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A spectral mapping theorem for Banach modules

by

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Abstract. Let G be a locally compact abelian group, $M(G)$ the convolution measure algebra, and X a Banach $M(G)$-module under the module multiplication $\mu\circ x$, $\mu \in M(G)$, $x \in X$. We show that if X is an essential $L^1(G)$-module, then $\sigma(T_\mu) = \overline{\widehat{\mu}(\mathrm{sp}(X))}$ for each measure μ in $\mathrm{reg}(M(G))$, where T_μ denotes the operator in $B(X)$ defined by $T_\mu x = \mu\circ x$, $\sigma(\cdot)$ the usual spectrum in $B(X)$, $\mathrm{sp}(X)$ the hull in $L^1(G)$ of the ideal $I_X = \{f \in L^1(G) \mid T_f = 0\}$, $\widehat{\mu}$ the Fourier–Stieltjes transform of μ, and $\mathrm{reg}(M(G))$ the largest closed regular subalgebra of $M(G)$; $\mathrm{reg}(M(G))$ contains all the absolutely continuous measures and discrete measures.

1. Introduction. Let G be a locally compact abelian group, $\widehat{G}$ its dual group, $L^1(G)$ the group algebra of G, and $M(G)$ the Banach algebra of all bounded regular complex Borel measures on G. It is well known that $M(G)$ is a commutative Banach algebra with the identity δ_0, where δ_0 is the Dirac measure concentrated in zero. It follows from Albrecht's theorem [1] that there exists a largest closed regular subalgebra of $M(G)$. As in [11] we denote this algebra by $\mathrm{reg}(M(G))$. Since the group algebra $L^1(G)$ and the discrete measure algebra $M_{\mathrm{d}}(G)$ are regular Banach subalgebras of $M(G)$, we have $L^1(G) + M_{\mathrm{d}}(G) \subset \mathrm{reg}(M(G))$. But in general, $L^1(G) + M_{\mathrm{d}}(G) \neq \mathrm{reg}(M(G))$ (see [11]). Furthermore, $\widehat{G}$ can be considered as a subset of the structure space of $\mathrm{reg}(M(G))$, and the restriction of the Gelfand transform of $\mu \in \mathrm{reg}(M(G))$ to $\widehat{G}$ coincides with the Fourier–Stieltjes transform $\widehat{\mu}$ of μ. Note also that $\mathrm{reg}(M(G))$ is a semisimple algebra with the identity δ_0.

Let X be a Banach space, $B(X)$ the algebra of all bounded linear operators on X, and 1_X the unit element of $B(X)$. For any $T \in B(X)$ we denote by $\sigma(T)$ the spectrum of T. For any (continuous) representation U of G by isometries on X and for any $\mu \in M(G)$ the generalized convolution operator $\pi(\mu) \in B(X)$ is defined by

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$$\pi(\mu) = \int_G U(g)\, d\mu(g).$$

The *Arveson spectrum* $\mathrm{sp}(U)$ of U (see [2]) is defined as the hull in $L^1(G)$ of the closed ideal $I_U = \{f \in L^1(G) \mid \pi(f) = 0\}$. In this setting, A. Connes [4] proved that for every Dirac measure μ the spectral mapping theorem

$$\sigma(\pi(\mu)) = \overline{\widehat{\mu}(\mathrm{sp}(U))}$$

holds. C.D'Antoni, R. Longo and L. Zsidó [5] proved the spectral mapping theorem for every $\mu \in L^1(G) + M_{\mathrm{d}}(G)$. Also, S.-E. Takahasi and J. Inoue [11] proved the spectral mapping theorem for any $\mu \in \mathrm{reg}(M(G))$ in the case that G is compact.

Since $\mathrm{reg}(M(G)) \supsetneq L^1(G) + M_{\mathrm{d}}(G)$, the Takahasi–Inoue theorem contains the D'Antoni–Longo–Zsidó spectral mapping theorem for the compact case. However, the spectral mapping theorem is not true for every $\mu \in M(G)$ ([5, Remark 1]).

