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On w-hyponormal operators

by

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Abstract. We study some properties of w-hyponormal operators. In particular we show that some w-hyponormal operators are subscalar. Also we state some theorems on invariant subspaces of w-hyponormal operators.

1. Introduction. Let $\mathbf{H}$ be a complex Hilbert space, and denote by $\mathcal{L}(\mathbf{H})$ the algebra of all bounded linear operators on $\mathbf{H}$. If $T \in \mathcal{L}(\mathbf{H})$, we write $\sigma(T)$, $\sigma_{\mathrm{ap}}(T)$, and $\sigma_{\mathrm{p}}(T)$ for the spectrum, approximate point spectrum, and point spectrum of T, respectively.

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to be *p-hyponormal*, $0 < p \le 1$, if $(T^*T)^p \ge (TT^*)^p$ where T^* is the adjoint of T. If $p = 1$, T is called *hyponormal*, and if $p = 1/2$, T is called *semi-hyponormal*. Semi-hyponormal operators were introduced by Xia (see [Xi]), and p-hyponormal operators for a general p, $0 < p < 1$, have been studied by Aluthge. Any p-hyponormal operator is q-hyponormal if $q \le p$ by Löwner's theorem (see [Lo]). But there are examples to show that the converse of the above statement is not true (see [Al]).

An arbitrary operator $T \in \mathcal{L}(\mathbf{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is the appropriate partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. Associated with T is a related operator $|T|^{1/2}U|T|^{1/2}$, called the *Aluthge transform* of T, and denoted by $\widetilde{T}$ throughout this paper.

An operator $T = U|T|$ (polar decomposition) in $\mathcal{L}(\mathbf{H})$ is *w-hyponormal* if $|\widetilde{T}| \ge |T| \ge |\widetilde{T}^*|$ where $|\widetilde{T}| = (\widetilde{T}^*\widetilde{T})^{1/2}$. This class of operators was introduced by Aluthge and Wang (see [AW 1] and [AW 2]).

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to satisfy the *single-valued extension property* if for any open subset U in $\mathbb{C}$, the function

$$z - T : \mathcal{O}(U, \mathbf{H}) \to \mathcal{O}(U, \mathbf{H})$$

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defined by the obvious pointwise multiplication is one-to-one, where $\mathcal{O}(U,\mathbf{H})$ denotes the Fréchet space of $\mathbf{H}$-valued analytic functions on U with respect to uniform topology. If T has the single-valued extension property, then for any $x \in \mathbf{H}$ there exists a unique maximal open set $\varrho_T(x)$ ($\supset \varrho(T)$, the resolvent set) and a unique $\mathbf{H}$-valued analytic function f defined in $\varrho_T(x)$ such that

$$(T-\lambda)f(\lambda) = x, \qquad \lambda \in \varrho_T(x).$$

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to have the *property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathbf{H}$ of $\mathbf{H}$-valued analytic functions such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G.

A bounded linear operator S on $\mathbf{H}$ is called *scalar of order* m if it has a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathbf{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \le m \le \infty$. An operator is *subscalar* if it is similar to the restriction of a scalar operator.

In this paper we study some properties of w-hyponormal operators. In particular we show that some w-hyponormal operators are subscalar. Also we study invariant subspaces of w-hyponormal operators.

2. Preliminaries. Let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space $\mathbf{H}$ and a bounded open disk D of $\mathbb{C}$. We shall denote by $L^2(D,\mathbf{H})$ the Hilbert space of measurable functions $f : D \to \mathbf{H}$ such that

$$\|f\|_{2,D} = \Big\{ \int_D \|f(z)\|^2 \, d\mu(z) \Big\}^{1/2} < \infty.$$

The space of functions $f \in L^2(D,\mathbf{H})$ which are analytic on D (i.e. $\overline{\partial} f = 0$) is denoted by

$$A^2(D,\mathbf{H}) = L^2(D,\mathbf{H}) \cap \mathcal{O}(D,\mathbf{H}).$$

$A^2(D,\mathbf{H})$ is called the *Bergman space* for D. Note that $A^2(D,\mathbf{H})$ is complete (i.e. $A^2(D,\mathbf{H})$ is a Hilbert space). We denote by P the orthogonal projection of $L^2(D,\mathbf{H})$ onto $A^2(D,\mathbf{H})$.

