Perturbation theorems of Miyadera type for locally Lipschitz continuous integrated semigroups

by

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Abstract. A class of perturbing operators for locally Lipschitz continuous integrated semigroups is introduced according to the idea of Miyadera. The paper gives perturbation theorems of Miyadera type for such integrated semigroups.

1. Introduction. This paper is devoted to perturbation theorems of Miyadera type for locally Lipschitz continuous integrated semigroups on a Banach space X with norm $\| \cdot \|$.

The study of abstract Cauchy problems for non-densely defined operators satisfying the Hille–Yosida conditions was initiated by Da Prato and Sinestrari [3], and it has recently been done by using the theory of locally Lipschitz continuous integrated semigroups. It is known [5, Proposition 3.3] that bounded perturbations preserve the property of being a generator for such integrated semigroups. This is an extension of a perturbation theorem for semigroups of class (C_0) due to Phillips.

Miyadera [7] introduced a class of perturbing operators for semigroups of class (C_0) and obtained interesting perturbation theorems for such semigroups by using a convergence theorem. We are interested in obtaining a theorem unifying the above-mentioned two results, and we discuss the perturbation problem for locally Lipschitz continuous integrated semigroups.

We recall the definition of locally Lipschitz continuous integrated semigroups. Let $B(X)$ be the set of all bounded linear operators on X. A family $\{U(t); \, t \geq 0\}$ in $B(X)$ is called a *locally Lipschitz continuous integrated semigroup* on X if it satisfies the following conditions:

(U1)   For each $\tau > 0$ there exists $L_\tau > 0$ such that
$$\|U(t) - U(s)\| \leq L_\tau |t - s| \quad \text{for } t, s \in [0, \tau].$$

(U2) $U(t)U(s)x = \int_s^{t+s} U(\tau)x \, d\tau - \int_0^t U(\tau)x \, d\tau$ for $x \in X$ and $t, s \geq 0$.

(U3)   If $U(t)x = 0$ for all $t > 0$ then $x = 0$.

It is known [5, Corollary 2.3] that every locally Lipschitz continuous integrated semigroup $\{U(t); t \geq 0\}$ on X is *exponentially bounded* in the sense that $\|U(t)\| \leq Me^{\omega t}$ for $t \geq 0$. In this case, there exists a unique closed linear operator A in X such that (ω, ∞) is contained in the resolvent set $\varrho(A)$ of A and

$$(1.1) \qquad (\lambda I - A)^{-1}x = \int_0^\infty \lambda e^{-\lambda t} U(t)x \, dt$$

for $x \in X$ and $\lambda > \omega$. The operator A is called the *generator* of $\{U(t); t \geq 0\}$ on X (see Arendt [1]). The following characterization of generators of locally Lipschitz continuous integrated semigroups was obtained in [5, Theorem 2.4]. A linear operator A is the generator of a locally Lipschitz continuous integrated semigroup on X if and only if there exist $M > 0$ and $\omega \geq 0$ such that $(\omega, \infty) \subset \varrho(A)$ and

$$(1.2) \qquad \|(\lambda I - A)^{-m}\| \leq M(\lambda - \omega)^{-m} \quad \text{for } m \geq 1 \text{ and } \lambda > \omega.$$

A simple example of a locally Lipschitz continuous integrated semigroup is the family $\{U(t); t \geq 0\}$ defined by

$$(1.3) \qquad U(t)x = \int_0^t T(s)x \, ds$$

for $x \in X$ and $t \geq 0$, where $\{T(t); t \geq 0\}$ is a semigroup of class (C_0) on X generated by A. In this case, the generator of $\{U(t); t \geq 0\}$ is equal to A.

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2. A class of perturbing operators. According to the idea of Miyadera [7], we discuss the perturbation problem for locally Lipschitz continuous integrated semigroups by using a convergence theorem for such integrated semigroups. For this purpose, we introduce a class of perturbing operators as follows.

