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Weighted integrability of double cosine series with nonnegative coefficients

by

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Abstract. Let $f_c(x,y) \equiv \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}(1-\cos jx)(1-\cos ky)$ with $a_{jk} \geq 0$ for all $j,k \geq 1$. We estimate the integral $\int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy$ in terms of the coefficients a_{jk}, where $\alpha,\beta \in \mathbb{R}$ and $\phi : [0,\infty] \to [0,\infty]$. Our results can be regarded as the trigonometric analogues of those of Mazhar and Móricz [MM]. They generalize and extend Boas [B, Theorem 6.7].

1. Introduction. Let $a_{jk} \geq 0$ for all $j,k \geq 1$. We consider the double cosine series

$$f_c(x,y) \equiv \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}(1-\cos jx)(1-\cos ky).$$

The purpose of this paper is to estimate the following integral:

$$\int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy,$$

where $\alpha,\beta \in \mathbb{R}$ and $\phi \in \Delta(p,q)$ for $p \geq q > 0$. As defined in [MP], $\phi \in \Delta(p,q)$ means that $\phi(t) \geq 0$ on $[0,\infty]$, $\phi(0)=0$, $\phi(\infty)=\infty$, $\phi(t)/t^p$ is nonincreasing and $\phi(t)/t^q$ is nondecreasing on $[0,\infty)$. We shall prove

THEOREM 1.1. *Let $p \geq q > 0$, $\phi \in \Delta(p,q)$, and $a_{jk} \geq 0$ $(j,k \geq 1)$. Then for $\alpha,\beta \in \mathbb{R}$,*

$$(1.1)\quad \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy \geq K_{p,\alpha,\beta}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-2p-\alpha-1}n^{-2p-\beta-1}\phi\Big(4\sum_{j=1}^{m}\sum_{k=1}^{n}(jk)^2 a_{jk}\Big).$$

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THEOREM 1.2. *Let* $1 \le p < \infty$, $p \ge q > 0$, $\phi \in \Delta(p,q)$, *and* $a_{jk} \ge 0$ $(j,k \ge 1)$. *Then for* $\alpha, \beta > -q$,

$$\int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy \le K_{p,q,\alpha,\beta}\sum_{m=1}^\infty\sum_{n=1}^\infty m^{p-q-\alpha-1}n^{p-q-\beta-1}\phi\Big(\sum_{j=m}^\infty\sum_{k=n}^\infty a_{jk}\Big). \tag{1.2}$$

The constants $K_{p,\alpha,\beta}$ and $K_{p,q,\alpha,\beta}$ in Theorems 1.1 and 1.2 depend on the indicated parameters only. We adopt the same convention whenever we mention constants of this type later on. Applying Theorems 1.1 and 1.2 to the case $\phi(t) = |t|^p$, we obtain

COROLLARY 1.3. *Let* $1 \le p < \infty$ *and* $a_{jk} \ge 0$ $(j,k \ge 1)$. *Then for* $-p < \alpha, \beta < 0$ *and* $s,t \ge 1$, *there exists a constant* $K \equiv K_{p,\alpha,\beta,s,t} > 0$ *such that*

$$\begin{aligned} K^{-1}\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-\alpha-1}n^{-\beta-1}\Big|\sum_{j=m}^\infty\sum_{k=n}^\infty a_{jk}\Big|^p &\le \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}|f_c(x,y)|^p\,dx\,dy \\ &\le K\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-\alpha-1}n^{-\beta-1}\Big|\sum_{j=m}^\infty\sum_{k=n}^\infty a_{jk}\Big|^p, \end{aligned} \tag{1.3}$$

$$\begin{aligned} K^{-1}\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-sp-\alpha-1}n^{-\beta-1}\Big|s\sum_{j=1}^m\sum_{k=n}^\infty j^s a_{jk}\Big|^p &\le \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}|f_c(x,y)|^p\,dx\,dy \\ &\le K\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-sp-\alpha-1}n^{-\beta-1}\Big|s\sum_{j=1}^m\sum_{k=n}^\infty j^s a_{jk}\Big|^p, \end{aligned} \tag{1.4}$$

$$\begin{aligned} K^{-1}\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-sp-\alpha-1}n^{-tp-\beta-1}\Big|st\sum_{j=1}^m\sum_{k=1}^n j^s k^t a_{jk}\Big|^p &\le \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}|f_c(x,y)|^p\,dx\,dy \\ &\le K\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-sp-\alpha-1}n^{-tp-\beta-1}\Big|st\sum_{j=1}^m\sum_{k=1}^n j^s k^t a_{jk}\Big|^p. \end{aligned} \tag{1.5}$$