Let us now define a Sobolev type space called $W^2(D,\mathbf{H})$ where D is a bounded disk in $\mathbb{C}$. $W^2(D,\mathbf{H})$ will be the space of those functions $f \in L^2(D,\mathbf{H})$ whose derivatives $\overline{\partial} f$, $\overline{\partial}^2 f$ in the sense of distributions still belong to $L^2(D,\mathbf{H})$. Endowed with the norm

$$\|f\|_{W^2}^2=\sum_{i=0}^{2}\|\overline{\partial}^i f\|_{2,D}^2,$$

$W^2(D,\mathbf{H})$ becomes a Hilbert space contained continuously in $L^2(D,\mathbf{H})$.

Now for $f\in C_0^2(\mathbb{C})$, let M_f denote the operator on $W^2(D,\mathbf{H})$ given by multiplication by f. It has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M: C_0^2(\mathbb{C})\to\mathcal{L}(W^2(D,\mathbf{H})),\qquad \Phi_M(f)=M_f.$$

Therefore, M_z is a scalar operator of order 2.

3. Single-valued extension property. In this section, we show that some w-hyponormal operators have the single-valued extension property. We also give an analogue of the single-valued extension property for $W^2(D,\mathbf{H})$ and some w-hyponormal operators T.

Recall that an operator $T\in\mathcal{L}(\mathbf{H})$ has *finite ascent* if for all $\lambda\in\mathbb{C}$ there is an $n\in\mathbb{N}$ such that $\ker(T-\lambda)^n=\ker(T-\lambda)^{n+1}$.

LEMMA 3.1. *An operator $|T|^{1/2}$ is one-to-one if and only if the operator $|\widetilde{T}|^{1/2}$ is one-to-one.*

Proof. Assume that $|T|^{1/2}$ is one-to-one. If $x\in\ker|\widetilde{T}|^{1/2}$, then $\widetilde{T}x=0$. Since $T(U|T|^{1/2})=(U|T|^{1/2})\widetilde{T}$, we have $|T|(U|T|^{1/2}x)=0$. Since $|T|^{1/2}$ is one-to-one, $x=0$.

Conversely, assume that $|\widetilde{T}|^{1/2}$ is one-to-one. If $x\in\ker|T|^{1/2}$, then $\widetilde{U}|\widetilde{T}|x=\widetilde{T}x=|T|^{1/2}U|T|^{1/2}x=0$. Since $|\widetilde{T}|^{1/2}$ is one-to-one, $x=0$. ∎

THEOREM 3.2. *If $T=U|T|$ (polar decomposition) is w-hyponormal with $0\notin\sigma_{\rm p}(|T|^{1/2})$, then T has finite ascent.*

Proof. Assume that T is w-hyponormal with $0\notin\sigma_{\rm p}(|T|^{1/2})$. Then $\widetilde{\widetilde{T}}$ is hyponormal from the definition of a w-hyponormal operator and [Al]. Since $\widetilde{\widetilde{T}}$ is hyponormal, $\ker(\widetilde{\widetilde{T}}-\lambda)=\ker(\widetilde{\widetilde{T}}-\lambda)^2$ for all $\lambda\in\mathbb{C}$. So it suffices to show that $\ker(\widetilde{T}-\lambda)\supset\ker(\widetilde{T}-\lambda)^2$. Let $\widetilde{T}=\widetilde{U}|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$ and let $x\in\ker(\widetilde{T}-\lambda)^2$. Since

$$(\widetilde{\widetilde{T}}-\lambda)^2|\widetilde{T}|^{1/2}x=|\widetilde{T}|^{1/2}(\widetilde{T}-\lambda)^2x=0,$$

it follows from the hypothesis that

$$|\widetilde{T}|^{1/2}x\in\ker(\widetilde{\widetilde{T}}-\lambda)^2=\ker(\widetilde{\widetilde{T}}-\lambda).$$

