G)$ is a commutative Banach algebra with the identity $\delta_0$, where $\delta_0$ is the Dirac measure concentrated in zero. It follows from Albrecht’s theorem [1] that there exists a largest closed regular subalgebra of $M(G)$. As in [11] we denote this algebra by $\text{reg}(M(G))$. Since the group algebra $L^1(G)$ and the discrete measure algebra $M_d(G)$ are regular Banach subalgebras of $M(G)$, we have $L^1(G) + M_d(G) \subset \text{reg}(M(G))$. But in general, $L^1(G) + M_d(G) \neq \text{reg}(M(G))$ (see [11]). Furthermore, $\widehat{G}$ can be considered as a subset of the structure space of $\text{reg}(M(G))$, and the restriction of the Gelfand transform of $\mu \in \text{reg}(M(G))$ to $\widehat{G}$ coincides with the Fourier–Stieltjes transform $\widehat{\mu}$ of $\mu$. Note also that $\text{reg}(M(G))$ is a semisimple algebra with the identity $\delta_0$.

Let $X$ be a Banach space, $B(X)$ the algebra of all bounded linear operators on $X$, and $1_X$ the unit element of $B(X)$. For any $T \in B(X)$ we denote by $\sigma(T)$ the spectrum of $T$. For any (continuous) representation $U$ of $G$ by isometries on $X$ and for any $\mu \in M(G)$ the generalized convolution operator $\pi(\mu) \in B(X)$ is defined by

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\[ \pi(\mu) = \int_{G} U(g) \, d\mu(g). \]

The Arveson spectrum \( \text{sp}(U) \) of \( U \) (see [2]) is defined as the hull in \( L^1(G) \) of the closed ideal \( I_U = \{ f \in L^1(G) \mid \pi(f) = 0 \} \). In this setting, A. Connes [4] proved that for every Dirac measure \( \mu \) the spectral mapping theorem
\[ \sigma(\pi(\mu)) = \overline{\mu(\text{sp}(U))} \]
holds. C.D’Antoni, R. Longo and L. Zsidó [5] proved the spectral mapping theorem for every \( \mu \in L^1(G) + M_d(G) \). Also, S.-E. Takahasi and J. Inoue [11] proved the spectral mapping theorem for any \( \mu \in \text{reg}(M(G)) \) in the case that \( G \) is compact.

Since \( \text{reg}(M(G)) \supseteq L^1(G) + M_d(G) \), the Takahasi–Inoue theorem contains the D’Antoni–Longo–Zsidó spectral mapping theorem for the compact case. However, the spectral mapping theorem is not true for every \( \mu \in M(G) \) ([5, Remark 1]).

Now, let \( X \) be a Banach \( M(G) \)-module under the module multiplication \( \mu \circ x, \mu \in M(G), x \in X \). Throughout this note we will assume that \( X \) is an essential \( L^1(G) \)-module, that is, the linear manifold spanned by \( \{ f \circ x \mid f \in L^1(G), x \in X \} \) is dense in \( X \). This is equivalent to the following ([8, Proposition 3.4]): If \( (e_\alpha) \) is a bounded approximate identity for \( L^1(G) \), then \( e_\alpha \circ x \to x \) for every \( x \in X \).

For any \( \mu \in M(G) \), define \( T_\mu \in B(X) \) by \( T_\mu x = \mu \circ x \) \( (x \in X) \). We define the spectrum \( \text{sp}(X) \) of \( X \) as the hull in \( L^1(G) \) of the ideal \( I_X = \{ f \in L^1(G) \mid T_f = 0 \} \). More precisely,
\[ \text{sp}(X) = \{ \chi \in \hat{G} \mid T_f = 0 \Rightarrow \hat{f}(\chi) = 0, f \in L^1(G) \}, \]
where \( \hat{f} \) denotes the Fourier transform of \( f \in L^1(G) \). It is easily seen that \( \text{sp}(X) \) is a nonempty closed subset of \( \hat{G} \) whenever \( X \neq \{0\} \).

2. Main result. With the above notations, our main theorem can be stated as follows.

**Theorem 2.1.** If \( X \) is a Banach \( M(G) \)-module and an essential \( L^1(G) \)-module, then
\[ \sigma(T_\mu) = \overline{\mu(\text{sp}(X))} \quad \text{for all} \ \mu \in \text{reg}(M(G)). \]

Note that the generalized convolution operators \( \pi(\mu), \mu \in M(G) \), define the \( M(G) \)-module multiplication on \( X \) given by \( \mu \circ x = \pi(\mu)x \). It is also evident that if \( (e_\alpha) \) is a bounded approximate identity for \( L^1(G) \), then \( \pi(e_\alpha)x \to x \) \( (x \in X) \). Hence \( X \) is an essential \( L^1(G) \)-module under the module multiplication defined above. Thus, the above theorem contains the preceding spectral mapping theorems ([4], [5], [11]).