Corollary 1.3 can be regarded as the trigonometric analogue of the result of Mazhar and Móricz [MM]. Whenever $a_{jk} \ge 0$ $(j, k \ge 1)$ and $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} < \infty$, the Weierstrass M-test theorem ensures $f_c(x,y) = f(0,0) - f(x,0) - f(0,y) + f(x,y)$, where $f(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\cos jx \cos ky$. In this case, the double integral in (1.3)–(1.5) can be replaced by

$$\int_0^{\pi}\int_0^{\pi} x^{\alpha-1}y^{\beta-1}|f(0,0) - f(x,0) - f(0,y) + f(x,y)|^p\,dx\,dy. \tag{1.6}$$

Moreover, one of the sums appearing on the right sides of (1.3)–(1.5) is finite if and only if

$$x^{\alpha-1}y^{\beta-1}|f(x,y) - f(x,0) - f(0,y) + f(0,0)|^p \in L^1([0,\pi]\times[0,\pi]). \tag{1.7}$$

On the other hand, if $1 < p < \infty$ and (1.7)–(1.9) are satisfied by some negative numbers α, α_*, β and β_*:

$$x^{\alpha_*-1}|f(x,0) - f(0,0)|^p \in L^1([0,\pi]), \tag{1.8}$$

$$y^{\beta_*-1}|f(0,y) - f(0,0)|^p \in L^1([0,\pi]), \tag{1.9}$$

then by the Hölder inequality, we get

$$\begin{aligned}
&\int_0^{\pi}\int_0^{\pi} x^{-1}y^{-1}|f(x,y) - f(x,0) - f(0,y) + f(0,0)|\,dx\,dy \\
&\quad \le \Big(\int_0^{\pi}\int_0^{\pi} x^{\alpha-1}y^{\beta-1}|f(x,y) - f(x,0) - f(0,y) + f(0,0)|^p\,dx\,dy\Big)^{1/p} \\
&\qquad \times \Big(\int_0^{\pi}\int_0^{\pi} x^{-1-\alpha/(p-1)}y^{-1-\beta/(p-1)}\,dx\,dy\Big)^{1-1/p} \\
&\quad < \infty.
\end{aligned}$$

This says that $x^{-1}y^{-1}|f(x,y)-f(x,0)-f(0,y)+f(0,0)| \in L^1([0,\pi]\times[0,\pi])$. Analogously, $x^{-1}|f(x,0) - f(0,0)| \in L^1([0,\pi])$ and $y^{-1}|f(0,y) - f(0,0)| \in L^1([0,\pi])$. Applying a multidimensional analogue of the Dini test (cf. [Z]) to this case, we conclude that $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} < \infty$. Hence, Corollary 1.3 generalizes [B, Theorem 6.7]. Boas's result corresponds to the case that $\alpha = -pr + 1, \beta = -1/2, s = t = 1$, $a_{jk} = \lambda_j$ for $k = 1$, and $a_{jk} = 0$ for $k > 1$.

The validity of (1.7) with $\alpha = \beta = 0$ and $p = 1$ has been investigated by several authors. For details, we refer the reader to [CH].

2. Preliminaries

LEMMA 2.1. *Let $p \ge q > 0$, $\phi \in \Delta(p,q)$, and $t_m \ge 0$ for $m \ge 1$. Then*

$$\theta^p\phi(t) \le \phi(\theta t) \le \theta^q\phi(t) \qquad (0 \le \theta \le 1;\ t \ge 0), \tag{2.1}$$

$$(2.2)\qquad \phi(\theta t) \le \theta^p \phi(t) \quad (\theta \ge 1;\, t \ge 0),$$

$$(2.3)\qquad \phi(x+y) \le 2^p(\phi(x)+\phi(y)) \quad (x, y \ge 0),$$

$$(2.4)\qquad \phi\Big(\sum_{m=1}^{\infty} t_m\Big) \le \Big\{\sum_{m=1}^{\infty} \phi^{1/p}(t_m)\Big\}^p.$$

