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On positive embeddings of $C(K)$ spaces

by

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Abstract. We investigate isomorphic embeddings $T : C(K) \to C(L)$ between Banach spaces of continuous functions. We show that if such an embedding T is a positive operator then K is the image of L under an upper semicontinuous set-function having finite values. Moreover we show that K has a π-base of sets whose closures are continuous images of compact subspaces of L. Our results imply in particular that if $C(K)$ can be positively embedded into $C(L)$ then some topological properties of L, such as countable tightness or Fréchetness, are inherited by K.

We show that some isomorphic embeddings $C(K) \to C(L)$ can be, in a sense, reduced to positive embeddings.

1. Introduction. For a compact space K we denote by $C(K)$ the Banach space of real–valued continuous functions with the usual supremum norm. In what follows, K and L always denote compact Hausdorff spaces.

Let $T : C(K) \to C(L)$ be an isomorphism of Banach spaces. By the classical Kaplansky theorem, if T is an order-isomorphism, i.e. $g \geq 0$ if and only if $Tg \geq 0$ for every $g \in C(K)$, then K and L are homeomorphic; see 7.8 in Semadeni's book [18] for further references. On the other hand, if $C(K)$ and $C(L)$ are isomorphic as Banach spaces then K may be topologically different from L. For example, by Milyutin's theorem, $C[0, 1]$ is isomorphic to $C(2^\omega)$ as well as to any $C(K)$, where K is uncountable metric space (see [18, 21.5.10] or [15]).

In the present paper we consider isomorphic embeddings $T : C(K) \to C(L)$ which are not necessarily onto but are *positive operators*, i.e.

$$\text{if} \quad g \in C(K) \text{ and } g \geq 0 \quad \text{then} \quad Tg \geq 0.$$

Elementary examples show that even if such an operator T is onto then K may not be homeomorphic to L (see e.g. Example 5.3 below).

Our main objective is to determine how K is related to L whenever $C(K)$ admits a positive embedding into $C(L)$. We show that in that case

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for some natural number p there is a function $\varphi : L \to [K]^{\le p}$ which is upper semicontinuous and onto (that is, K is the union of values of φ). This implies that some topological properties of L, such as countable tightness or Fréchetness, are inherited by K. We moreover prove that K has a π-base of sets with closures being continuous images of subspaces of L. Our results offer partial generalizations of a theorem due to Jarosz [7], who proved that if $T : C(K) \to C(L)$ is an isomorphic embedding which is not necessarily positive but satisfies $\|T\| \cdot \|T^{-1}\| < 2$ then K is a continuous image of a compact subspace of L.

Let us recall the following open problem related to isomorphic embeddings of $C(K)$ spaces and the class of Corson compacta (see the next section for the terminology):

PROBLEM 1.1. *Suppose that $T : C(K) \to C(L)$ is an isomorphic embedding and L is Corson compact. Is L necessarily Corson compact?*

The answer to 1.1 is positive under Martin's axiom and the negation of continuum hypothesis (see Argyros et al. [2]). However, the problem remains open in ZFC even if the operator T in question is onto (see Negrepontis [12, 6.45] or Koszmider [10, Question 1]). In [11] we proved that the answer to 1.1 is positive under a certain additional measure-theoretic assumption on K.

We have not been able to fully resolve Problem 1.1 even for positive embeddings but we show in Section 4 that the answer is affirmative whenever K is homogeneous.

Our approach to analysing embeddings $T : C(K) \to C(L)$ follows Cambern [3] and Pełczyński [15]: we consider the conjugate operator $T^* : C(L)^* \to C(K)^*$ and a mapping $L \ni y \mapsto T^*\delta_y$ which sends points of L to measures on K and is weak* continuous. The main advantage of dealing with positive T here is that measures of the form $T^*\delta_y$ are nonnegative. In fact, as explained in the final section, what is crucial here is the continuity of the mapping $L \ni y \mapsto \|T^*\delta_y\|$. In a recent preprint [17] we were able to extend some of the results presented here to the case of arbitrary isomorphisms between spaces of continuous functions.

The paper is organized as follows. In the next section we collect some standard facts on operators and the weak* topology of $C(K)^*$, and we recall some concepts from general topology. In Section 3, we consider several properties of compact spaces that are preserved by taking images under upper semicontinuous finite-valued maps.