Let A be the generator of a locally Lipschitz continuous integrated semigroup $\{U(t); t \geq 0\}$ on X. A linear operator B is said to *belong to the class* $\mathfrak{P}(A)$ if there exists a sequence $\{B_n\}$ in $B(X)$ such that

(P1) $D(A) \subset D(B)$ and $\lim_{n \to \infty} B_n x = Bx$ for every $x \in D(A)$,

(P2) $\sup_{n \geq 1} \text{TV}(B_n U(\cdot)x; [0, 1]) < \infty$ for every $x \in X$,

where $\text{TV}(f; [0, 1])$ denotes the total variation of f on $[0, 1]$.

We give two subclasses of $\mathfrak{P}(A)$.

PROPOSITION 2.1. (a) *Let A be the generator of a semigroup $\{T(t);\, t \geq 0\}$ of class (C_0) on X. If a linear operator B in X satisfies the two conditions*

(a-i) $D(A) \subset D(B)$ *and* $B(\lambda_0 I - A)^{-1} \in B(X)$ *for some* λ_0,
(a-ii) *there exists $K > 0$ such that*

$$\int_0^1 \|BT(t)x\|\, dt \leq K\|x\| \quad \text{for } x \in D(A),$$

then $B \in \mathfrak{P}(A)$.

(b) *Let A be the generator of a locally Lipschitz continuous integrated semigroup $\{U(t);\, t \geq 0\}$ on X. If a closed linear operator B in X satisfies the three conditions*

(b-i) $D(B) \supset \bigcup_{t \geq 0} U(t)(X)$,
(b-ii) *for each $x \in X$, $BU(\cdot)x \colon [0, \infty) \to X$ is continuous,*
(b-iii) *there exists $K > 0$ such that*

$$\mathrm{TV}(BU(\cdot)x; [0, 1]) \leq K\|x\| \quad \text{for } x \in X,$$

then $B \in \mathfrak{P}(A)$. In particular, if $B \in B(\overline{D(A)}, X)$ then $B \in \mathfrak{P}(A)$.

REMARK 2.1. The perturbation theory for semigroups of class (C_0) was initiated and a class of perturbing operators ([4, Section 13.4, Definition 13.3.5]) was introduced by Phillips. The class of linear operators B satisfying (a-i) and (a-ii) is the class of perturbing operators introduced by Miyadera [7]. It was proved in [7, Section 4] that this class properly includes the above-mentioned class of perturbing operators due to Phillips.

Proof of Proposition 2.1. To prove (a), we first notice that the operator A is the generator of the locally Lipschitz continuous integrated semigroup $\{U(t);\, t \geq 0\}$ defined by (1.3). Consider $B_n \in B(X)$ defined by $B_n = B(I - (1/n)A)^{-1}$ for sufficiently large n. This definition is unambiguous because of condition (a-i) and the resolvent identity. Since

$$B(I - (1/n)A)^{-1}(\lambda_0 I - A)^{-1}x = B(\lambda_0 I - A)^{-1}(I - (1/n)A)^{-1}x$$
$$\to B(\lambda_0 I - A)^{-1}x$$

for every $x \in X$, we see that condition (P1) is satisfied. To check condition (P2), let $0 = t_0 < t_1 < \ldots < t_N = 1$ and $x \in X$. By (1.3) we have

$$\sum_{i=1}^N \|B_n U(t_i)x - B_n U(t_{i-1})x\| \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|BT(s)(I - (1/n)A)^{-1}x\|\, ds,$$

so that $\mathrm{TV}(B_n U(\cdot)x; [0, 1]) \leq K\|(I - (1/n)A)^{-1}\|\, \|x\|$ by condition (a-ii). This means that condition (P2) is satisfied.