Proof. For (2.1) and (2.4), we refer the reader to [MP, Lemma 1]. Let $\theta \ge 1$ and $t > 0$. Then $\theta t \ge t$, and consequently, $\phi(\theta t)/(\theta t)^p \le \phi(t)/t^p$. This leads to (2.2). For $x, y \ge 0$, we have

$$\phi(x+y) \le \{\phi^{1/p}(x) + \phi^{1/p}(y)\}^p \le 2^p \max\{\phi(x), \phi(y)\} \le 2^p(\phi(x)+\phi(y)).$$

This proves (2.3). ∎

Let $(1, n)$ denote the interval $\{1 < x < n\}$ and let $\overline{A}_n$ denote the closure of A_n in $\mathbb{R}$. The relation $k \in \overline{A}_n$ means that k runs over all positive integers in $\overline{A}_n$. The following lemma generalizes [CL, Lemma 3.2]. The result of Chen and Luor corresponds to the case $\phi(t) = t^p$.

LEMMA 2.2. *Let $\lambda_{mn} \ge 0$ $(m, n = 1, 2, \ldots)$ and let each of $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be of the form $\{(1,k)\}_{k=1}^{\infty}$ or $\{(k,\infty)\}_{k=1}^{\infty}$. Assume there exists $0 \le \tau < \infty$ such that*

$$(2.5)\qquad \sum_{j \in \overline{A}_m} \lambda_{jn} \le \tau m \lambda_{mn} \quad (m, n \ge 1),$$

$$(2.6)\qquad \sum_{k \in \overline{B}_n} \lambda_{mk} \le \tau n \lambda_{mn} \quad (m, n \ge 1).$$

Then for $1 \le p < \infty$, $\phi \in \Delta(p,q)$ with $p \ge q > 0$, and $a_{mn} \ge 0$ $(m, n \ge 1)$, we have

$$(2.7)\qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} \phi\Big(\sum_{j \in \overline{(1,\infty)\setminus A_m}} \sum_{k \in \overline{(1,\infty)\setminus B_n}} a_{jk}\Big) \le \tau^{2p} p^{2p} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^p \lambda_{mn}\, \phi(a_{mn}).$$

Proof. This lemma can be verified by using (2.4) and [CL, Lemma 3.2]:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} \phi\Big(\sum_{j \in \overline{(1,\infty)\setminus A_m}} \sum_{k \in \overline{(1,\infty)\setminus B_n}} a_{jk}\Big) \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn}\Big(\sum_{j \in \overline{(1,\infty)\setminus A_m}} \sum_{k \in \overline{(1,\infty)\setminus B_n}} \phi^{1/p}(a_{jk})\Big)^p \le \tau^{2p} p^{2p} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^p \lambda_{mn} \phi(a_{mn}).$$ ∎

Employing Lemma 2.2, we can easily derive the following lemma, which generalizes [L, Lemma 3] and will be used to establish Theorem 1.2 and Corollary 1.3.

LEMMA 2.3. *Let λ_{mn}, τ, p, q, ϕ, and a_{mn} be as in Lemma 2.2.*

(i) *If both* (2.5) *and* (2.6) *are satisfied by $A_k = B_k = (1, k)$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big) \le \tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{p-(s+1)q}n^{p-(t+1)q} \times \lambda_{mn}\phi\Big(st\sum_{j=1}^{m}\sum_{k=1}^{n}j^s k^t a_{jk}\Big) \quad (s,t>0). \tag{2.8}$$

(ii) *If* (2.5) *is satisfied by $A_k = (1, k)$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big) \le \tau^{p}p^{p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{p-(s+1)q}\lambda_{mn}\phi\Big(s\sum_{j=1}^{m}\sum_{k=n}^{\infty}j^s a_{jk}\Big) \quad (s>0). \tag{2.9}$$

(iii) *If both* (2.5) *and* (2.6) *are satisfied by $A_k = B_k = (k, \infty)$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-(s-1)p}n^{-(t-1)p}\lambda_{mn}\phi\Big(st\sum_{j=1}^{m}\sum_{k=1}^{n}j^s k^t a_{jk}\Big) \le (st)^p\tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^p\lambda_{mn}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big) \quad (s,t\ge 1). \tag{2.10}$$

(iv) *If* (2.5) *is satisfied by $A_k = (k, \infty)$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-(s-1)p}\lambda_{mn}\phi\Big(s\sum_{j=1}^{m}\sum_{k=n}^{\infty}j^s a_{jk}\Big) \le s^p\tau^{p}p^{p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^p\lambda_{mn}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big) \quad (s\ge 1). \tag{2.10}$$

