(E, F)-**Schur multipliers and applications**

by

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Abstract. For two given symmetric sequence spaces E and F we study the (E, F)-multiplier space, that is, the space of all matrices M for which the Schur product $M * A$ maps E into F boundedly whenever A does. We obtain several results asserting continuous embedding of the (E, F)-multiplier space into the classical (p, q)-multiplier space (that is, when $E = l_p$, $F = l_q$). Furthermore, we present many examples of symmetric sequence spaces E and F whose projective and injective tensor products are not isomorphic to any subspace of a Banach space with an unconditional basis, extending classical results of S. Kwapień and A. Pełczyński (1970) and of G. Bennett (1976, 1977) for the case when $E = l_p$, $F = l_q$.

1. Introduction. For an infinite scalar-valued (real or complex) matrix $A = (a_{ij})_{i,j=1}^{\infty}$ and $n \in \mathbb{N}$ we set

$$T_n(A) := (t_{ij}^{(n)})_{i,j=1}^{\infty}, \quad \text{where} \quad t_{ij}^{(n)} = a_{ij} \quad \text{for } 1 \leq j \leq i \leq n$$

and $t_{ij}^{(n)} = 0$ otherwise. The operator T_n is called the nth *main triangle projection*. S. Kwapień and A. Pełczyński [KP] studied the norms of the operators $(T_n)_{n \geq 1}$ acting on the space $B(l_p, l_q)$ of all bounded linear operators and showed that $\sup_n \|T_n\|_{B(l_p,l_q) \to B(l_p,l_q)} = \infty$ for $1 \leq q \leq p \leq \infty$, $q \neq \infty$, $p \neq 1$. Moreover, as an application, they established that for $1 < p \leq \infty$, $1 < q \leq \infty$ and $1/p + 1/q \leq 1$ (respectively, $1 \leq p < \infty$, $1 \leq q < \infty$ and $1/p + 1/q \geq 1$) the projective (respectively, injective) tensor product of the spaces l_p and l_q is not isomorphic to any subspace of a Banach space with an unconditional basis. In the same paper [KP] it is asked (Problem 1) whether the sequence $(\|T_n\|_{B(l_p,l_q) \to B(l_p,l_q)})_{n \geq 1}$ is bounded for $1 < p < q < \infty$. The positive answer to that question was obtained by G. Bennett [B1], who established that the main triangle projection T defined on an element $A = (a_{ij})_{i,j=1}^{\infty} \in B(l_p, l_q)$ by $T(A) := (t_{ij})_{i,j=1}^{\infty}$, where $t_{ij} = a_{ij}$ for $1 \leq j \leq i$ and $t_{ij} = 0$ otherwise, is bounded for $1 < p < q < \infty$. G. Bennett obtained this result in the framework of the general theory of Schur multipliers on $B(l_p, l_q)$ (briefly,

(p, q)-multipliers). For a deep study and applications of this notion in analysis and operator theory we refer to [B1, B2, P].

The classical Banach space l_p $(1 \leq p \leq \infty)$ is an important representative of the class of symmetric sequence spaces (see e.g. [LT1]). The present paper extends results from [B1], [B2] and [KP] to a wider class of symmetric sequence spaces satisfying certain convexity conditions. In particular, we present sufficient conditions in terms of p-convexity and q-concavity of symmetric sequence spaces guaranteeing that their projective and injective tensor products are not isomorphic to any subspace of a Banach space with an unconditional basis (Theorem 6.5). Our methods are based on the study of general Schur multipliers on $B(E, F)$ (briefly, (E, F)-multipliers), extending and generalizing several results from [B2]. In particular, we establish a number of results concerning the embedding of an (E, F)-multiplier space into a (p, q)-multiplier space and their coincidence (Theorems 4.13 and 4.17).

An important technical tool used in this paper is the theory of generalized Köthe duality (Section 3), which (to the best of our knowledge) was first introduced by Hoffman [H] and presented in a detailed manner in [MP] (see also recent papers [CDS] and [DS]).

In Section 5, we give sufficient conditions for boundedness and unboundedness of the main triangle projection on the space of all bounded operators between symmetric sequence spaces E and F (Proposition 5.2), which is a generalization of the results mentioned above from [B1] and [KP].

In the final section (Section 6), we present an extension of Kwapień and Pełczyński's results for l_p-spaces to a wide class of Orlicz–Lorentz sequence spaces (Theorem 6.7).

2. Preliminaries and notation. Let c_0 be a linear space of all real sequences converging to zero. For every $x = (x_i)_{i=1}^{\infty} \in c_0$, we denote by $|x|$ the sequence $(|x_i|)_{i=1}^{\infty}$ and by x^* the non-increasing rearrangement of $|x|$, that is, $x^* = (x_i^*)_{i=1}^{\infty} \in c_0$, where

$$x_i^* = |x_{n_i}| \quad (i = 1, 2, \dots),$$

where $(n_i)_{i=1}^{\infty}$ is a permutation of the natural numbers such that the sequence $(|x_{n_i}|)$ is non-increasing.

In this paper, we work with symmetric sequence spaces which are a 'close relative' of the classical l_p-spaces, $1 \leq p \leq \infty$ (see [LT1, LT2]).

Recall that a linear space $E \subset c_0$ equipped with a Banach norm $\| \cdot \|$ is said to be a *symmetric sequence space* if the following conditions hold:

(i) if $x, y \in E$ and $|x| \leq |y|$, then $\|x\| \leq \|y\|$;
(ii) if $x \in E$, then $x^* \in E$ and $\|x^*\| = \|x\|$.

Without loss of generality we shall assume that $\|(1, 0, 0, \dots)\| = 1$.

A symmetric sequence space E is said to be *p-convex* $(1 \leq p \leq \infty)$, respectively, *q-concave* $(1 \leq q \leq \infty)$, if

$$\left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p},$$

respectively,

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq C \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|$$

(with a natural modification in the case $p = \infty$ or $q = \infty$) for some constant $C > 0$ and every choice of vectors $x_1, \ldots, x_n$ in X. The least such constant is denoted by $M^{(p)}(E)$ (respectively, $M_{(q)}(E)$) (see e.g. [LT2]).

REMARK 2.1. Any symmetric sequence space is 1-convex and ∞-concave with constants equal to 1.

The following proposition links p-convex and q-concave sequence spaces to classical l_p-spaces.

PROPOSITION 2.2 ([LT2, p. 132]). *If a symmetric sequence space E is p-convex and q-concave, then*

$$(2.1) \qquad\qquad l_p \subset E \subset l_q$$

and

$$(2.2) \qquad\qquad \| \cdot \|_q / M_{(q)}(E) \leq \| \cdot \| \leq M^{(p)}(E) \| \cdot \|_p.$$

Without loss of generality we shall assume that the embedding constants in (2.2) are both equal to 1 [LT2, Proposition 1.d.8].

Below, we restate the result of [LT2, Proposition 1.d.4(iii)] for the case of symmetric sequence spaces. (Here, E^* denotes the Banach dual of E.)