To prove (b), we need (ii-2) of Lemma 2.2 which will be proved below. By (ii-2) there exist $K_0 > 0$ and $\beta \geq 0$ such that $\|BU(t)x\| \leq K_0 e^{\beta t}\|x\|$ for $x \in X$ and $t \geq 0$. This fact together with (1.1) implies that $D(A) \subset D(B)$ by the closedness of B, so that $B_n = B(I - (1/n)A)^{-1} \in B(X)$ is well defined for sufficiently large n. Since $U(t)$ commutes with the resolvent of A and $\|(I - (1/n)A)^{-1}\| \leq M(1 - \omega/n)^{-1}$ for some $M \geq 1$ and $\omega \geq 1$ (by (1.2)), condition (P2) is a direct consequence of (b-iii). By (1.1) and property (U2) we find

$$B_n U(t)x = n^2 \int_0^\infty e^{-n\tau} BU(\tau)U(t)x \, d\tau$$

$$= n \int_0^\infty e^{-n\tau} B(U(\tau + t)x - U(\tau)x) \, d\tau$$

for $x \in X$ and sufficiently large n, where we have used integration by parts to obtain the last equality. The right-hand side of the above equality is equal to $\int_0^\infty e^{-s} B(u(s/n + t)x - U(s/n)x) \, ds$, which tends to $BU(t)x$ as $n \to \infty$, by condition (b-ii). Notice that $\|B_n U(t)x\| \leq MK_0 e^{\beta t}(1 - \omega/n)^{-1}\|x\|$ for $x \in X$ and $n > \omega$. Condition (P1) follows readily from the above fact and (1.1), with the help of Lebesgue's convergence theorem.

Finally, let $B \in B(\overline{D(A)}, X)$ and $\|Bx\| \leq M_B\|x\|$ for $x \in \overline{D(A)}$. Since $U(t)(X) \subset \overline{D(A)}$ for $t \geq 0$ (by the inversion formula [4, Theorem 6.3.5]), we see that conditions (b-i) and (b-ii) are satisfied. To check condition (b-iii), let $0 = t_0 < t_1 < \ldots < t_N = 1$. Then, by property (U1),

$$\|BU(t_i)x - BU(t_{i-1})x\| \leq M_B\|U(t_i)x - U(t_{i-1})x\|$$
$$\leq M_B L_1(t_i - t_{i-1})\|x\| \quad \text{for } i = 1, \ldots, N,$$

so that $\mathrm{TV}(BU(\cdot)x; [0,1]) \leq M_B L_1\|x\|$. It follows that $B \in \mathfrak{P}(A)$. ∎

The following lemma will play an essential role in later arguments.

Let A be the generator of a locally Lipschitz continuous integrated semigroup $\{U(t); t \geq 0\}$ on X. Then there exist $M \geq 1$ and $\omega \geq 0$ such that

$$(2.1) \qquad \|U(t + h) - U(t)\| \leq M e^{\omega(t+h)}h \quad \text{for } t, h \geq 0.$$

This fact is obtained by combining [5, Theorem 2.4] and [9, Theorem 3.1].

LEMMA 2.2. (i) *If a sequence* $\{B_n\}$ *in* $B(X)$ *satisfies condition* (P2), *then there exists* $K > 0$ *such that*

$$\mathrm{TV}(B_n U(\cdot)x; [0,1]) \leq K\|x\| \quad \text{for } x \in X \text{ and } n \geq 1.$$

(ii) *Let* B *be a closed linear operator in* X *satisfying conditions* (b-i) *through* (b-iii). *Let* $\lambda > \omega$ *and* $0 \leq a = t_0 < t_1 < \ldots < t_N = b$. *Then:*

(ii-1) *For $x \in X$ we have*

$$\sum_{l=1}^{N} e^{-\lambda t_l} \|BU(t_l)x - BU(t_{l-1})x\|$$

$$\leq \begin{cases} (K + MKe^{-(\lambda-\omega)}(1 - e^{-(\lambda-\omega)})^{-1})\|x\| & \text{for } 0 \leq a < 1, \\ MKe^{-(\lambda-\omega)[a]}(1 - e^{-(\lambda-\omega)})^{-1}\|x\| & \text{for } a \geq 1, \end{cases}$$

where $[a]$ denotes the integer part of a.