Hence

$$|\widetilde{T}|^{1/2}(\widetilde{T}-\lambda)x=(\widetilde{\widetilde{T}}-\lambda)|\widetilde{T}|^{1/2}x=0.$$

Since $|\widetilde{T}|^{1/2}$ is one-to-one by Lemma 3.1, $(\widetilde{T}-\lambda)x = 0$. Hence $x \in \ker(\widetilde{T}-\lambda)$. Thus $\widetilde{T}$ has finite ascent. By a similar method we deduce that T has finite ascent. ■

Corollary 3.3. *If $T = U|T|$ (polar decomposition) is a w-hyponormal operator with $0 \notin \sigma_{\rm p}(|T|^{1/2})$, then T has the single-valued extension property.*

Proof. This follows from Theorem 3.2 and [La]. ■

Corollary 3.4. *Let $T=U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\rm p}(|T|^{1/2})$. If $f : G \to \mathbb{C}$ is an analytic function nonconstant on every component of G where G is open and $G \supset \sigma(T)$, then $f(T)$ has the single-valued extension property.*

Proof. Since T has the single-valued extension property by Corollary 3.3, the assertion follows from [CF, Theorem 1.1.5]. ■

Recall that an $X \in \mathcal{L}(\mathbf{H}, \mathbf{K})$ is called a *quasi-affinity* if it has trivial kernel and dense range. An operator $A \in \mathcal{L}(\mathbf{H})$ is said to be a *quasi-affine transform* of an operator $T \in \mathcal{L}(\mathbf{K})$ if there is a quasi-affinity $X \in \mathcal{L}(\mathbf{H}, \mathbf{K})$ such that $XA = TX$.

Corollary 3.5. *Let $T=U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\rm p}(|T|^{1/2})$. If A is any quasi-affine transform of T, then A has the single-valued extension property.*

Proof. From [La], it suffices to show that $\ker(A-\lambda)^2 \subset \ker(A-\lambda)$ for all $\lambda \in \mathbb{C}$. Let X be a quasi-affinity such that $XA = TX$. If $x \in \ker(A-\lambda)^2$, then $X(A-\lambda)^2 x = 0$. Hence $(T-\lambda)^2 Xx = 0$. Since $\ker(T-\lambda)^2 = \ker(T-\lambda)$ from Theorem 3.2, $(T-\lambda)Xx = 0$. Hence $X(A-\lambda)x = 0$. Since X is one-to-one, $x \in \ker(A-\lambda)$. ■

The next result gives an analogue of the single-valued extension property for $W^2(D, \mathbf{H})$ and some w-hyponormal operators T.

Theorem 3.6. *Let $T = U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\rm p}(|T|^{1/2})$ in $\mathcal{L}(\mathbf{H})$ and let D be an arbitrary bounded disk in $\mathbb{C}$. Then the operator*

$$T - z : W^2(D, \mathbf{H}) \to W^2(D, \mathbf{H})$$

is one-to-one.

Proof. Let $f \in W^2(D, \mathbf{H})$ be such that $(T-z)f = 0$. Then

$$(\widetilde{T} - z)|T|^{1/2} f = 0. \tag{1}$$

Let $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$. Then from (1) we get

$$(\widetilde{\widetilde{T}} - z)|\widetilde{T}|^{1/2}|T|^{1/2} f = 0. \tag{2}$$

Since $\widetilde{\widetilde{T}}$ is hyponormal from the definition of a w-hyponormal operator and [Al], [Pu, Corollary 2.2] implies that

$$|\widetilde{T}|^{1/2}|T|^{1/2}f = P(|\widetilde{T}|^{1/2}|T|^{1/2}f) \tag{3}$$

where P is the orthogonal projection of $L^2(D,\mathbf{H})$ onto $A^2(D,\mathbf{H})$. From (2) and (3), we have

$$(\widetilde{\widetilde{T}} - z)P(|\widetilde{T}|^{1/2}|T|^{1/2}f) = 0.$$

Since $\widetilde{\widetilde{T}}$ has the single-valued extension property,

$$|\widetilde{T}|^{1/2}|T|^{1/2}f = P(|\widetilde{T}|^{1/2}|T|^{1/2}f) = 0.$$

Since $|T|^{1/2}$ is one-to-one, $|\widetilde{T}|^{1/2}$ is also one-to-one from Lemma 3.1. Hence $f = 0$. ∎

Corollary 3.7. *Let $T = U|T|$ (polar decomposition) be any w-hyponormal operator in $\mathcal{L}(\mathbf{H})$. If T has no nontrivial invariant subspace, then the operator*

$$T - z : W^2(D,\mathbf{H}) \to W^2(D,\mathbf{H})$$

is one-to-one.