Proof. Let $A_{mn} = \sum_{j=1}^{m}\sum_{k=1}^{n} j^s k^t a_{jk}$. Summation by parts implies

$$\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk} \le st\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}j^{-s-1}k^{-t-1}A_{jk}.$$

We know that ϕ is nondecreasing and $m^{-s-1}n^{-t-1} \le 1$. By (2.1) and (2.7), we get

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big)&\\ &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\phi\Big(st\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}j^{-s-1}k^{-t-1}A_{jk}\Big)\\ &\le \tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^p\lambda_{mn}\,\phi(stm^{-s-1}n^{-t-1}A_{mn})\\ &\le \tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{p-(s+1)q}n^{p-(t+1)q}\lambda_{mn}\,\phi\Big(st\sum_{j=1}^{m}\sum_{k=1}^{n}j^sk^ta_{jk}\Big).\end{aligned}$$

This is (2.8). For (2.10), let $A_{mn}=\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}$. We have

$$\sum_{j=1}^{m}\sum_{k=1}^{n}j^sk^ta_{jk}\le m^{s-1}n^{t-1}\sum_{j=1}^{m}\sum_{k=1}^{n}jka_{jk}\le m^{s-1}n^{t-1}\sum_{j=1}^{m}\sum_{k=1}^{n}A_{jk}.$$

By (2.2) and (2.7), we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-(s-1)p}n^{-(t-1)p}\lambda_{mn}\phi\Big(st\sum_{j=1}^{m}\sum_{k=1}^{n}j^sk^ta_{jk}\Big)&\\ &\le (st)^p\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\phi\Big(\sum_{j=1}^{m}\sum_{k=1}^{n}A_{jk}\Big)\\ &\le (st)^p\tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^p\lambda_{mn}\phi(A_{mn})\\ &= (st)^p\tau^{2p}p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^p\lambda_{mn}\,\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}\Big).\end{aligned}$$

This verifies (2.10). By modifying the above proofs, we can easily derive (2.9) and (2.11) by using the one-dimensional version of (2.7). We leave the details to the reader. ∎

3. Proofs of the main results

Proof of Theorem 1.1. Let $x,y\in(0,\pi]$. We know that $1-\cos\theta\ge 2\theta^2/\pi^2$ for $0\le\theta\le\pi$, so $f_c(x,y)\ge 4(x/\pi)^2(y/\pi)^2\sum_{j=1}^{[\pi/x]}\sum_{k=1}^{[\pi/y]}(jk)^2a_{jk}$. By (2.1), we obtain

$$\phi(f_c(x,y))\ge\Big(\frac{x}{\pi}\Big)^{2p}\Big(\frac{y}{\pi}\Big)^{2p}\phi\Big(4\sum_{j=1}^{[\pi/x]}\sum_{k=1}^{[\pi/y]}(jk)^2a_{jk}\Big).$$

Therefore,

$$\text{(3.1)}\qquad \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy \ge \pi^{-4p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}^{2p,2p}\phi\Big(4\sum_{j=1}^{m}\sum_{k=1}^{n}(jk)^2a_{jk}\Big),$$

where

$$\text{(3.2)}\qquad \gamma_{mn}^{st} \equiv \int_{\pi/(m+1)}^{\pi/m}\int_{\pi/(n+1)}^{\pi/n} x^{s+\alpha-1}y^{t+\beta-1}\,dx\,dy.$$

We have $\gamma_{mn}^{2p,2p} \ge K_{p,\alpha,\beta}m^{-2p-\alpha-1}n^{-2p-\beta-1}$. Plugging this into (3.1) yields (1.1). ∎

Proof of Theorem 1.2. We have $|1-\cos kt| \le \min(k|t|,2)$. Let $0 < x, y \le \pi$. Then

$$f_c(x,y) \le xy\sum_{j=1}^{[\pi/x]}\sum_{k=1}^{[\pi/y]} jka_{jk} + 2x\sum_{j=1}^{[\pi/x]}\sum_{k=[\pi/y]}^{\infty} ja_{jk} + 2y\sum_{j=[\pi/x]}^{\infty}\sum_{k=1}^{[\pi/y]} ka_{jk} + 4\sum_{j=[\pi/x]}^{\infty}\sum_{k=[\pi/y]}^{\infty} a_{jk}.$$