(ii-2) *There exist $K_0 > 0$ and $\beta \geq 0$ such that $\|BU(t)x\| \leq K_0 e^{\beta t}\|x\|$ for $x \in X$ and $t \geq 0$.*

Proof. The proof is based on the idea of Miyadera [7, Lemma 1], but more delicate arguments are needed here. The functional p defined by $p(x) = \sup_{n \geq 1} \text{TV}(B_n U(\cdot)x; [0,1])$ for $x \in X$ is a lower semicontinuous seminorm on X. If we set $X_n = \{x \in X; p(x) \leq n\}$ for each $n \geq 1$, then each X_n is closed in X by the lower semicontinuity of p, and the union of all sets X_n is equal to X. The Baire category theorem asserts that at least one X_{n_0} contains an open ball with center x_0 and radius R. Since p is a seminorm on X, we have $p(x) \leq (2\|x\|/R)\{p(x_0 + Rx/(2\|x\|)) + p(x_0)\} \leq (4n_0/R)\|x\|$ for all $x \in X$ with $x \neq 0$, whence (i) follows.

To prove (ii), let $P = \{[a] = \tau_0 < \tau_1 < \ldots\}$ be the partition of $[[a], \infty)$ which is constructed by appending the points $\{k\}_{k=[a]}^{\infty}$ to the sequence $\{t_i\}_{i=0}^{N}$. By the construction of P, for each integer $k \geq [a]$ there exists an integer $i(k)$ such that $\tau_{i(k)} = k$. Let $x \in X$. Then

$$\sum_{l=1}^{N} e^{-\lambda t_l} \|BU(t_l)x - BU(t_{l-1})x\| \leq \sum_{j=1}^{\infty} e^{-\lambda \tau_j} \|BU(\tau_j)x - BU(\tau_{j-1})x\|$$

$$= \sum_{k=[a]+1}^{\infty} \sum_{j=i(k-1)+1}^{i(k)} e^{-\lambda \tau_j} \|BU(\tau_j)x - BU(\tau_{j-1})x\|.$$

If $i(k-1) + 1 \leq j \leq i(k)$ then $\tau_j \geq k - 1$, so that the above inequality shows that

$$(2.2) \quad \sum_{l=1}^{N} e^{-\lambda t_l} \|BU(t_l)x - BU(t_{l-1})x\|$$

$$\leq \sum_{k=[a]+1}^{\infty} e^{-\lambda(k-1)} \sum_{j=i(k-1)+1}^{i(k)} \|BU(\tau_j)x - BU(\tau_{j-1})x\|.$$

By (b-iii) we have $\sum_{j=i(k-1)+1}^{i(k)} \|BU(\tau_j)x - BU(\tau_{j-1})x\| \leq K\|x\|$ for $k = 1$. Let $k \geq 2$. By property (U2) we find

$$(U(s+\xi) - U(s))(U(t+h)x - U(t)x)/h$$
$$= \frac{1}{h} \int_{t+s+\xi}^{t+s+\xi+h} U(\sigma)x \, d\sigma - \frac{1}{h} \int_{t+s}^{t+s+h} U(\sigma)x \, d\sigma$$

for $s, t, \xi, h \geq 0$. Applying this equality with $s = \tau_{j-1} - (k-1)$, $\xi = \tau_j - \tau_{j-1}$ and $t = k - 1$ we have by the closedness of B and (b-ii),

$$\sum_{j=i(k-1)+1}^{i(k)} \left\| \frac{1}{h} \int_{k-1}^{k-1+h} (BU(\sigma + \tau_j - (k-1))x - BU(\sigma + \tau_{j-1} - (k-1))x) \, d\sigma \right\|$$

$$= \sum_{j=i(k-1)+1}^{i(k)} \|(BU(\tau_j - (k-1)) - BU(\tau_{j-1} - (k-1)))$$
$$\times (U((k-1) + h)x - U(k-1)x)/h\|.$$

We estimate the right-hand side by (b-iii) and (2.1), and let $h \downarrow 0$ in the resulting inequality. This yields $\sum_{j=i(k-1)+1}^{i(k)} \|BU(\tau_j)x - BU(\tau_{j-1})x\| \leq KMe^{\omega(k-1)}\|x\|$. The desired estimate is obtained by substituting the two estimates shown above into (2.2).