PROPOSITION 2.3. *Let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q = 1$. A separable symmetric space E is p-convex (respectively, concave) if and only if E^* is q-concave (respectively, convex).*

REMARK 2.4. Let E be q-concave $(q < \infty)$. If E is not separable, then E does not have order-continuous norm. It follows from [KA, Chapter 10, §4] that there exists a pairwise disjoint sequence $(z_n)_n \subset E$ such that $z_n \geq 0$ and $\|z_n\|_E = 1$, $n = 1, 2, \ldots$, and $x = \sum_{n=1}^{\infty} z_n \in E$, and this contradicts the q-concavity of E. So if E is q-concave, then E is separable and hence E^* is a symmetric sequence space.

3. Generalized Köthe duality. For a symmetric sequence space E, we denote by $E^{\times}$ its *Köthe dual*, that is,

$$E^{\times} := \left\{ y \in l_{\infty} : \sum_{n=1}^{\infty} |x_n y_n| < \infty \text{ for every } x \in E \right\},$$

and for $y \in E^{\times}$ we set

$$\|y\|_{E^{\times}} := \sup\Big\{\sum_{n=1}^{\infty} |x_n y_n| : \|x\|_E \leq 1\Big\}.$$

Then $(E^{\times}, \|\cdot\|_{E^{\times}})$ is a symmetric sequence space [KPS, Chapter II, §3].

We say that a symmetric sequence space E has the *Fatou property* if the conditions $x_n \uparrow x$, $(x_n)_{n=1}^{\infty} \subset E$, $0 \leq x_n \in E$ and $\sup_n \|x_n\|_E < \infty$ imply that $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$. It is known (see [LT2, Chapter I, b, p. 30]) that E has the Fatou property if and only if $\|x\|_E = \|x\|_{E^{\times\times}}$ for every $x \in E$, where $E^{\times\times} = (E^{\times})^{\times}$.

For a pair of sequences $x = (x_n)_{n=1}^{\infty}, y = (y_n)_{n=1}^{\infty} \in l_{\infty}$ we denote by $x \cdot y$ the sequence $(x_n y_n)_{n=1}^{\infty}$.

For any two symmetric sequence spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we set

$$(3.1) \qquad E^F := \{x \in c_0 : x \cdot y \in F \text{ for every } y \in E\},$$

and for $x \in E^F$,

$$(3.2) \qquad \|x\|_{E^F} := \sup_{\|y\|_E \leq 1} \|x \cdot y\|_F.$$

The fact that the supremum in (3.2) above is finite for every $x \in E^F$ is explained below.

REMARK 3.1. Any $x \in E^F$ can be considered as a bounded linear operator from E into F. So the supremum in (3.2) is finite by the closed graph theorem applied to the operator $x(y) = x \cdot y$ for every $y \in E$.

The proof of the following proposition is routine and is therefore omitted.

PROPOSITION 3.2 ([BCLS, Theorem 4.4]). $(E^F, \|\cdot\|_{E^F})$ *is a symmetric sequence space.*

The symmetric sequence space E^F is called the *generalized Köthe dual space* of the spaces E and F.

REMARK. Another suggestive notation for the space E^F introduced above would be $F : E$ (see e.g. [H]) or $M(E, F)$ (see [Sch]). We use the notation E^F since it is in line with the notations from [MP] and [CDS] which are widely used in this section.

Analyzing the definitions of the Köthe dual and generalized Köthe dual spaces, it is not difficult to see that the spaces E^{l_1} and $E^{\times}$ coincide (see also [MP]).

In the following proposition we collect a number of known results from [MP].

PROPOSITION 3.3.

(i) ([MP, p. 326, item (f)]) $l_\infty^E = E$.
(ii) ([MP, Proposition 3]) *If* $1 \leq r < p \leq \infty$ *and* $1/p + 1/q = 1/r$, *then* $l_p^{l_r} = l_q$.
(iii) ([MP, Theorem 2]) *If* $1 \leq p \leq r \leq \infty$, *then* $l_p^{l_r} = l_\infty$.

It is known (see e.g. [MP, Theorem 2]) that in the general setting of Banach function spaces the space E^F may be trivial, that is, $E^F = \{0\}$. The following proposition shows that this is not the case in the setting of symmetric sequence spaces.

PROPOSITION 3.4. $E^F \supset F$.

Proof. Let $x \in l_\infty^F$ and $y \in E$. Then obviously $y \in l_\infty$ and thus we have
$$\|x \cdot y\|_F \leq \|x\|_{l_\infty^F} \|y\|_E.$$
In particular $x \in E^F$ and $\|x\|_{E^F} \leq \|x\|_{l_\infty^F}$. That is, $E^F \supset l_\infty^F$.

Since $F = l_\infty^F$ (see Proposition 3.3(i)), the claim follows. ∎

Let E, F be sets of sequences. Then we denote
$$E \cdot F := \{x = y \cdot z : y \in E, z \in F\}.$$
The general result of the following proposition can also be found in the earlier paper [L].

PROPOSITION 3.5 ([JR, Theorem 1]). $E \cdot E^\times = l_1$.

The second generalized Köthe dual is defined by $E^{FF} := (E^F)^F$.

REMARK 3.6. Since any symmetric sequence space E is a solid subspace of l_∞, we have $l_\infty \cdot E = E$.

THEOREM 3.7 ([Sch, Theorem 3.8]). *Let* E *and* F *be symmetric sequence spaces such that there exists* $1 < p < \infty$ *such that* E *is* p-*convex and* F *is* p-*concave and* E *has the Fatou property. Then:*

(i) $E \cdot E^F$ *is complete with the norm*
$$\|f\|_{E \cdot E^F} = \inf\{\|g\|_E \|h\|_{E^F} : |f| = gh, \, 0 \leq g \in E, \, 0 \leq h \in E^F\}$$
and $E \cdot E^F = F$.
(ii) $E = E^{FF}$.

4. Schur multipliers. Let E and F be symmetric sequence spaces. For every $A \in B(E, F)$, we set for brevity
$$\|A\|_{E,F} := \|A\|_{B(E,F)} = \sup_{\|x\|_E \leq 1} \|A(x)\|_F$$
and
$$\|A\|_{1,F} := \|A\|_{B(l_1, F)}, \quad \|A\|_{E,\infty} := \|A\|_{B(E, l_\infty)}.$$

If the dual space E^* of E coincides with its Köthe dual $E^\times$, then any such operator A can be identified with the matrix $A = (a_{ij})_{i,j=1}^\infty$, each of whose rows represents an element from $E^\times$ and each column represents an element from F. For a sequence $x = (x_n)_{n \geq 1} \in E$, we have $A(x) = (\sum_j a_{ij} x_j)_i \in F$.