Now, let X be a Banach $M(G)$-module under the module multiplication $\mu \circ x$, $\mu \in M(G)$, $x \in X$. Throughout this note we will assume that X is an *essential* $L^1(G)$-*module*, that is, the linear manifold spanned by $\{f \circ x \mid f \in L^1(G),\ x \in X\}$ is dense in X. This is equivalent to the following ([8, Proposition 3.4]): If (e_α) is a bounded approximate identity for $L^1(G)$, then $e_\alpha \circ x \to x$ for every $x \in X$.

For any $\mu \in M(G)$, define $T_\mu \in B(X)$ by $T_\mu x = \mu \circ x$ $(x \in X)$. We define the *spectrum* $\mathrm{sp}(X)$ of X as the hull in $L^1(G)$ of the ideal $I_X = \{f \in L^1(G) \mid T_f = 0\}$. More precisely,

$$\mathrm{sp}(X) = \{\chi \in \widehat{G} \mid T_f = 0 \Rightarrow \widehat{f}(\chi) = 0,\ f \in L^1(G)\},$$

where $\widehat{f}$ denotes the Fourier transform of $f \in L^1(G)$. It is easily seen that $\mathrm{sp}(X)$ is a nonempty closed subset of $\widehat{G}$ whenever $X \neq \{0\}$.

2. Main result. With the above notations, our main theorem can be stated as follows.

THEOREM 2.1. *If X is a Banach $M(G)$-module and an essential $L^1(G)$-module*, *then*

$$\sigma(T_\mu) = \overline{\widehat{\mu}(\mathrm{sp}(X))} \quad \text{for all } \mu \in \mathrm{reg}(M(G)).$$

Note that the generalized convolution operators $\pi(\mu)$, $\mu \in M(G)$, define the $M(G)$-module multiplication on X given by $\mu \circ x = \pi(\mu)x$. It is also evident that if (e_α) is a bounded approximate identity for $L^1(G)$, then $\pi(e_\alpha)x \to x$ $(x \in X)$. Hence X is an essential $L^1(G)$-module under the module multiplication defined above. Thus, the above theorem contains the preceding spectral mapping theorems ([4], [5], [11]).

For the proof of the theorem we need some preliminary results.

Let A be a (complex) commutative, regular and semisimple Banach algebra, $\Delta(A)$ the structure space of A, and $\widehat{a}$ the Gelfand transform of $a \in A$. It is well known that for a closed subset S of $\Delta(A)$, $I(S) = \{a \in A \mid \widehat{a}(\varphi) = 0, \varphi \in S\}$ is the largest and $J(S) = \mathrm{cl}\{a \in A \mid \operatorname{supp}\widehat{a}$ is compact and $\operatorname{supp}\widehat{a} \cap S = \emptyset\}$ the smallest closed ideal of A whose hull is S. For brevity, the structure space of $\mathrm{reg}(M(G))$ will be denoted by Δ_{reg}. The hull in $\mathrm{reg}(M(G))$ of the ideal $K = \{\mu \in \mathrm{reg}(M(G)) \mid T_\mu = 0\}$ will be denoted by $h(K)$. Also the symbol μ^v will be used to denote the Gelfand transform of any $\mu \in \mathrm{reg}(M(G))$.

LEMMA 2.2. *Suppose the hypotheses of Theorem* 2.1 *are satisfied. Then*, *under the above notations*,

$$\sigma(T_\mu) = \mu^v(h(K)) \quad \textit{for all } \mu \in \mathrm{reg}(M(G)).$$