To prove (ii-2), let $\beta > \omega$. Since $U(0) = 0$, the desired assertion is obtained by using (ii-1) with $a = 0$, $b = t$ and $\lambda = \beta$. ■

3. Perturbation theorems of Miyadera type for locally Lipschitz continuous integrated semigroups. According to the idea of Miyadera [7], we first introduce the number r defined by (3.2) below. This number is used to obtain a lower estimate for perturbing operators.

Let A be the generator of a locally Lipschitz continuous integrated semigroup $\{U(t); t \geq 0\}$ on X satisfying (2.1), and let $B \in \mathfrak{P}(A)$. Then there exists a sequence $\{B_n\}$ in $B(X)$ such that conditions (P1) and (P2) are satisfied.

Let $x \in X$, $\lambda > \omega$ and $R > 0$. For each partition $P = \{0 = t_0 < t_1 < \ldots < t_N = R\}$, consider the number

$$(3.1) \qquad r_{\lambda,R}^{(n)}(x) = \liminf_{|P| \to 0} \sum_{l=1}^{N} e^{-\lambda t_l} \|B_n U(t_l)x - B_n U(t_{l-1})x\|$$

for $n = 1, 2, \ldots$, where $|P| = \max_{1 \leq l \leq N}(t_l - t_{l-1})$, and define

$$r_\lambda = \sup_{\|x\| \leq 1} r_\lambda(x),$$

where $r_\lambda(x) = \sup_{n \geq 1} r_\lambda^{(n)}(x)$ and $r_\lambda^{(n)}(x) = \sup_{R>0} r_{\lambda,R}^{(n)}(x)$. By Lemma 2.2 we have $r_{\lambda,R}^{(n)}(x) \leq (K + MKe^{-(\lambda-\omega)}(1 - e^{-(\lambda-\omega)})^{-1})\|x\|$ for $n = 1, 2, \ldots$, so that r_λ is finite. Since r_λ is a nonnegative, nonincreasing function in

$\lambda \in (\omega, \infty)$, the limit

(3.2)
$$r = \lim_{\lambda \to \infty} r_\lambda$$

exists and $0 \leq r < \infty$.

LEMMA 3.1. *Let* $n \geq 1$, $\lambda > \omega$ *and* $x \in X$. *The Riemann–Stieltjes integral* $\int_0^\infty e^{-\lambda t} \, dB_n U(t)x := \lim_{R \to \infty} \int_0^R e^{-\lambda t} \, dB_n U(t)x$ *exists, and*

(3.3)
$$B_n(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} \, dB_n U(t)x.$$

Proof. Let $n \geq 1$, $\lambda > \omega$ and $x \in X$. Since $B_n \in B(X)$ and $U(\cdot)x$ is locally Lipschitz continuous in X on $[0, \infty)$, we see that $B_n U(\cdot)x \colon [a, b] \to X$ is of bounded variation for $0 \leq a < b < \infty$. It follows that

$$\lim_{|P| \to 0} \sum_{l=1}^N e^{-\lambda t_l}(B_n U(t_l)x - B_n U(t_{l-1})x) = \int_a^b e^{-\lambda t} \, dB_n U(t)x$$

for $0 \leq a < b < \infty$, where $P = \{a = t_0 < t_1 < \ldots < t_N = b\}$ is a partition of $[a, b]$. By Lemma 2.2 we have

$$\left\| \int_a^b e^{-\lambda t} \, dB_n U(t)x \right\| \leq MK e^{-(\lambda-\omega)[a]}(1 - e^{-(\lambda-\omega)})^{-1}\|x\| \to 0$$

as $b > a \to \infty$. This shows that the integral $\int_0^\infty e^{-\lambda t} \, dB_n U(t)x$ exists.

To prove (3.3) we notice that

$$\int_0^R e^{-\lambda t} \, dB_n U(t)x = e^{-\lambda R} B_n U(R)x + \int_0^R \lambda e^{-\lambda t} B_n U(t)x \, dt$$

for $R > 0$. The desired result is obtained by letting $R \to \infty$. ∎

The main result of the present paper is the following theorem, which extends the result of Miyadera [7, Theorem 1].