PROPOSITION 4.1. *If E and F are symmetric sequence spaces with the Fatou property, then $\|A\|_{E,F} = \|A^T\|_{F^\times, E^\times}$, where A^T is the transpose matrix of A.*

Proof. Indeed,

$$\|A\|_{E,F} = \sup_{\|x\|_E \leq 1} \|A(x)\|_F = \sup\{\langle A(x), y \rangle : \|x\|_E \leq 1, \|y\|_{F^\times} \leq 1\}$$

$$= \sup\{\langle x, A^T(y) \rangle : \|x\|_E \leq 1, \|y\|_{F^\times} \leq 1\}$$

$$= \sup\{\langle x, A^T(y) \rangle : \|x\|_{E^{\times\times}} \leq 1, \|y\|_{F^\times} \leq 1\}$$

$$= \sup_{\|y\|_{F^\times} \leq 1} \|A^T(y)\|_{E^\times} = \|A^T\|_{F^\times, E^\times}. \quad \blacksquare$$

The following proposition presents formulae for computing the norm of $A = (a_{ij})_{i,j=1}^\infty \in B(E, F)$ in some special cases.

PROPOSITION 4.2. *If E is a symmetric sequence space with the Fatou property, then*

(i) $\|A\|_{1,E} = \sup_j \|(a_{ij})_i\|_E$;
(ii) $\|A\|_{E,\infty} = \sup_i \|(a_{ij})_j\|_{E^\times}$.

Proof. (i) By definition we have

$$\|A\|_{1,E} = \sup_{\|x\|_1 \leq 1} \|A(x)\|_E = \sup_{\|x\|_1 \leq 1} \left\|\left(\sum_j a_{ij} x_j\right)_i\right\|_E.$$

Hence, if $x = e_j$, where $e_j = (e_j^k)_k \in c_0$ is such that $e_j^k = 1$ for $j = k$ and $e_j^k = 0$ for $j \neq k$, then $\|A\|_{1,E} \geq \|(a_{ij})_i\|_E$ for $j = 1, 2, \ldots$, that is, $\|A\|_{1,E} \geq \sup_j \|(a_{ij})_i\|_E$.

Using the triangle inequality for the norm, we obtain the converse inequality

$$\|A(x)\|_E = \left\|\left(\sum_j a_{ij} x_j\right)_i\right\|_E \leq \sum_j |x_j| \, \|(a_{ij})_i\|_E$$

$$\leq \sup_j \|(a_{ij})_i\|_E \sum_j |x_j| = \sup_j \|(a_{ij})_i\|_E \|x\|_1.$$

Hence, $\|A\|_{1,E} \leq \sup_j \|(a_{ij})_i\|_E$.

(ii) Applying Propositions 4.1 and 4.2(i) we obtain $\|A\|_{E,\infty} = \|A^T\|_{1,E^\times}$ $= \sup_i \|(a_{ij})_j\|_{E^\times}$. $\quad \blacksquare$

With every bounded sequence $x = (x_k)_{k=1}^{\infty}$, we associate a linear (diagonal) operator D_x (say on c_0) given by the matrix

$$D_x := \begin{pmatrix} x_1 & 0 & 0 & 0 & \dots \\ 0 & x_2 & 0 & 0 & \dots \\ 0 & 0 & x_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In other words, D_x acts on an element $y = (y_k)_{k=1}^{\infty} \in c_0$ by the formula $D_x(y) = x \cdot y = (x_k y_k)_{k=1}^{\infty}$.

REMARK 4.3. For any element $x = (x_k)_{k=1}^{\infty} \in E^F$ the diagonal operator D_x (restricted to E) is a bounded linear operator from E into F.

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same size (finite or infinite), their *Schur product* is defined to be the matrix of elementwise products $A * B = (a_{ij}b_{ij})$.

DEFINITION 4.4. An infinite matrix $M = (m_{ij})$ is called an (E, F)-*Schur multiplier* (or (E, F)-*multiplier*) if $M * A \in B(E, F)$ for every $A \in B(E, F)$.

The set of all (E, F)-multipliers is denoted by

(4.1) $$\mathcal{M}(E, F) := \{M : M * A \in B(E, F), \forall A \in B(E, F)\}.$$

It is a normed space with respect to the norm

(4.2) $$\|M\|_{(E,F)} := \sup_{\|A\|_{E,F} \leq 1} \|M * A\|_{E,F}$$

(when $E = l_p$ and $F = l_q$, we use the notation $\mathcal{M}(p, q)$ for (4.1) and $\|M\|_{(p,q)}$ for (4.2)).

REMARK 4.5. (i) Viewing $M \in \mathcal{M}(E, F)$ as a linear operator $M : B(E, F) \to B(E, F)$, one easily checks that the supremum in (4.2) is finite via the closed graph theorem.

(ii) Since $\|M\|_{(E,F)} = \sup_{\|A\|_{E,F} \leq 1} \|M * A\|_{E,F} \geq \|M * u_{jk}\|_{E,F} = |m_{jk}|$ for every $j, k = 1, 2, \dots$, where $u_{jk} = (u_{jk}^{(nm)})_{nm}$ is such that $u_{jk}^{(nm)} = 1$ if $n = j$, $m = k$ and $u_{jk}^{(nm)} = 0$ otherwise, for $j, k, n, m = 1, 2, \dots$, we have $\|M\|_{(E,F)} \geq \sup_{j,k} |m_{jk}|$.

The proofs of Theorem 4.6 and Lemma 4.7 below are routine and incorporated here for the convenience of the reader.

THEOREM 4.6. *The normed space* $(\mathcal{M}(E, F), \|\cdot\|_{(E,F)})$ *is complete.*

Proof. Since $M(E,F) \subseteq B(B(E,F))$, all we need to see is that $M(E,F)$ is closed. Take $M_n \in M(E,F)$, $n \geq 1$. Let

$$M_n = (m_{jk}^{(n)})_{j,k=1}^{\infty}, \quad n \geq 1.$$

Assume that $\lim_{n \to \infty} \|M_n - T\|_{B(B(E,F))} = 0$, for some $T \in B(B(E,F))$.

Fix $j,k \geq 0$ and fix a matrix unit $u_{jk} \in B(E,F)$. The sequence $(M_n)_{n \geq 1}$ is Cauchy in $B(B(E,F))$. Consequently, $(M_n(u_{jk}))_{n \geq 1}$ is Cauchy in $B(E,F)$. Thus, $(m_{jk}^{(n)})_{n \geq 0}$ is Cauchy in $\mathbb{R}$. Hence, for every $j,k = 1,2,\dots$ there is a number m_{jk} such that

$$\lim_{n \to \infty} |m_{jk}^{(n)} - m_{jk}| = 0.$$

Since

$$\|(M_n(u_{jk}) - m_{jk}u_{jk})(x)\|_F = \|(m_{jk}^{(n)}u_{jk} - m_{jk}u_{jk})(x)\|_F$$
$$= |m_{jk}^{(n)} - m_{jk}|\,|x_k|\,\|e_k\|_E \leq |m_{jk}^{(n)} - m_{jk}|\,\|x\|_E$$

for every $x \in E$, we have

$$\|M_n(e_{jk}) - m_{jk}e_{jk}\|_{E,F} \leq |m_{jk}^{(n)} - m_{jk}|.$$

Hence

$$(4.3) \qquad \lim_{n \to \infty} \|M_n(u_{jk}) - m_{jk}u_{jk}\|_{E,F} = 0.$$

On the other hand, the assumption

$$\lim_{n \to \infty} \|M_n - T\|_{B(B(E,F))} = 0$$

implies that

$$(4.4) \qquad \lim_{n \to \infty} \|M_n(u_{jk}) - T(u_{jk})\|_{E,F} = 0.$$

Combining (4.3) with (4.4) yields

$$T(u_{jk}) = m_{jk}u_{jk} \quad \text{for every } j,k = 1,2,\dots.$$

That is, T is a Schur multiplier. ∎

LEMMA 4.7. *Let $A \in B(E,F)$ and $M = (m_{ij})$ be such that $\sup_{i,j} |m_{ij}| < \infty$. If $M * A$ maps E into F, then $M * A \in B(E,F)$.*