By (2.1) and (2.3), we get

$$\phi(f_c(x,y)) \le K_{p,q}\Big\{(xy)^q\phi\Big(\sum_{j=1}^{[\pi/x]}\sum_{k=1}^{[\pi/y]} jka_{jk}\Big) + x^q\phi\Big(\sum_{j=1}^{[\pi/x]}\sum_{k=[\pi/y]}^{\infty} ja_{jk}\Big) + y^q\phi\Big(\sum_{j=[\pi/x]}^{\infty}\sum_{k=1}^{[\pi/y]} ka_{jk}\Big) + \phi\Big(\sum_{j=[\pi/x]}^{\infty}\sum_{k=[\pi/y]}^{\infty} a_{jk}\Big)\Big\}.$$

This implies

$$\text{(3.3)}\qquad \int_0^\pi\int_0^\pi x^{\alpha-1}y^{\beta-1}\phi(f_c(x,y))\,dx\,dy$$
$$\le K_{p,q}\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}^{qq}\phi\Big(\sum_{j=1}^{m}\sum_{k=1}^{n} jka_{jk}\Big) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}^{q0}\phi\Big(\sum_{j=1}^{m}\sum_{k=n}^{\infty} ja_{jk}\Big)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}^{0q}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=1}^{n} ka_{jk}\Big) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}^{00}\phi\Big(\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} a_{jk}\Big)\Big\}$$
$$= K_{p,q}\{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\}, \quad \text{say},$$

where γ_{mn}^{st} is defined by (3.2). An elementary calculation gives

$$\gamma_{mn}^{st} \le K_{\alpha,\beta,s,t} m^{-s-\alpha-1} n^{-t-\beta-1} \qquad (s,t \ge 0;\ m,n \ge 1).$$

On the other hand, the hypothesis $\alpha, \beta > -q$ indicates that (2.5) and (2.6) are satisfied by $A_k = B_k = (k, \infty)$, $\lambda_{mn} = m^{-q-\alpha-1} n^{-q-\beta-1}$, and

$$\tau = \max\Bigg(\max\left(\frac{2^{\alpha+q}}{\alpha+q}, 1 + \frac{1}{\alpha+q} \right), \max\left(\frac{2^{\beta+q}}{\beta+q}, 1 + \frac{1}{\beta+q} \right) \Bigg).$$

Applying (2.10) to the case $s = t = 1$, we get

$$\Sigma_1 \le K_{p,q,\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{p-q-\alpha-1} n^{p-q-\beta-1} \phi\Big(\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} \Big). \tag{3.4}$$

Similarly, it follows from (2.11) that

$$\Sigma_2 \le K_{p,q,\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{p-q-\alpha-1} n^{-\beta-1} \phi\Big(\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} \Big), \tag{3.5}$$

$$\Sigma_3 \le K_{p,q,\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-\alpha-1} n^{p-q-\beta-1} \phi\Big(\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} \Big). \tag{3.6}$$

We have $p - q \ge 0$. Putting (3.3)–(3.6) together gives (1.2). ∎

Proof of Corollary 1.3. Obviously, (2.5) and (2.6) are satisfied by $A_k = B_k = (1, k)$, $\lambda_{mn} = m^{-\alpha-1} n^{-\beta-1}$, and $\tau = \max(-1/(2^\alpha \alpha), -1/(2^\beta \beta))$. Hence, (2.8) with $p = q$ leads to

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-\alpha-1} n^{-\beta-1} \phi\Big(\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} \Big) \\ &\qquad \le K_{p,\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-sp-\alpha-1} n^{-tp-\beta-1} \phi\Big(st \sum_{j=1}^{m} \sum_{k=1}^{n} j^s k^t a_{jk} \Big). \end{aligned} \tag{3.7}$$

Conversely, applying (2.10) to the case $\lambda_{mn} = m^{-p-\alpha-1} n^{-p-\beta-1}$, we obtain

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-sp-\alpha-1} n^{-tp-\beta-1} \phi\Big(st \sum_{j=1}^{m} \sum_{k=1}^{n} j^s k^t a_{jk} \Big) \\ &\qquad \le K_{p,\alpha,\beta,s,t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-\alpha-1} n^{-\beta-1} \phi\Big(\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} \Big). \end{aligned} \tag{3.8}$$

Consider $\phi(t) = |t|^p$. Putting (1.1), (1.2), (3.7), and (3.8) together yields (1.3) and (1.5). As for (1.4), it follows from (2.9) and (2.11). ∎

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