THEOREM 3.2. *Let* A *be the generator of a locally Lipschitz continuous integrated semigroup on* X *and let* $B \in \mathfrak{P}(A)$. *Then for each* ε *with* $|\varepsilon| < 1/r$, *where* $1/r = \infty$ *if* $r = 0$, $A + \varepsilon B$ *is the generator of a locally Lipschitz continuous integrated semigroup on* X.

Proof. Let ε satisfy $|\varepsilon| < 1/r$. We first show that for each $n \geq 1$, $A + \varepsilon B_n$ is the generator of a locally Lipschitz continuous integrated semigroup on X. For this purpose, let $n \geq 1$ and inductively define a sequence $\{V_n^{(k)}\}_{k=1}^\infty$ in $\text{Lip}_{\text{loc}}([0, \infty); B(X))$ by

$$\begin{cases} V_n^{(0)}(t) = U(t) & \text{for } t \geq 0, \\ V_n^{(k)}(t)x = \int_0^t V_n^{(k-1)}(t-s) \, d\varepsilon B_n U(s)x & \text{for } x \in X \text{ and } t \geq 0. \end{cases}$$

Since $B_n U(\cdot)x \colon [0, \infty) \to X$ is of bounded variation on every compact subinterval of $[0, \infty)$, $V_n^{(k)}(t)$ is well defined. Clearly, $V_n^{(k)}(0) = 0$ for $k = 1, 2, \ldots$ We shall show inductively that $V_n^{(k)} \in \mathrm{Lip}_{\mathrm{loc}}([0, \infty); B(X))$ for each $k \geq 1$. By the definition (3.2) of r, there exists β $(> \omega)$ such that $|\varepsilon| r_\beta < 1$, where ω is a number satisfying (2.1). Let $x \in X$ and $t, h \geq 0$. By the definition of $V_n^{(k)}(t)$ we have

$$V_n^{(k)}(t+h)x - V_n^{(k)}(t)x = \int\limits_{t}^{t+h} V_n^{(k-1)}(t+h-s)\, d\varepsilon B_n U(s)x$$

$$+ \int\limits_{0}^{t} (V_n^{(k-1)}(t+h-s) - V_n^{(k-1)}(t-s))\, d\varepsilon B_n U(s)x.$$

If $\|V_n^{(k-1)}(\tau + \delta) - V_n^{(k-1)}(\tau)\| \leq M e^{\beta(\tau+\delta)} \delta(|\varepsilon| r_\beta)^{k-1}$ for $\tau, \delta \geq 0$, then

$$\left\| \int\limits_{t}^{t+h} V_n^{(k-1)}(t+h-s)\, d\varepsilon B_n U(s)x \right\|$$

$$\leq M e^{\beta(t+h)} h(|\varepsilon| r_\beta)^{k-1} |\varepsilon| \liminf_{|P| \to 0} \sum_{l=1}^{N} e^{-\beta t_l} \|B_n U(t_l)x - B_n U(t_{l-1})x\|$$

and

$$\left\| \int\limits_{0}^{t} (V_n^{(k-1)}(t+h-s) - V_n^{(k-1)}(t-s))\, d\varepsilon B_n U(s)x \right\|$$

$$\leq \liminf_{|Q| \to 0} \sum_{l=1}^{\widehat{N}} M e^{\beta(t+h-s_l)} h(|\varepsilon| r_\beta)^{k-1} |\varepsilon|\, \|B_n U(s_l)x - B_n U(s_{l-1})x\|,$$

where P and Q are partitions of $[t, t+h]$ and $[0, t]$ respectively. Since $P \cup Q$ is a partition of $[0, t+h]$, we find by (3.1) that

$$\|V_n^{(k)}(t+h)x - V_n^{(k)}(t)x\| \leq M e^{\beta(t+h)} h(|\varepsilon| r_\beta)^{k-1} |\varepsilon| r_{\beta, t+h}^{(n)}(x),$$

and inductively, $\|V_n^{(k)}(t+h) - V_n^{(k)}(t)\| \leq M e^{\beta(t+h)} h(|\varepsilon| r_\beta)^k$ for $t, h \geq 0$ and $k = 1, 2, \ldots$