Proof. Since T has no nontrivial invariant subspace for T, $\ker T = \{0\}$. Hence $\ker |T|^{1/2} = \{0\}$. By Theorem 3.6, $T - z$ is one-to-one. ∎

Corollary 3.8. *Let T_1 and T_3 be w-hyponormal operators with $0 \notin \sigma_{\rm p}(|T_1|^{1/2}) \cup \sigma_{\rm p}(|T_3|^{1/2})$. Then*

$$A - z = \begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} : W^2(D,\mathbf{H}) \oplus W^2(D,\mathbf{H}) \to W^2(D,\mathbf{H}) \oplus W^2(D,\mathbf{H})$$

is one-to-one.

Proof. Let $f = f_1 \oplus f_2 \in W^2(D,\mathbf{H}) \oplus W^2(D,\mathbf{H})$ be such that $(A-z)f = 0$. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1 + T_2 f_2 \\ (T_3 - z)f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we have

$$(T_1 - z)f_1 + T_2 f_2 = 0, \tag{4}$$

$$(T_3 - z)f_2 = 0. \tag{5}$$

By Theorem 3.6 and (5), $f_2 = 0$. Hence from (4) we have $(T_1 - z)f_1 = 0$. Again by Theorem 3.6, $f_1 = 0$. Thus $f = 0$. ∎

4. Subscalarity. In this section we show that some w-hyponormal operators have scalar extensions.

Lemma 4.1. *An operator $|T|^{1/2}$ is bounded below if and only if the operator $|\widetilde{T}|^{1/2}$ is bounded below.*

Proof. If $|T|^{1/2}$ is bounded below, then there exists $c > 0$ such that $\||T|^{1/2}x\| \geq c\|x\|$ for all $x \in \mathbf{H}$. An easy calculation shows that $\|\widetilde{T}x\| = \||T|^{1/2}U|T|^{1/2}x\| \geq c^2\|x\|$ for all $x \in \mathbf{H}$. Hence $\||\widetilde{T}|x\| \geq c^2\|x\|$ for all $x \in \mathbf{H}$. Thus $|\widetilde{T}|^{1/2}$ is bounded below.

Conversely, if $|\widetilde{T}|^{1/2}$ is bounded below, then it is clear that $\widetilde{T}$ is bounded below. Since $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(\widetilde{T})$ by [JKP], T is bounded below. Hence $|T|$ is bounded below. So we conclude that $|T|^{1/2}$ is bounded below. ∎

Corollary 4.2. *An operator $|T|^{1/2}$ has closed range if and only if the operator $|\widetilde{T}|^{1/2}$ has closed range.*

Proof. This is clear from Lemma 4.1. ∎

Lemma 4.3. *Let $T \in \mathcal{L}(\mathbf{H})$ be a semi-hyponormal operator. If $\{f_n\}$ is a sequence in $L^2(D, \mathbf{H})$ such that $\lim_{n\to\infty} \|(T - z)f_n\|_{2,D} = 0$ for all $z \in D$, then $\lim_{n\to\infty} \|(T - z)^* f_n\|_{2,D} = 0$.*

Proof. Assume that $\{f_n\}$ is as in the hypothesis. Let $Q = |T| - |T^*|$, $z = \varrho e^{i\theta}$, $0 < \varrho$, and $|e^{i\theta}| = 1$ where $|T^*| = (TT^*)^{1/2}$. Since T is semi-hyponormal, [Xi, Lemma 2.1] implies

$$\begin{cases} \lim_{n\to\infty} \|(|T| - \varrho)f_n\|_{2,D} = 0, \\ \lim_{n\to\infty} \varrho\|(|T|^{1/2}(U - e^{i\theta})^* f_n\|_{2,D} = 0, \\ \lim_{n\to\infty} \varrho\langle Qf_n, f_n\rangle = 0. \end{cases}$$