For the proof of the theorem we need some preliminary results.
Let $A$ be a (complex) commutative, regular and semisimple Banach algebra, $\Delta(A)$ the structure space of $A$, and $\hat{a}$ the Gelfand transform of $a \in A$. It is well known that for a closed subset $S$ of $\Delta(A)$, $I(S) = \{ a \in A \mid \hat{a}(\varphi) = 0, \varphi \in S \}$ is the largest and $J(S) = cl\{ a \in A \mid \text{supp} \hat{a} \text{ is compact and } \text{supp} \hat{a} \cap S = \emptyset \}$ the smallest closed ideal of $A$ whose hull is $S$. For brevity, the structure space of $\text{reg}(M(G))$ will be denoted by $\Delta_{\text{reg}}$. The hull in $\text{reg}(M(G))$ of the ideal $K = \{ \mu \in \text{reg}(M(G)) \mid T_\mu = 0 \}$ will be denoted by $h(K)$. Also the symbol $\mu^v$ will be used to denote the Gelfand transform of any $\mu \in \text{reg}(M(G))$.

**Lemma 2.2.** Suppose the hypotheses of Theorem 2.1 are satisfied. Then, under the above notations,

$$\sigma(T_\mu) = \mu^v(h(K)) \quad \text{for all } \mu \in \text{reg}(M(G)).$$

**Proof.** Denote by $A$ the (operator-norm) closure of $\{ T_\mu \mid \mu \in \text{reg}(M(G)) \}$. Since $X$ is an essential $L^1(G)$-module, from the equality $T_{\delta_0}T_f = T_f$ we get $T_{\delta_0} = 1_X$. Thus, $A$ is a commutative unital subalgebra of $B(X)$. Consider the mapping $\theta : \Delta(A) \to h(K)$ defined by $\theta(\varphi)(\mu) = \hat{T}_\mu(\varphi)$. First we show that $\theta$ is onto (since $\theta$ is one-to-one, this means that $\theta$ is a homeomorphism). Suppose on the contrary that there exists $\varphi_0 \in h(K)$ but $\varphi_0 \notin \theta(\Delta(A))$. Let $U$ and $V$ be disjoint neighborhoods of $\varphi_0$ and $\theta(\Delta(A))$ respectively. By regularity of $\text{reg}(M(G))$, there exist elements $\mu, \lambda \in \text{reg}(M(G))$ such that $\mu^v(\varphi_0) = 1$, $\mu^v(\Delta_{\text{reg}} \setminus U) = 0$, $\lambda^v(\theta(\Delta(A))) = 1$ and $\lambda^v(\Delta_{\text{reg}} \setminus V) = 0$. It can be seen that $\mu^v \cdot \lambda^v = 0$ on $\Delta_{\text{reg}}$. This clearly implies that $\mu \cdot \lambda = 0$ and so $T_\mu T_\lambda = 0$. Since $\hat{T}_\lambda(\Delta(A)) = \lambda^v(\theta(\Delta(A))) = 1$, $T_\lambda$ is invertible in $A$ and hence $T_\mu = 0$. Also, since $\varphi_0 \in h(K)$ we have $\mu^v(\varphi_0) = 0$. This contradicts the fact that $\mu^v(\varphi_0) = 1$. Thus $\theta(\Delta(A)) = h(K)$, from which it follows that

$$\sigma_A(T_\mu) = \mu^v(h(K)) \quad \text{for all } \mu \in \text{reg}(M(G)).$$

It remains to show that $A$ is a full subalgebra of $B(X)$. Let $a \in A$ be such that $a \in B(X)^{-1}$ and let $\hat{A}$ be the smallest closed subalgebra of $B(X)$ that contains $a^{-1}$ and $A$. It is easily seen that $\hat{A}$ is commutative and $A$ is a regular subalgebra of $\hat{A}$. By the Shilov theorem ([7, p. 249]) any $\varphi \in \Delta(A)$ can be extended to some $\tilde{\varphi} \in \Delta(\hat{A})$. Hence since $a \in \hat{A}^{-1}$ we have $\varphi(a) = \tilde{\varphi}(a) \neq 0$ for all $\varphi \in \Delta(A)$ and so $a \in A^{-1}$.

Let $\text{sp}(X)$ denote the closure of $\text{sp}(X)$ in the usual topology of $\Delta_{\text{reg}}$. Recall that $I(\text{sp}(X))$ is the largest and $J(\text{sp}(X))$ the smallest closed ideal of $\text{reg}(M(G))$ whose hull is $\text{sp}(X)$.