Now, define a family $\{V_n(t); t \geq 0\}$ in $B(X)$ by $V_n(t)x = \sum_{k=0}^{\infty} V_n^{(k)}(t)x$ for $x \in X$ and $t \geq 0$, with the help of the estimate shown above. Clearly,

(3.4) $\|V_n(t+h) - V_n(t)\| \leq M e^{\beta(t+h)} h(1 - |\varepsilon| r_\beta)^{-1}$

for all $t \geq 0$ and $h \geq 0$. Since $(d/dt)\int_0^t V_n^{(k-1)}(t-s) B_n U(s)x\, ds = \int_0^t V_n^{(k-1)}(t-s)\, dB_n U(s)x$, we inductively show

$$\lambda \int\limits_0^\infty e^{-\lambda t} V_n^{(k)}(t) x \, dt = (\lambda I - A)^{-1} (\varepsilon B_n (\lambda I - A)^{-1})^k x$$

for $x \in X$ and $\lambda > \beta$, by using (1.1) and the theorem about the Laplace transform of a convolution. By (3.3) together with the definition of r_β, we find

$$(3.5) \qquad \|\varepsilon B_n (\lambda I - A)^{-1}\| \le |\varepsilon| r_\beta < 1$$

for $\lambda > \beta$. Here we have used the fact that r_λ is nonincreasing in λ. It follows that $\int_0^\infty \lambda e^{-\lambda t} V_n(t) x \, dt = (\lambda I - A)^{-1} (I - \varepsilon B_n (\lambda I - A)^{-1})^{-1} x = (\lambda I - (A + \varepsilon B_n))^{-1} x$ for $x \in X$ and $\lambda > \beta$. By [1, Theorem 3.1] we see that $\{V_n(t); t \ge 0\}$ is an integrated semigroup on X, and its generator is $A + \varepsilon B_n$.

The desired claim will be derived from a convergence theorem (Theorem 4.1 in Appendix) for locally Lipschitz continuous integrated semigroups as follows: By condition (P1), the estimate that $\|\varepsilon B(\lambda I - A)^{-1}\| \le |\varepsilon| r_\beta$ for $\lambda > \beta$ follows from (3.5). This estimate implies that $(\lambda I - (A + \varepsilon B))^{-1}$ exists as an element of $B(X)$. Clearly, $\lim_{n\to\infty} (A + \varepsilon B_n) x = (A + \varepsilon B) x$ for $x \in D(A)$, by condition (P1). Notice that the family $\{V_n(t); t \ge 0\}$ satisfies (3.4). Then Theorem 4.1 asserts that $A + \varepsilon B$ is the generator of a locally Lipschitz continuous integrated semigroup on X. ∎

Combining Proposition 2.1 and Theorem 3.2 we have the following two results.

COROLLARY 3.3 [7, Theorem 2]. *Let A be the generator of a semigroup of class (C_0) on X. If a linear operator B in X satisfies conditions* (a-i) *and* (a-ii), *then there exists an $\varepsilon_0 > 0$ which is finite or ∞ such that for each ε with $|\varepsilon| < \varepsilon_0$, $A + \varepsilon B$ is the generator of a semigroup of class (C_0) on X.*

REMARK 3.1. (1) By Remark 2.1, Corollary 3.3 is an extension of the result of Phillips [4, Section 13.4, Corollary 1].

(2) Voigt [10] obtained a similar result under weaker assumptions. We do not know whether his result can be obtained from ours.

COROLLARY 3.4. *Let A be the generator of a locally Lipschitz continuous integrated semigroup on X. If a closed linear operator B in X satisfies conditions* (b-i) *through* (b-iii), *then there exists an $\varepsilon_0 > 0$ which is finite or ∞ such that for each ε with $|\varepsilon| < \varepsilon_0$, $A + \varepsilon B$ is the generator of a locally Lipschitz continuous integrated semigroup on X.*

The following result is a special case of Corollary 3.4.

COROLLARY 3.5 [5, Proposition 3.3]. *If A is the generator of a locally Lipschitz continuous integrated semigroup on X and $B \in B(\overline{D(A)}, X)$ then*

$A+B$ is the generator of a locally Lipschitz continuous integrated semigroup on X.