Since

$$(T - z)^* f_n = |T|^{1/2}[|T|^{1/2}(U - e^{i\theta})^* f_n] + e^{-i\theta}[(|T| - \varrho)f_n],$$

we have

$$\|(T - z)^* f_n\|_{2,D} \leq \||T|^{1/2}\| \cdot \|(|T|^{1/2}(U - e^{i\theta})^* f_n\|_{2,D} + \|(|T| - \varrho)f_n\|_{2,D}.$$

This completes the proof. ∎

Lemma 4.4. *Let $T = U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$, and let D be a bounded disk which contains $\sigma(T)$. Then the map $V : \mathbf{H} \to H(D)$ defined by*

$$Vh = 1 \widetilde{\otimes} h (\equiv 1 \otimes h + \overline{(T - z)W^2(D, \mathbf{H})})$$

is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h and $H(D) := W^2(D, \mathbf{H})/\overline{(T - z)W^2(D, \mathbf{H})}$.

Proof. Let $h_n \in \mathbf{H}$ and $f_n \in W^2(D, \mathbf{H})$ be sequences such that

$$\lim_{n\to\infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0. \tag{6}$$

Then by the definition of the norm of Sobolev space, (6) implies

$$\lim_{n\to\infty}\|(U|T|-z)\overline{\partial}^i f_n\|_{2,D}=0 \tag{7}$$

for $i=1,2$. Since $\widetilde{T}=|T|^{1/2}U|T|^{1/2}$,

$$\lim_{n\to\infty}\|(\widetilde{T}-z)\overline{\partial}^i(|T|^{1/2}f_n)\|_{2,D}=0 \tag{8}$$

for $i=1,2$. Let $\widetilde{T}=\widetilde{U}|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$. Then from (8) we have, for $i=1,2$,

$$\lim_{n\to\infty}\|(\widetilde{\widetilde{T}}-z)\overline{\partial}^i(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)\|_{2,D}=0. \tag{9}$$

Since $\widetilde{\widetilde{T}}$ is hyponormal, by [Pu, Corollary 2.2],

$$\lim_{n\to\infty}\|(I-P)(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)\|_{2,D}=0 \tag{10}$$

where P denotes the orthogonal projection of $L^2(D,\mathbf{H})$ onto $A^2(D,\mathbf{H})$. From (6) and (10) we get

$$\lim_{n\to\infty}\|(\widetilde{\widetilde{T}}-z)P(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)+1\otimes|\widetilde{T}|^{1/2}|T|^{1/2}h_n\|_{2,D}=0. \tag{11}$$

Let Γ be a curve in D surrounding $\sigma(T)$ $(=\sigma(\widetilde{T})=\sigma(\widetilde{\widetilde{T}})$ by [JKP]). Then for $z\in\Gamma$,

$$\lim_{n\to\infty}\|P(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)(z)+(\widetilde{\widetilde{T}}-z)^{-1}(1\otimes|\widetilde{T}|^{1/2}|T|^{1/2}h_n)\|=0$$

uniformly, from (11). Hence

$$\lim_{n\to\infty}\left\|\frac{1}{2\pi i}\int_{\Gamma}P(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)(z)\,dz+|\widetilde{T}|^{1/2}|T|^{1/2}h_n\right\|=0.$$

But by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{\Gamma}P(|\widetilde{T}|^{1/2}|T|^{1/2}f_n)(z)\,dz=0.$$

Hence $\lim_{n\to\infty}|\widetilde{T}|^{1/2}|T|^{1/2}h_n=0$. Since $0\notin\sigma_{\mathrm{ap}}(|T|^{1/2})$, by Lemma 4.1, $|\widetilde{T}|^{1/2}|T|^{1/2}$ is bounded below. Hence $\lim_{n\to\infty}h_n=0$. Thus the map V is one-to-one and has closed range. ∎

THEOREM 4.5. *If $T=U|T|$ (polar decomposition) is a w-hyponormal operator with $0\notin\sigma_{\mathrm{ap}}(|T|^{1/2})$, then T is a subscalar operator of order* 2.