Proof. Denote by A the (operator-norm) closure of $\{T_\mu \mid \mu \in \mathrm{reg}(M(G))\}$. Since X is an essential $L^1(G)$-module, from the equality $T_{\delta_0}T_f = T_f$ we get $T_{\delta_0} = 1_X$. Thus, A is a commutative unital subalgebra of $B(X)$. Consider the mapping $\theta : \Delta(A) \to h(K)$ defined by $\theta(\varphi)(\mu) = \widehat{T}_\mu(\varphi)$. First we show that θ is onto (since θ is one-to-one, this means that θ is a homeomorphism). Suppose on the contrary that there exists $\varphi_0 \in h(K)$ but $\varphi_0 \notin \theta(\Delta(A))$. Let U and V be disjoint neighborhoods of φ_0 and $\theta(\Delta(A))$ respectively. By regularity of $\mathrm{reg}(M(G))$, there exist elements $\mu, \lambda \in \mathrm{reg}(M(G))$ such that $\mu^v(\varphi_0) = 1$, $\mu^v(\Delta_{\mathrm{reg}} \backslash U) = 0$, $\lambda^v(\theta(\Delta(A))) = 1$ and $\lambda^v(\Delta_{\mathrm{reg}} \backslash V) = 0$. It can be seen that $\mu^v \cdot \lambda^v = 0$ on Δ_{reg}. This clearly implies that $\mu_* \lambda = 0$ and so $T_\mu T_\lambda = 0$. Since $\widehat{T}_\lambda(\Delta(A)) = \lambda^v(\theta(\Delta(A))) = 1$, T_λ is invertible in A and hence $T_\mu = 0$. Also, since $\varphi_0 \in h(K)$ we have $\mu^v(\varphi_0) = 0$. This contradicts the fact that $\mu^v(\varphi_0) = 1$. Thus $\theta(\Delta(A)) = h(K)$, from which it follows that

$$\sigma_A(T_\mu) = \mu^v(h(K)) \quad \text{for all } \mu \in \mathrm{reg}(M(G)).$$

It remains to show that A is a full subalgebra of $B(X)$. Let $a \in A$ be such that $a \in B(X)^{-1}$ and let $\widetilde{A}$ be the smallest closed subalgebra of $B(X)$ that contains a^{-1} and A. It is easily seen that $\widetilde{A}$ is commutative and A is a regular subalgebra of $\widetilde{A}$. By the Shilov theorem ([7, p. 249]) any $\varphi \in \Delta(A)$ can be extended to some $\widetilde{\varphi} \in \Delta(\widetilde{A})$. Hence since $a \in \widetilde{A}^{-1}$ we have $\varphi(a) = \widetilde{\varphi}(a) \neq 0$ for all $\varphi \in \Delta(A)$ and so $a \in A^{-1}$. ∎

Let $\overline{\mathrm{sp}(X)}$ denote the closure of $\mathrm{sp}(X)$ in the usual topology of Δ_{reg}. Recall that $I(\overline{\mathrm{sp}(X)})$ is the largest and $J(\overline{\mathrm{sp}(X)})$ the smallest closed ideal of $\mathrm{reg}(M(G))$ whose hull is $\overline{\mathrm{sp}(X)}$.

LEMMA 2.3. *Under the hypotheses of Theorem* 2.1,

$$h(K) = \overline{\mathrm{sp}(X)}.$$

Proof. It is enough to show that

$$J(\overline{\mathrm{sp}(X)}) \subset K \subset I(\overline{\mathrm{sp}(X)}).$$

Let $\mu \in K$. Then $T_\mu = 0$, which implies that $T_{\mu * f} = T_\mu T_f = 0$ for all $f \in L^1(G)$. However since $\mu_* f \in L^1(G)$, we have $\widehat{\mu * f} = \widehat{\mu} \cdot \widehat{f} = 0$ on $\mathrm{sp}(X)$ for all $f \in L^1(G)$, which can clearly be valid only if $\widehat{\mu} = 0$ on $\mathrm{sp}(X)$. It follows that $\mu^v = 0$ on $\overline{\mathrm{sp}(X)}$ and consequently $\mu \in I(\overline{\mathrm{sp}(X)})$. Thus we have $K \subset I(\overline{\mathrm{sp}(X)})$.

To prove $J(\overline{\mathrm{sp}(X)}) \subset K$, let W be an open set in Δ_{reg} that contains $\overline{\mathrm{sp}(X)}$. Assume that μ^v vanishes on W for some $\mu \in \mathrm{reg}(M(G))$. We have to show that $T_\mu = 0$. First we observe that the usual topology of $\widehat{G}$ is a base for the relative topology induced in $\widehat{G}$ by Δ_{reg}. For this fix $\chi_0 \in \widehat{G}$, $\varepsilon > 0$ and $\{\mu_1, \ldots, \mu_n\} \subset \mathrm{reg}(M(G))$. Since $\mathrm{reg}(M(G)) \supset L^1(G)$ it suffices to show that