Proof. It suffices to show that the number r defined by (3.2) is zero. Let $M \geq 1$ and $\omega \geq 0$ be constants satisfying (2.1). As was shown in Proposition 2.1, the operator B is an element of $\mathfrak{P}(A)$ by considering a sequence $\{B_n\}$ in $B(X)$ defined by $B_n = B(I-(1/n)A)^{-1}$ for $n \geq n_0$, where $n_0 = [\omega] + 1$. Let $\lambda > \omega$ and $n \geq n_0$. If $P = \{0 = t_0 < t_1 < \ldots < t_N = R\}$ is a partition of $[0, R]$ then by (2.1) we have

$$\|B_n U(t_l)x - B_n U(t_{l-1})x\| \leq M_B M^2 e^{\omega t_l}(t_l - t_{l-1})(1 - \omega/n)^{-1}\|x\|,$$

where $M_B > 0$ is a constant such that $\|Bx\| \leq M_B\|x\|$ for $x \in \overline{D(A)}$. This estimate shows that $r_{\lambda,R}^{(n)}(x) \leq M_B M^2 (1-\omega/n)^{-1}(\int_0^R e^{-(\lambda-\omega)t}\,dt)\|x\|$, which implies that $r = 0$. ∎

4. Appendix. This section is devoted to a convergence theorem for locally Lipschitz continuous integrated semigroups. Although such theorems may be known, we give the proof for completeness. It should be noticed that generalizations of the classical Trotter–Kato theorem are found in [2] and [6].

THEOREM 4.1. *For each $n \geq 1$, let A_n be the generator of a locally Lipschitz continuous integrated semigroup $\{V_n(t); t \geq 0\}$ on X satisfying the condition*

$$(4.1) \qquad \|V_n(t+h) - V_n(t)\| \leq Me^{\omega(t+h)}h \quad \text{for } t, h \geq 0,$$

where $M \geq 1$ and $\omega \geq 0$ are independent of n. Assume that A is a closed linear operator in X satisfying the following two conditions:

(a) *For each $x \in D(A)$, there exist $x_n \in D(A_n)$ such that $x_n \to x$ and $A_n x_n \to Ax$ as $n \to \infty$.*
(b) *For some $\lambda_0 > \omega$, the range $R(\lambda_0 I - A)$ of $\lambda_0 I - A$ is dense in X.*

Then A is the generator of a locally Lipschitz continuous integrated semigroup $\{V(t); t \geq 0\}$ on X. Moreover, for $x \in X$ we have

$$(4.2) \qquad \lim_{n\to\infty} V_n(t)x = V(t)x$$

uniformly on every compact subinterval of $[0, \infty)$.

Proof. By (4.1) we have $(\omega, \infty) \subset \varrho(A_n)$ and $\|(\lambda I - A_n)^{-k}\| \leq M(\lambda - \omega)^{-k}$ for $\lambda > \omega$ and $k = 1, 2, \ldots$ As in the proof of [8, Theorem 2.1] we see that A is the generator of a locally Lipschitz continuous integrated semigroup $\{V(t); t \geq 0\}$ on X and

$$(4.3) \qquad \lim_{n\to\infty} (\lambda I - A_n)^{-k}x = (\lambda I - A)^{-k}x$$

for $x \in X$, $\lambda > \omega$ and $k = 1, 2, \ldots$ By [8, Lemma 2.2] we have

$$\|V(t)x - V_n(t)x\| \leq 2M(1 - \lambda\omega)^{-[t/\lambda]} e^{\omega t}(([t/\lambda]\lambda - t)^2 + [t/\lambda]\lambda^2)^{1/2}\|x\|$$
$$+ \sum_{l=1}^{[t/\lambda]} \lambda \|(I - \lambda A_n)^{-l}x - (I - \lambda A)^{-l}x\|$$

for $n \geq 1$, $x \in X$, $t \geq 0$ and $\lambda > 0$ with $\lambda\omega < 1$. This estimate together with (4.3) implies (4.2). ∎

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