Proof. Since $A = (a_{ij}) \in B(E,F)$, we have

$$\|(a_{ij})_j\|_{E^{\times}} < \infty \quad \text{for all } i = 1,2,\dots.$$

Assume that $x = (x_i)$, $x_n = (x_i^{(n)}) \in E$, $y = (y_i) \in F$ are such that $x_n \to x$ and $(M * A)(x_n) \to y$. The claim of the lemma follows from the closed graph theorem if we show that $y = (M * A)(x)$. To that end, using

the Hölder inequality, we have

$$\left| \sum_j m_{ij} a_{ij} (x_j^{(n)} - x_j) \right| \leq \sum_j |m_{ij}| \, |a_{ij}| \, |x_j^{(n)} - x_j|$$

$$\leq \left(\sup_{i,j} |m_{ij}| \right) \sum_j |a_{ij}| \, |x_j^{(n)} - x_j| \leq \left(\sup_{i,j} |m_{ij}| \right) \|x_n - x\|_E \|(a_{ij})_j\|_{E^\times} \to 0$$

for every $i = 1, 2, \ldots$. It follows that

$$y_i = \lim_{n \to \infty} \sum_j m_{ij} a_{ij} x_j^{(n)} = \sum_j m_{ij} a_{ij} x_j \quad \text{for every } i = 1, 2, \ldots.$$

So we conclude that $y = (M * A)(x)$. ∎

PROPOSITION 4.8. *If E and F are symmetric sequence spaces with the Fatou property and $M \in \mathcal{M}(E,F)$, then $M^T \in \mathcal{M}(F^\times, E^\times)$ and $\|M^T\|_{(F^\times, E^\times)} = \|M\|_{(E,F)}$.*

Proof. Using Proposition 4.1, we obtain

$$\|M\|_{(E,F)} = \sup_{\|A\|_{E,F} \leq 1} \|M * A\|_{E,F} = \sup_{\|A^T\|_{F^\times, E^\times} \leq 1} \|(M * A)^T\|_{F^\times, E^\times}$$

$$= \sup_{\|A^T\|_{F^\times, E^\times} \leq 1} \|M^T * A^T\|_{F^\times, E^\times} = \sup_{\|B\|_{F^\times, E^\times} \leq 1} \|M^T * B\|_{F^\times, E^\times}$$

$$= \|M^T\|_{(F^\times, E^\times)}. \quad ∎$$

PROPOSITION 4.9. *For every symmetric sequence space E with the Fatou property, the following equations hold:*

(i) $\|M\|_{(1,E)} := \|M\|_{(l_1, E)} = \sup_{i,j} |m_{ij}|$;
(ii) $\|M\|_{(E,\infty)} := \|M\|_{(E, l_\infty)} = \sup_{i,j} |m_{ij}|$.

Proof. (i) As we have seen above, $\|M\|_{(1,E)} \geq \sup_{i,j} |m_{ij}|$ (Remark 4.5). Let us prove the converse inequality. For every operator $A = (a_{ij}) \in B(l_1, E)$, using Proposition 4.2(i), we have

$$\|M * A\|_{1,E} = \sup_j \|(m_{ij} a_{ij})_i\|_E \leq \sup_j \left(\sup_i |m_{ij}| \, \|(a_{ij})_i\|_E \right)$$

$$\leq \sup_j \sup_i |m_{ij}| \sup_j \|(a_{ij})_i\|_E = \sup_{i,j} |m_{ij}| \, \|A\|_{1,E}.$$

Hence, $\|M\|_{(1,E)} \leq \sup_{i,j} |m_{ij}|$.

(ii) The claim follows from Proposition 4.8 and (i) above. ∎

Now, we are well equipped to consider the question of embedding of the (E,F)-multiplier space into an (p,q)-multiplier space. We start by recalling the following result from [B2].

Analyzing the proof of [B2, Theorem 6.1], we restate its result as follows.

THEOREM 4.10.

(i) *If* $1 \leq p_2 \leq p_1 \leq \infty$ *and* $1 \leq q_1 \leq q_2 \leq \infty$, *then* $\mathcal{M}(p_1, q_1) \subseteq \mathcal{M}(p_2, q_2)$.
(ii) *If* $1 \leq q_1, q_2 \leq \infty$ *and* $p_1 = p_2 = 1$, *then* $\mathcal{M}(p_1, q_1) = \mathcal{M}(p_2, q_2)$.
(iii) *If* $1 \leq p_1, p_2 \leq \infty$ *and* $q_1 = q_2 = \infty$, *then* $\mathcal{M}(p_1, q_1) = \mathcal{M}(p_2, q_2)$.
(iv) *If* $1 \leq p_1, q_1 \leq \infty$ *and* $q_2 \leq 2 \leq p_2$, *then* $\mathcal{M}(p_2, q_2) \subseteq \mathcal{M}(p_1, q_1)$.

The following corollary follows immediately from Theorem 4.10(i), (iv).

COROLLARY 4.11. $\mathcal{M}(2, 2) = \mathcal{M}(\infty, 1)$.

Further we will consider symmetric sequence spaces with the Fatou property only.

The following result is an immediate corollary of Proposition 4.9. It extends Theorem 4.10(ii), (iii).

COROLLARY 4.12. *The multiplier spaces* $\mathcal{M}(1, E)$ *and* $\mathcal{M}(F, \infty)$ *are isometrically isomorphic and do not depend on the choice of the spaces* E *and* F.

The following theorem is one of the main results of this section. It significantly extends Theorem 4.10(i) (excluding the extreme values for p and q).

THEOREM 4.13. *Let* $1 < p, q < \infty$. *Suppose that* E_1, E_2, F_1, F_2 *are symmetric sequence spaces with the Fatou property such that* E_1 *is* p-*convex,* F_1 *is* q-*concave,* E_2 *is* p-*concave, and* F_2 *is* q-*convex. Then* $\mathcal{M}(E_1, F_1) \subseteq \mathcal{M}(E_2, F_2)$.

Proof. Let $A = (a_{ij})_{i,j=1}^{\infty} \in B(E_2, F_2)$ and $M \in \mathcal{M}(E_1, F_1)$. We need to show that $M * A \in B(E_2, F_2)$.