$$\begin{aligned} U &= \{\chi \in \widehat{G} \mid \sup_{g \in K} |\chi(g) - \chi_0(g)| < \delta\} \\ &\subset \{\chi \in \widehat{G} \mid |\widehat{\mu}_i(\chi) - \widehat{\mu}_i(\chi_0)| < \varepsilon,\ i = 1, \ldots, n\} \end{aligned}$$

for some compact $K \subset G$ and $\delta > 0$. Choose a compact set K in G so that $|\mu_i|(G \backslash K) < \varepsilon/4$ and $0 < \delta < \varepsilon/(2 \max_i \|\mu_i\|)$, $i = 1, \ldots, n$. If $\chi \in U$, then

$$\begin{aligned} |\widehat{\mu}_i(\chi) - \widehat{\mu}_i(\chi_0)| &\le \int_K |\chi(g) - \chi_0(g)|\, d|\mu_i| + \int_{G-K} |\chi(g) - \chi_0(g)|\, d|\mu_i| \\ &\le \sup_{g \in K} |\chi(g) - \chi_0(g)| (\max_i \|\mu_i\|) + 2|\mu_i|(G \backslash K) < \varepsilon, \qquad i = 1, \ldots, n. \end{aligned}$$

It follows that $W \cap \widehat{G}$ is an open set in $\widehat{G}$ (in the usual topology of $\widehat{G}$) that contains $\mathrm{sp}(X)$ $(= \overline{\mathrm{sp}(X)} \cap \widehat{G})$. On the other hand since $\mu^v = 0$ on W, we see that $\widehat{\mu}$ vanishes on $W \cap \widehat{G}$. Now, let (e_α) be an approximate identity for $L^1(G)$ such that $\mathrm{supp}\, \widehat{e}_\alpha$ is compact. Notice that $\mu_* e_\alpha$ belongs to the smallest ideal of $L^1(G)$ whose hull is $\mathrm{sp}(X)$. From this we deduce that $0 = T_{\mu * e_\alpha} = T_\mu T_{e_\alpha}$. Since $T_{e_\alpha} x \to x$ for all $x \in X$, we conclude that $T_\mu = 0$. ∎

Now, we can prove the main result of this note.

Proof of Theorem 2.1. Let $\mu \in \mathrm{reg}(M(G))$. Then by Lemma 2.2, we have $\sigma(T_\mu) = \mu^v(h(K))$. On the other hand by Lemma 2.3, since $h(K) = \overline{\mathrm{sp}(X)}$ we get $\sigma(T_\mu) = \mu^v(\overline{\mathrm{sp}(X)})$. Further, from the continuity of μ^v on Δ_{reg} we deduce that

$$\mu^v(\overline{\mathrm{sp}(X)}) \subset \overline{\mu^v(\mathrm{sp}(X))} = \overline{\widehat{\mu}(\mathrm{sp}(X))}.$$

Also since $\overline{\mathrm{sp}(X)}$ is a compact subset of Δ_{reg}, it follows that $\mu^v(\overline{\mathrm{sp}(X)})$ is closed and consequently

$$\mu^v(\overline{\mathrm{sp}(X)}) \supset \overline{\mu^v(\mathrm{sp}(X))} = \overline{\widehat{\mu}(\mathrm{sp}(X))}.$$

Thus, we obtain

$$\sigma(T_\mu) = \overline{\widehat{\mu}(\mathrm{sp}(X))}.$$

The proof is complete. ∎

Let Y be a Banach $M(G)$-submodule of X. Define $\mathrm{sp}(Y)$ as the hull in $L^1(G)$ of the ideal $I_Y = \{f \in L^1(G) \mid T_f y = 0,\, y \in Y\}$.

COROLLARY 2.4. *Assume the hypotheses of Theorem* 2.1 *are satisfied. If Y is a Banach $M(G)$-submodule of X, then*

$$\sigma(T_\mu|Y) = \overline{\widehat{\mu}(\mathrm{sp}(Y))} \quad \textit{for all } \mu \in \mathrm{reg}(M(G)).$$

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