Proof. Suppose that $T=U|T|$ (polar decomposition) is a w-hyponormal operator with $0\notin\sigma_{\mathrm{ap}}(|T|^{1/2})$. Consider an arbitrary bounded open disk D in the complex plane $\mathbb{C}$ and the quotient space

$$H(D)=W^2(D,\mathbf{H})/\overline{(T-z)W^2(D,\mathbf{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{A}$. Let M be the operator of multiplication by z on $W^2(D,\mathbf{H})$. As noted at the end of Section 2, M is a scalar operator of order 2 and has a spectral distribution Φ. Let $S \equiv \widetilde{M}$. Since $\overline{(T-z)W^2(D,\mathbf{H})}$ is invariant under every operator M_f, $f \in C^2(D)$, we infer that S is a scalar operator of order 2 with spectral distribution $\widetilde{\Phi}$.

Consider the natural map $V : \mathbf{H} \to H(D)$ defined by $Vh = (1 \otimes h)^{\sim}$ for $h \in \mathbf{H}$, where $1 \otimes h$ denotes the constant function identically equal to h. Note that $VT = SV$. In particular $\operatorname{ran} V$ is an invariant subspace for S. Since V is one-to-one and has closed range by Lemma 4.4, T is a subscalar operator of order 2. ∎

Corollary 4.6. *Invertible w-hyponormal operators are subscalar of order* 2.

Proof. Let $T = U|T|$ (polar decomposition) be any invertible w-hyponormal operator. Then $|T|$ is invertible and U is unitary. By [Ru, Thm. 12.33], $|T|^{1/2}$ is invertible. Since $|T|^{1/2}$ is positive, $\sigma(|T|^{1/2}) = \sigma_{\mathrm{ap}}(|T|^{1/2})$. Hence $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$. By Theorem 4.5, T is a subscalar operator of order 2. ∎

Corollary 4.7. *If $T = U|T|$ (polar decomposition) is a w-hyponormal operator with $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$, then T has Bishop's property (β).*

Corollary 4.8. *Let $T = U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$. If $A \in \mathcal{L}(\mathbf{H})$ is any quasi-affine transform of T, then $\sigma(T) \subseteq \sigma(A)$.*

Proof. This follows from Corollary 4.7 and [Ko 1, Theorem 3.2]. ∎

Corollary 4.9. *Let $T = U|T|$ (polar decomposition) be a w-hyponormal operator with $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$ and let f be a function analytic in a neighborhood of $\sigma(T)$. With the notation of the proof of Theorem 4.5, $Vf(T) = f(S)V$, where $f \mapsto f(T)$ is the functional calculus morphism.*

Proof. This follows from a general property of the analytic functional calculus. ∎

5. Theorems on invariant subspaces. In this section we study invariant subspaces of w-hyponormal operators. Recall that if U is a nonempty open set in $\mathbb{C}$ and if $\Omega \subset U$ has the property that

$$\sup_{\lambda \in \Omega} |f(\lambda)| = \sup_{\beta \in U} |f(\beta)|$$

for every function f in $H^\infty(U)$ (i.e. for all f bounded and holomorphic on U), then Ω is said to be *dominating* for U.

The next theorem is a generalization of Scott Brown's theorem.

THEOREM 5.1. *Suppose that T is an arbitrary w-hyponormal operator and there exists a nonempty open set U in $\mathbb{C}$ such that $\sigma(T) \cap U$ is dominating for U. Then T has a nontrivial invariant subspace.*

Proof. If T is not a quasi-affinity, then $0 \in \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)$. So it is trivial that T has a nontrivial invariant subspace. Let T be a quasi-affinity. Since $\widetilde{T}$ is semi-hyponormal from the definition of a w-hyponormal operator, [JKP, Theorem 1.24] implies that $\widetilde{T}$ has a nontrivial invariant subspace. By [JKP, Theorem 1.15], T has a nontrivial invariant subspace. ∎

The following theorem is a generalization of Berger's theorem.