Since $D_x \in B(E_1, E_2)$ and $D_y \in B(F_2, F_1)$ for $x \neq 0 \in E_1^{E_2}$, $y \neq 0 \in F_2^{F_1}$ (see Remark 4.3), $D_y \circ A \circ D_x$ is a bounded linear operator from E_1 to F_1.

Since $M \in \mathcal{M}(E_1, F_1)$, we obtain $M * (D_y \circ A \circ D_x) \in B(E_1, F_1)$. It is easy to see that $M * (D_y \circ A \circ D_x) = D_y \circ (M * A) \circ D_x$. Therefore $D_y \circ (M * A) \circ D_x \in B(E_1, F_1)$.

The next step is to prove that $M * A$ maps E_2 into F_2. Let $z_0 \in E_2$. Since F_1 is q-concave and F_2 is q-convex, Theorem 3.7(ii) shows that $F_2 = F_2^{F_1 F_1}$. It follows that

(4.5)   if $D_y((M * A)(z_0)) \in F_1$ for every $y \in F_2^{F_1}$, then $(M * A)(z_0) \in F_2$.

Since E_1 is p-convex and E_2 is p-concave, by Theorem 3.7(i) we have $E_1 \cdot E_1^{E_2} = E_2$. Therefore for $z_0 \in E_2$ there exist $x_1 \in E_1$ and $x_2 \in E_1^{E_2}$ such that $z_0 = x_1 \cdot x_2$.

Since $D_y \circ (M * A) \circ D_x \in B(E_1, F_1)$ for every $x \in E_1^{E_2}$ and $y \in F_2^{F_1}$, we have $(D_y \circ (M * A) \circ D_{x_2})(x_1) \in F_1$ for every $y \in F_2^{F_1}$. At the same time, it is easy to see that

$$(D_y \circ (M * A) \circ D_{x_2})(x_1) = D_y \circ ((M * A)(x_1 \cdot x_2)) = D_y \circ ((M * A)(z_0)).$$

By (4.5) we conclude $(M * A)(z_0) \in F_2$, that is, $M * A : E_2 \to F_2$.

Since $M \in \mathcal{M}(E_1, F_1)$, we have $\sup_{i,j} |m_{ij}| < \infty$. By Lemma 4.7 we conclude that $M \in \mathcal{M}(E_2, F_2)$. ∎

The following theorem is a generalization of Theorem 4.10(i) for the case when p or q is 1 or ∞.

THEOREM 4.14. *Let E and F be symmetric sequence spaces with the Fatou property. Then the following continuous embeddings hold:*

(i) $\qquad \mathcal{M}(\infty, 1) \overset{(1)}{\subseteq} \mathcal{M}(\infty, F) \overset{(2)}{\subseteq} \mathcal{M}(E, F) \overset{(3)}{\subseteq} \mathcal{M}(1, F) \overset{(4)}{=} \mathcal{M}(1, \infty),$

(ii) $\qquad \mathcal{M}(\infty, 1) \overset{(5)}{\subseteq} \mathcal{M}(E, 1) \overset{(6)}{\subseteq} \mathcal{M}(E, F) \overset{(7)}{\subseteq} \mathcal{M}(E, \infty) \overset{(8)}{=} \mathcal{M}(1, \infty),$

where equalities (4) and (8) signify isometric isomorphism.

Proof. (1) follows from (6) (proved below), by taking $E = l_\infty$.

(2) Let $A = (a_{ij})_{i,j=1}^\infty \in B(E, F)$ and $M \in \mathcal{M}(\infty, F)$. We need to show that $M * A \in B(E, F)$.

For $x \in E = l_\infty^E$ the operator $A \circ D_x$ is bounded from l_∞ to F, since $D_x \in B(l_\infty, E)$, according to Remark 4.3.

As $M \in \mathcal{M}(\infty, F)$, we obtain $M * (A \circ D_x) \in B(l_\infty, F)$. It is easy to see that $M * (A \circ D_x) = (M * A) \circ D_x$. Therefore $(M * A) \circ D_x \in B(l_\infty, F)$ for any $x \in E$. For given $z_0 \in E$, denoting $I = (1, 1, \dots) \in l_\infty$, we have

$$(M * A)(z_0) = (M * A)(D_{z_0}(I)) = ((M * A) \circ D_{z_0})(I) \in F.$$

Since $M \in \mathcal{M}(\infty, F)$, we have $\sup_{i,j} |m_{ij}| < \infty$. Lemma 4.7 implies that $M \in \mathcal{M}(E, F)$.

(3) Let $A = (a_{ij})_{i,j=1}^\infty \in B(l_1, F)$ and $M \in \mathcal{M}(E, F)$. We need to show that $M * A \in B(l_1, F)$.

For $x \in E^{l_1}$ the operator $A \circ D_x$ is bounded from E to F, since $D_x \in B(E, F)$, according to Remark 4.3.

As $M \in \mathcal{M}(E, F)$, we obtain $M * (A \circ D_x) \in B(E, F)$. It is easy to see that $M * (A \circ D_x) = (M * A) \circ D_x$. Therefore $(M * A) \circ D_x \in B(E, F)$ for any $x \in E^{l_1}$.

For given $z_0 \in l_1$, by Proposition 3.5 there exist $z_1 \in E$ and $z_2 \in E^{l_1}$ such that $z_0 = z_1 \cdot z_2$. Hence we obtain

$$(M*A)(z_0) = (M*A)(z_2 \cdot z_1) = (M*A)(D_{z_2}(z_1)) = ((M*A) \circ D_{z_2})(z_1) \in F.$$

Since $M \in \mathcal{M}(E, F)$, we have $\sup_{i,j} |m_{ij}| < \infty$. Lemma 4.7 implies that $M \in \mathcal{M}(1, F)$.

(4) has already been established in Proposition 4.9(i).

(5) immediately follows from (2) by taking $F = l_1$.

(6) follows from (2) by applying transposition (see Proposition 4.8).

(7) follows from (3) by applying transposition (see Proposition 4.8).

(8) has already been proved in Proposition 4.9(ii). ∎

COROLLARY 4.15. *If E is p-convex and F is q-concave, then $\mathcal{M}(E,F) \subseteq \mathcal{M}(p_1,q_1)$ for every $1 \leq p_1 \leq p \leq \infty$ and $1 \leq q \leq q_1 \leq \infty$.*

Proof. By the assumptions and by Theorem 4.13 we have $\mathcal{M}(E,F) \subseteq \mathcal{M}(p,q)$ and Theorem 4.10(i) yields $\mathcal{M}(p,q) \subseteq \mathcal{M}(p_1,q_1)$. ∎

COROLLARY 4.16. *If E is p-concave and F is q-convex, then $\mathcal{M}(p_1,q_1) \subseteq \mathcal{M}(E,F)$ for every $1 \leq p \leq p_1 \leq \infty$ and $1 \leq q_1 \leq q \leq \infty$.*

Proof. Since E is p-concave and F is q-convex, it follows from Theorem 4.13 that $\mathcal{M}(p,q) \subseteq \mathcal{M}(E,F)$. On the other hand, Theorem 4.10(i) yields $\mathcal{M}(p_1,q_1) \subseteq \mathcal{M}(p,q)$. ∎

The following theorem gives sufficient conditions on E and F guaranteeing the equality $\mathcal{M}(\infty,1) = \mathcal{M}(E,F)$.