**Lemma 2.3.** Under the hypotheses of Theorem 2.1,

$$h(K) = \text{sp}(X).$$
Proof. It is enough to show that
\[ J(\text{sp}(X)) \subset K \subset I(\text{sp}(X)). \]

Let \( \mu \in K \). Then \( T_\mu = 0 \), which implies that \( T_{\mu^*f} = T_\mu T_f = 0 \) for all \( f \in L^1(G) \). However since \( \mu^*f \in L^1(G) \), we have \( \mu^*f = \hat{\mu} \cdot \hat{f} = 0 \) on \( \text{sp}(X) \) for all \( f \in L^1(G) \), which can clearly be valid only if \( \hat{\mu} = 0 \) on \( \text{sp}(X) \). It follows that \( \mu^v = 0 \) on \( \text{sp}(X) \) and consequently \( \mu \in I(\text{sp}(X)) \). Thus we have
\[ K \subset I(\text{sp}(X)). \]

To prove \( J(\text{sp}(X)) \subset K \), let \( W \) be an open set in \( \Delta_{\text{reg}} \) that contains \( \text{sp}(X) \). Assume that \( \mu^v \) vanishes on \( W \) for some \( \mu \in \text{reg}(M(G)) \). We have to show that \( T_\mu = 0 \). First we observe that the usual topology of \( \hat{G} \) is a base for the relative topology induced in \( \hat{G} \) by \( \Delta_{\text{reg}} \). For this fix \( \chi_0 \in \hat{G} \), \( \varepsilon > 0 \) and \( \{ \mu_1, \ldots, \mu_n \} \subset \text{reg}(M(G)) \). Since \( \text{reg}(M(G)) \supset L^1(G) \) it suffices to show that
\[ U = \{ \chi \in \hat{G} | \sup_{g \in K} |\chi(g) - \chi_0(g)| < \delta \} \]
\[ \subset \{ \chi \in \hat{G} | |\hat{\mu}_i(\chi) - \hat{\mu}_i(\chi_0)| < \varepsilon, \ i = 1, \ldots, n \} \]
for some compact \( K \subset G \) and \( \delta > 0 \). Choose a compact set \( K \) in \( G \) so that
\[ |\mu_i|(G \setminus K) < \varepsilon/4 \text{ and } 0 < \delta < \varepsilon/(2 \max_i ||\mu_i||), \ i = 1, \ldots, n. \]
If \( \chi \in U \), then
\[ |\hat{\mu}_i(\chi) - \hat{\mu}_i(\chi_0)| \leq \int_K |\chi(g) - \chi_0(g)| d|\mu_i| + \int_{G \setminus K} |\chi(g) - \chi_0(g)| d|\mu_i| \]
\[ \leq \sup_{g \in K} |\chi(g) - \chi_0(g)| (\max_i ||\mu_i||) + 2 |\mu_i|(G \setminus K) < \varepsilon, \ i = 1, \ldots, n. \]
It follows that \( W \cap \hat{G} \) is an open set in \( \hat{G} \) (in the usual topology of \( \hat{G} \)) that contains \( \text{sp}(X) \) (= \( \text{sp}(X) \cap \hat{G} \)). On the other hand since \( \mu^v = 0 \) on \( W \), we see that \( \hat{\mu} \) vanishes on \( W \cap \hat{G} \). Now, let \( (e_\alpha) \) be an approximate identity for \( L^1(G) \) such that \( \text{supp} \hat{e}_\alpha \) is compact. Notice that \( \hat{\mu} e_\alpha \) belongs to the smallest ideal of \( L^1(G) \) whose hull is \( \text{sp}(X) \). From this we deduce that \( 0 = T_{\mu^*e_\alpha} = T_\mu T_{e_\alpha} \). Since \( T_{e_\alpha} x \to x \) for all \( x \in X \), we conclude that \( T_\mu = 0 \). \( \blacksquare \)

Now, we can prove the main result of this note.

Proof of Theorem 2.1. Let \( \mu \in \text{reg}(M(G)) \). Then by Lemma 2.2, we have \( \sigma(T_\mu) = \mu^v(h(K)) \). On the other hand by Lemma 2.3, since \( h(K) = \text{sp}(X) \) we get \( \sigma(T_\mu) = \mu^v(\text{sp}(X)) \). Further, from the continuity of \( \mu^v \) on \( \Delta_{\text{reg}} \) we deduce that
\[ \mu^v(\text{sp}(X)) \subset \mu^v(\text{sp}(X)) = \hat{\mu}(\text{sp}(X)). \]
Also since \( \text{sp}(X) \) is a compact subset of \( \Delta_{\text{reg}} \), it follows that \( \mu^v(\text{sp}(X)) \) is closed and consequently
\[ \mu^v(\text{sp}(X)) \supset \mu^v(\text{sp}(X)) = \hat{\mu}(\text{sp}(X)). \]
Thus, we obtain
\[ \sigma(T_\mu) = \bar{\mu}(\text{sp}(X)). \]
The proof is complete. ■

Let \( Y \) be a Banach \( M(G) \)-submodule of \( X \). Define \( \text{sp}(Y) \) as the hull in \( L^1(G) \) of the ideal \( I_Y = \{ f \in L^1(G) \mid T f y = 0, \ y \in Y \} \).

**Corollary 2.4.** Assume the hypotheses of Theorem 2.1 are satisfied. If \( Y \) is a Banach \( M(G) \)-submodule of \( X \), then
\[ \sigma(T_\mu|Y) = \bar{\mu}(\text{sp}(Y)) \quad \text{for all} \ \mu \in \text{reg}(M(G)). \]

**References**


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