THEOREM 5.2. *Let T be an arbitrary w-hyponormal operator. Then there exists a positive integer K such that for all positive integers $k \geq K$, T^k has a nontrivial invariant subspace.*

Proof. If T is not a quasi-affinity, then the result is trivial. Suppose that T is a quasi-affinity. Since $\widetilde{T}$ is semi-hyponormal from the definition of a w-hyponormal operator, by [JKP, Theorem 1.25] there exists a positive integer K such that for all positive integers $k \geq K$, $(\widetilde{T})^k$ has a nontrivial invariant subspace $\mathcal{M}_k$. Since $U|T|^{1/2}(\widetilde{T})^j = T^j U|T|^{1/2}$ and $\mathcal{M}_k \in \mathrm{Lat}((\widetilde{T})^k)$ for $k \geq K$,

$$T^k U|T|^{1/2}\mathcal{M}_k = U|T|^{1/2}(\widetilde{T})^k\mathcal{M}_k \subset U|T|^{1/2}\mathcal{M}_k, \quad k \geq K.$$

By [JKP, Theorem 1.15],

$$\{0\} \neq (U|T|^{1/2}\mathcal{M}_k)^- \neq \mathbf{H}.$$

Therefore, $(U|T|^{1/2}\mathcal{M}_k)^-$ is the desired invariant subspace for T^k. ∎

Recall that a closed subspace of $\mathbf{H}$ is said to be *hyperinvariant* for T if it is invariant under every operator in the commutant $\{T\}'$ of T.

THEOREM 5.3. *Suppose that T is an arbitrary w-hyponormal operator and*

$$\lim_{n\to\infty} \|T^n h\|^{1/n} < \|T\|$$

for some nonzero $h \in \mathbf{H}$. Then T has a nontrivial hyperinvariant subspace.

Proof. If T is an arbitrary w-hyponormal operator, then by [AW 2],

$$\|Th\|^2 \leq \|T^2h\| \cdot \|h\|$$

for all $h \in \mathbf{H}$. Hence [Bo, Remark] implies that T has a nontrivial hyperinvariant subspace. ∎

Recall that an operator $T \in \mathcal{L}(\mathbf{H})$ is *decomposable* provided that, for each open cover $\{U, V\}$ of $\mathbb{C}$, there exist closed T-invariant subspaces Y, Z of $\mathbf{H}$ such that $\mathbf{H} = Y + Z$, $\sigma(T|_Y) \subset U$, and $\sigma(T|_Z) \subset V$. Here, $T|_Y$ denotes the restriction of T to Y.

LEMMA 5.4 ([LW, Lemma 3.6.1]). *If T is subscalar, then for all closed F in $\mathbb{C}$, $H_T(F)$ is the linear span of all manifolds Z in $\mathbf{H}$ satisfying $(\lambda-T)Z = Z$ for all $\lambda \notin F$, where $H_T(F) = \{x \in \mathbf{H} : x = (\lambda - T)f(\lambda)$ for some analytic $f : \mathbb{C} \setminus F \to \mathbf{H}\}$.*

THEOREM 5.5. *Let T be a w-hyponormal operator with $0 \notin \sigma_{\mathrm{ap}}(|T|^{1/2})$ and let $T \neq \lambda I$ for all $\lambda \in \mathbb{C}$. If S is a decomposable quasi-affine transform of T, then T has a nontrivial hyperinvariant subspace.*

Proof. Assume that X is a quasi-affinity such that $XS = TX$ where S is decomposable. If T has no nontrivial hyperinvariant subspace, we may assume that $\sigma_{\mathrm{p}}(T) = \emptyset$ and $H_T(F) = \{0\}$ for each closed F proper in $\sigma(T)$ by Lemma 5.4. Let $\{U, V\}$ be an open cover of $\mathbb{C}$ with $\sigma(T) \setminus \overline{U} \neq \emptyset$ and $\sigma(T) \setminus \overline{V} \neq \emptyset$. Then

$$X\mathbf{H} = XH_S(\overline{U}) + XH_S(\overline{V}) \subseteq H_T(\overline{U}) + H_T(\overline{V}) = \{0\}.$$

So we have a contradiction. ∎

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