THEOREM 4.17. *If E is 2-convex, F is 2-concave and $1 \leq q \leq 2 \leq p \leq \infty$, then*
$$\mathcal{M}(E,F) = \mathcal{M}(p,q).$$

Proof. It is sufficient to prove the assertion for the case $p = q = 2$. Indeed, for $q \leq 2 \leq p$ the embeddings $\mathcal{M}(p,q) \subseteq \mathcal{M}(2,2)$ and $\mathcal{M}(2,2) \subseteq \mathcal{M}(p,q)$ follow from Theorem 4.10(i) and (iv), respectively.

By Theorem 4.13 we have $\mathcal{M}(E,F) \subseteq \mathcal{M}(2,2)$. Using Theorem 4.10(iv) we obtain $\mathcal{M}(2,2) \subseteq \mathcal{M}(\infty,1)$. Theorem 4.14 yields the converse embedding, that is, $\mathcal{M}(\infty,1) \subseteq \mathcal{M}(E,F)$. The proof is complete. ∎

The following corollaries follow immediately from [P, Theorem 5.1(i)] and Theorem 4.17.

COROLLARY 4.18. *Let E be 2-convex and F be 2-concave. A matrix $M = (m_{ij})_{i,j=1}^{\infty}$ is an element of the space $\mathcal{M}(E,F)$ if and only if there is a Hilbert space H and families $(y_i)_{i=1}^{\infty}$, $(x_j)_{j=1}^{\infty}$ of elements of H such that $m_{ij} = \langle y_i, x_j \rangle$ for every $(i,j) \in \mathbb{N} \times \mathbb{N}$ and $\sup_i \|y_i\| \sup_j \|x_j\| < \infty$.*

We shall denote by $E \otimes F$ the algebraic tensor product of E and F. We introduce the tensor norm γ_2^* as follows. For all u in $E \otimes F$ we define
$$\gamma_2^*(u) := \inf\Big\{ \Big(\sum_j \|x_j\|_E^2\Big)^{1/2} \Big(\sum_i \|\xi_i\|_F^2\Big)^{1/2} \Big\},$$

where the infimum runs over all finite sequences $(x_j)_{j=1}^n$ in E and $(\xi_i)_{i=1}^n$ in F such that $u = \sum_{i=1}^n x_i \otimes \xi_i$. It is not difficult to check that γ_2^* is a norm on $E \otimes F$. We will denote by $E \widehat{\otimes}_{\gamma_2^*} F$ the completion of $E \otimes F$ with respect to that norm.

The next result follows from Theorem 4.17 and [P, Theorems 5.1(ii) and 5.3].

COROLLARY 4.19. *Let E be 2-convex and F be 2-concave. Then*

$$\mathcal{M}(E, F) = (l_1 \widehat{\otimes}_{\gamma_2^*} l_1)^*.$$

We complete this section with the following observation.

PROPOSITION 4.20. *The embeddings in Theorems and Corollaries* 4.10–4.17 *are continuous.*

Proof. For example, we will prove the claim for Theorem 4.10(i). Let p_1, q_1 be as in Theorem 4.10(i) and let $I : \mathcal{M}(p_1, q_1) \to \mathcal{M}(p_2, q_2)$ be the embedding operator, that is, $I(M) = M$ for every $M \in \mathcal{M}(p_1, q_1)$. Let $(M_n)_{n \geq 1} \subset \mathcal{M}(p_1, q_1)$ be such that $M_n \to 0$ in $\mathcal{M}(p_1, q_1)$ as $n \to \infty$ and $I(M_n) = M_n \to M$ in $\mathcal{M}(p_2, q_2)$ as $n \to \infty$. Using the notations from the proof of Proposition 4.2, we have

$$\langle M_n(u_{jk}), e_k \rangle = m_{jk}^{(n)} e_j, \quad \langle M(u_{jk}), e_k \rangle = m_{jk} e_j,$$

for every $j, k, n = 1, 2, \ldots$. Since $M_n \to 0$ in $\mathcal{M}(p_1, q_1)$, we obtain $m_{jk}^{(n)} \to 0$ as $n \to \infty$ for $j, k = 1, 2, \ldots$. Since $M_n \to M$ in $\mathcal{M}(p_2, q_2)$ we have $m_{jk}^{(n)} \to m_{jk}$. Hence, $m_{jk} = 0$ for all $j, k = 1, 2, \ldots$, that is, $M = 0$. Thanks to Theorem 4.6 we can apply the closed graph theorem to conclude that the operator I is bounded. ∎

5. The main triangle projection. As before, the nth main triangle projection is denoted by T_n ($n \in \mathbb{N}$). The question when the sequence $(\|T_n\|_{B(l_p, l_q) \to B(l_p, l_q)})_{n \geq 1}$ is (un)bounded was completely answered in [B1] and [KP].

PROPOSITION 5.1.

(i) ([KP, Proposition 1.2]) *Let $p \neq 1$, $q \neq \infty$ and $q \leq p$. Then*

$$\|T_n\|_{(p,q)} \geq C(p, q) \ln n, \quad \forall n \geq 1.$$

(ii) ([B1, Theorem 5.1]) *Let $1 \leq p < q \leq \infty$. Then*

$$\|T_n\|_{(p,q)} \leq C(p, q), \quad \forall n \geq 1.$$

Here $C(p, q)$ is a constant dependent only on p and q.

The following proposition extends the result of Proposition 5.1 to a wider class of symmetric sequence spaces.

PROPOSITION 5.2. *Let E and F be symmetric sequence spaces.*

(i) *If E is p-convex and F is q-concave for $p \neq 1$, $q \neq \infty$ and $q \leq p$, then*
$$\|T_n\|_{(E,F)} \geq C(p,q) \ln n, \quad \forall n \geq 1.$$

(ii) *If E is p-concave and F is q-convex for $1 \leq p < q \leq \infty$, then*
$$\|T_n\|_{(E,F)} \leq C(p,q), \quad \forall n \geq 1.$$

Here $C(p,q)$ is a constant dependent only on p and q.

Proof. (i) Since T_n is a finite rank operator, it belongs to the space of multipliers $\mathcal{M}(E,F)$ and also to $\mathcal{M}(p,q)$. Since E is p-convex and F is q-concave, by Theorem 4.13 we have $\mathcal{M}(E,F) \subset \mathcal{M}(p,q)$. By Proposition 4.20, there exists a constant C_1 such that
$$\|T_n\|_{(E,F)} \geq C_1 \|T_n\|_{(p,q)}, \quad \forall n \geq 1.$$
Applying Proposition 5.1(i), we conclude
$$\|T_n\|_{(E,F)} \geq C_1 \|T_n\|_{(p,q)} \geq C_1 C(p,q) \ln n, \quad \forall n \geq 1.$$

(ii) The proof is similar to the proof of (i). Theorem 4.13 yields the embedding $\mathcal{M}(p,q) \subset \mathcal{M}(E,F)$. It remains to appeal to Proposition 5.1(ii) and use Proposition 4.20 as above. ■

6. Projective and injective tensor products of symmetric sequence spaces. We briefly recall some notions and notations from [KP].

Let $\mathcal{M}_0$ be the set of scalar-valued (real or complex) infinite matrices such that if $A = (a_{ij}) \in \mathcal{M}_0$, then $a_{ij} \neq 0$ for all but finitely many $(i,j) \in \mathbb{N} \times \mathbb{N}$.

A non-negative function $\|\cdot\|_{\mathcal{M}_0}$ on $\mathcal{M}_0$ is called a *matrix norm* if it satisfies the following conditions:

(i) for every $A, B \in \mathcal{M}_0$ and for any scalar α,
- $\|A\|_{\mathcal{M}_0} = 0$ iff $A = 0$;
- $\|\alpha A\|_{\mathcal{M}_0} = |\alpha| \, \|A\|_{\mathcal{M}_0}$;
- $\|A + B\|_{\mathcal{M}_0} \leq \|A\|_{\mathcal{M}_0} + \|B\|_{\mathcal{M}_0}$;

(ii) $\|u_{jk}\|_{\mathcal{M}_0} = 1$ for all $j, k \geq 1$ (see the definition of the matrix unit u_{jk} in Remark 4.5(ii));

(iii) $\|P_{nm}(A)\|_{\mathcal{M}_0} \leq \|A\|_{\mathcal{M}_0}$ for all $A \in \mathcal{M}_0$, $n, m = 1, 2, \ldots$, where P_{nm} is the projection on the first n lines and m columns.

A matrix norm is called *unconditional* if

(iv) $\|A\|_{\mathcal{M}_0} = \|(x_{ij} a_{ij})_{ij}\|_{\mathcal{M}_0}$ for all $A \in \mathcal{M}_0$, where $|x_{ij}| = 1$, $i, j = 1, 2, \ldots$.

An unconditional matrix norm is called *symmetric* if

(v) $\|A\|_{\mathcal{M}_0} = \|(a_{\varphi(i)\psi(j)})_{ij}\|_{\mathcal{M}_0}$ for all $A \in \mathcal{M}_0$ and for all permutations φ, ψ of the positive integers.

If $\|\cdot\|_{\mathcal{M}_0}$ is a matrix norm, then the conjugate norm is defined by

$$\|A\|_{\mathcal{M}_0}^* := \sup\Big\{\Big|\sum_{i,j} a_{ij}b_{ij}\Big| : B \in \mathcal{M}_0, \|B\|_{\mathcal{M}_0} \le 1\Big\}.$$

We have $\|A\|_{\mathcal{M}_0}^{**} = \|A\|_{\mathcal{M}_0}$.

We denote

$$\|T_n\|_{(\mathcal{M}_0)} := \sup\{\|T_n(A)\|_{\mathcal{M}_0} : \|A\|_{\mathcal{M}_0} \le 1\}.$$

It is known (see [KP, equation (1.1)]) that

(6.1) $\quad \|T_n\|_{(\mathcal{M}_0)}^* := \sup\{\|T_n(A)\|_{\mathcal{M}_0}^* : A \in \mathcal{M}_0, \|A\|_{\mathcal{M}_0}^* \le 1\} = \|T_n\|_{(\mathcal{M}_0)}.$

The following theorem connects the boundedness of the norms of the main triangle projections to the possibility of embedding a matrix space into a Banach space with an unconditional basis.

THEOREM 6.1 ([KP, Theorem 2.3]). *Let* $\|\cdot\|_{\mathcal{M}_0}$ *be a symmetric matrix norm. If the sequence* $\{\|T_n\|_{(\mathcal{M}_0)}\}_n$ *is unbounded, then the space* $(\mathcal{M}_0, \|\cdot\|_{\mathcal{M}_0})$ *is not isomorphic to any subspace of a Banach space with an unconditional basis.*

In the present paper, we consider only two types of matrix spaces, projective and injective tensor products. Recall the definitions of these spaces (see e.g. [R]).

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be Banach spaces over the field $\mathbb{K}$ (of real or complex numbers). We denote by $E \otimes F$ the algebraic tensor product of E and F.

For every $u \in E \otimes F$ we define the *projective tensor norm*

$$\pi(u) := \inf\Big\{\sum_{i=1}^{n} \|x_i\|_E\|y_i\|_F : u = \sum_{i=1}^{n} x_i \otimes y_i\Big\}$$

and the *injective tensor norm*

$$\varepsilon(u) := \sup\Big\{\Big|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\Big| : \varphi \in E^*, \|\varphi\|_{E^*} \le 1, \psi \in F^*, \|\psi\|_{F^*} \le 1\Big\}.$$

The completion of $E \otimes F$ with respect to the norm π (respectively, ε) is denoted by $E\widehat{\otimes}F$ (respectively, $E\widehat{\overline{\otimes}}F$) and called the *projective* (respectively, *injective*) *tensor product* of the Banach spaces E and F.

For convenience, we denote the norm π (respectively, ε) on $E \otimes F$ by $\pi_{E,F}$ (respectively, $\varepsilon_{E,F}$).

Let c_{00} be the linear space of all finitely supported sequences. The tensor product $c_{00} \otimes c_{00}$ can be identified with the space $\mathcal{M}_0$ of matrices on $\mathbb{K}$. The tensor product basis $\{e_j \otimes e_k\}_{j,k=1}^{\infty}$ corresponds to the standard basis in $\mathcal{M}_0$ (see [R, §1.5] and [KP, §3]).

If E is a separable and p-convex symmetric sequence space and F is a q-concave symmetric sequence space, then E^* and F^* are symmetric spaces too (see Remark 2.4), and their dual spaces coincide with $E^\times$ and $F^\times$, respectively (see [KA, Part I, Chapter X, §4, Theorem 1]). Therefore $\varepsilon_{E,F}$ and $\pi_{E,F}$ are symmetric matrix norms on the space $c_{00} \otimes c_{00}$. For this reason, below we shall only consider separable symmetric sequence spaces.

The following proposition explains the connection between tensor product norms and the operator norm in $B(E, F)$ (see [R, §2.2 and 3.1]).

PROPOSITION 6.2.

(i) *The norm $\varepsilon_{E,F}$ coincides with the operator norm on $B(E^\times, F)$.*
(ii) *The conjugate norm to $\pi_{E,F}$ coincides with the operator norm on $B(E, F^\times)$.*

REMARK 6.3. In particular, Proposition 6.2(i) shows that for $A = (a_{ij}) \in B(E^\times, F)$, we have

$$\|A\|_{\varepsilon_{E,F}} = \sup\Big\{\Big|\sum_{i,j} a_{ij} x_i y_j\Big| : \|x\|_{E^\times} \leq 1, \|y\|_{F^\times} \leq 1\Big\}.$$

Another important observation is

(6.2) $$\|T_n\|_{(\pi_{E,F})} = \|T_n\|^*_{(\varepsilon_{F^\times, E^\times})} \quad \text{for every } n \geq 1.$$

We can reformulate Proposition 5.2 as follows:

PROPOSITION 6.4. *Let E and F be symmetric sequence spaces.*

(i) *If E is p-concave and F is q-concave for $p \neq \infty$, $q \neq \infty$ and $q \leq p^*$, then*
$$\|T_n\|_{(\varepsilon_{E,F})} \geq C(p, q) \ln n.$$
(ii) *If E is p-convex and F is q-convex for $p \neq 1$, $q \neq 1$ and $p^* \leq q$, then*
$$\|T_n\|_{(\pi_{E,F})} \geq C(p, q) \ln n.$$

Proof. (i) Since the norm $\varepsilon_{E,F}(\cdot)$ coincides with the norm $\|\cdot\|_{E^\times, F}$ (see Proposition 6.2(i)) and $E^\times$ is p^*-convex (see Proposition 2.3), by Proposition 5.2 we have
$$\|T_n\|_{(\varepsilon_{E,F})} = \|T_n\|_{(E^\times, F)} \geq C(p, q) \ln n.$$

(ii) Applying (6.2) and (6.1), we have

(6.3) $$\|T_n\|_{(\pi_{E,F})} = \|T_n\|^*_{(\varepsilon_{F^\times, E^\times})} = \|T_n\|_{(\varepsilon_{F^\times, E^\times})}.$$

Since E (respectively, F) is p-convex (respectively, q-convex), it follows that $E^\times$ (respectively, $F^\times$) is p^*-concave (respectively, q^*-concave) (see Proposition 2.3). By (i),

$$(6.4) \qquad \|T_n\|_{(\varepsilon_{F^\times, E^\times})} \geq C(p, q) \ln n, \quad \forall n \geq 1,$$

whenever $p^* \neq \infty$, $q^* \neq \infty$ and $q^* \leq p$. Applying (6.3) and (6.4), we obtain

$$\|T_n\|_{(\pi_{E,F})} = \|T_n\|_{(\varepsilon_{F^\times, E^\times})} \geq C(p, q) \ln n, \quad \forall n \geq 1,$$

for $p \neq 1$, $q \neq 1$ and $p^* \leq q$. ∎

The following theorem is the main result of this section.

THEOREM 6.5. *Let E and F be symmetric sequence spaces.*

(i) *If E is p-concave and F is q-concave for $p \neq \infty$, $q \neq \infty$ and $q \leq p^*$, then $E \widehat{\widehat{\otimes}} F$ is not isomorphic to any subspace of a Banach space with an unconditional basis.*

(ii) *If E is p-convex and F is q-convex for $p \neq 1$, $q \neq 1$ and $p^* \leq q$, then $E \widehat{\otimes} F$ is not isomorphic to any subspace of a Banach space with an unconditional basis.*

Proof. (i) By Proposition 6.4(i), with $p \neq \infty$, $q \neq \infty$ and $q \leq p^*$, the sequence $\{\|T_n\|_{(\varepsilon_{E,F})}\}_n$ is unbounded. By Theorem 6.1 the space $c_{00} \otimes c_{00}$ with the norm $\varepsilon_{E,F}$ is not isomorphic to any subspace of a Banach space with an unconditional basis. Since $(c_{00} \otimes c_{00}, \varepsilon_{E,F})$ is a linear subspace in $E \widehat{\widehat{\otimes}} F$, we conclude that $E \widehat{\widehat{\otimes}} F$ is not isomorphic to any subspace of a Banach space with an unconditional basis.

(ii) Similar to (i), using Proposition 6.4(ii) instead of Proposition 6.4(i). ∎

Now we consider a class of symmetric sequence spaces, called Orlicz–Lorentz sequence spaces, generalizing the class of l_p-spaces. For detailed studies of this class of spaces we refer to [HKM], [K1] and [K2].

We recall that $G : [0, \infty) \to [0, \infty)$ is an *Orlicz function*, that is, a convex function which assumes value zero only at zero, and $w = (w_k)$ is a *weight sequence*, a non-increasing sequence of positive reals such that $\sum_{k=1}^\infty w_k = \infty$.

The *Orlicz–Lorentz sequence space* $\lambda_{w,G}$ is defined by

$$\lambda_{w,G} := \left\{ x = (x_k) : \sum_{k=1}^\infty G(\lambda x_k^*) w_k < \infty \text{ for some } \lambda > 0 \right\}.$$

It is easy to check that $\lambda_{G,w}$ is a symmetric sequence space, equipped with the norm

$$\|x\|_{w,G} := \inf\left\{ \lambda > 0 : \sum_{k=1}^\infty G(\lambda x_k^*) w_k \leq 1 \right\}.$$

Two Orlicz functions G_1 and G_2 are said to be *equivalent* if there exists a constant $c < \infty$ such that

$$G_1(c^{-1}t) \leq G_2(t) \leq G_1(ct) \quad \text{for every } t \in [0, \infty).$$

The following theorem indicates sufficient conditions under which the space $\lambda_{w,G}$ is p-convex or q-concave (see also [KR]).

THEOREM 6.6 ([MS, Theorem 5.1]). *Let G be an Orlicz function, $w = (w_k)$ be a weight sequence and $1 < p, q < \infty$. Then the following claims hold:*

(i) *If $G \circ t^{1/p}$ is equivalent to a convex function and $\sum_{k=1}^{n} w_k$ is concave, then $\lambda_{w,G}$ is p-convex.*

(ii) *If $G \circ t^{1/q}$ is equivalent to a concave function and $\sum_{k=1}^{n} w_k$ is convex, then $\lambda_{w,G}$ is q-concave.*

According to Theorem 6.6, we can reformulate Theorem 6.5 for Orlicz–Lorentz sequence spaces as follows.

THEOREM 6.7. *Let G_1 and G_2 be Orlicz functions and $w_1 = (w_k^{(1)})$, $w_2 = (w_k^{(2)})$ be weight sequences such that the spaces λ_{w_1,G_1} and λ_{w_2,G_2} are separable. If $G_1 \circ t^{1/p}$ and $G_2 \circ t^{1/q}$ are equivalent to concave (convex, respectively) functions for $p \neq \infty$, $q \neq \infty$ and $q \leq p^*$ ($p \neq 1$, $q \neq 1$ and $p^* \leq q$, respectively), and $\sum_{k=1}^{n} w_k^{(1)}$, $\sum_{k=1}^{n} w_k^{(2)}$ are convex (concave, respectively) functions, then the tensor product $\lambda_{w_1,G_1} \widehat{\overline{\otimes}} \lambda_{w_2,G_2}$ ($\lambda_{w_1,G_1} \widehat{\overline{\otimes}} \lambda_{w_2,G_2}$, respectively) is not isomorphic to any subspace of a Banach space with an unconditional basis.*

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