In Section 4 we show that a positive embedding $T : C(K) \to C(L)$ gives rise to a natural map $L \to [K]^{<\omega}$ and deduce from this our main results. Section 5 contains a few comments on the results; in Section 6 we consider arbitrary embeddings $C(K) \to C(L)$ for which the above mentioned function $y \mapsto \|T^*\delta_y\|$ is continuous.

2. Preliminaries. Throughout this paper we tacitly assume that K and L denote compact Hausdorff spaces. The dual space $C(K)^*$ of the Banach space $C(K)$ is identified with $M(K)$, the space of all signed Radon measures of finite variation; we use $M_1(K)$ to denote the unit ball of $M(K)$, and $P(K)$ for the space of Radon probability measures on K; every $\mu \in P(K)$ is an inner regular probability measure defined on the Borel σ-algebra $\mathrm{Bor}(K)$ of K. The spaces $M_1(K)$ and $P(K)$ are always equipped with the weak* topology inherited from $C(K)^*$, i.e. the topology making all the functionals $\mu \mapsto \int g\,d\mu$ continuous for $g \in C(K)$. We usually write $\mu(g)$ rather than $\int_K g\,d\mu$.

We shall frequently use the following simple remark: for every closed set $F \subseteq K$, the set

$$\{\mu \in P(K) : \mu(F) < r\}$$

is weak* open in $P(K)$.

For any $x \in K$ we write $\delta_x \in P(K)$ for the corresponding Dirac measure; recall that $\Delta(K) = \{\delta_x : x \in K\}$ is a subspace of $P(K)$ which is homeomorphic to K.

A linear operator $T : C(K) \to C(L)$ is an isomorphic embedding if there are positive constants m_1, m_2 such that

$$m_1\|g\| \le \|Tg\| \le m_2\|g\| \quad \text{for every } g \in C(K)$$

(so that $\|T\| \le m_2$ and $\|T^{-1}\| \le 1/m_1$). By 'embedding' we always mean an isomorphic embedding which is not necessarily surjective.

To every bounded operator $T : C(K) \to C(L)$ we can associate a conjugate operator T^*, where

$$T^* : M(L) \to M(K), \quad T^*(\nu)(f) = \nu(Tf);$$

T^* is surjective whenever T is an isomorphic embedding.

A set $M \subseteq M(K)$ is said to be *m-norming*, where $m > 0$, if for every $g \in C(K)$ there is $\mu \in M$ such that $|\mu(g)| \ge m\|g\|$.

LEMMA 2.1. *If $T : C(K) \to C(L)$ is an embedding then $L \ni y \mapsto T^*\delta_y \in M(K)$ is a continuous mapping, and the set*

$$T^*[\Delta_L] = \{T^*\delta_y : y \in L\}$$

is a weak compact and m-norming subset of $M(K)$, where $m = 1/\|T^{-1}\|$.*

Proof. If $(y_t)_t$ is a net in L converging to y then the measures δ_{y_t} converge to δ_y in the weak* topology of $M(L)$, and $T^*\delta_{y_t} \to T^*\delta_y$ since the conjugate operator is weak*-weak* continuous.

If $g \in C(K)$ and $\|g\| = 1$, then $m \le \|Tg\|$, so there is $y \in L$ such that $|Tg(y)| \ge m$, so $|Tg(y)| = |T^*\delta_y(g)|$; this shows that $T^*[\Delta_L]$ is m-norming. ∎

Every signed measure $\mu \in M(K)$ can be written as $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are nonnegative mutually singular measures. We write $|\mu| = \mu^+ + \mu^-$ for the total variation $|\mu|$ of μ; the natural norm of $\mu \in M(K) = C(K)^*$ is defined as $\|\mu\| = |\mu|(K)$. The mapping $\mu \mapsto |\mu|$ is not weak* continuous, but the following holds.

LEMMA 2.2. *Let $(\mu_i)_{i \in I}$ be a net in $M_1(K)$ converging to $\mu \in M_1(K)$. Then*

$$|\mu|(g) \le \liminf_i |\mu_i|(g)$$

for $g \in C(K)$, $g \ge 0$ (that is, the maping $\mu \mapsto |\mu|$ is lower semicontinuous). Moreover

$$|\mu|(g) \ge |\mu|(K) + \limsup_i |\mu_i|(g) - 1,$$

whenever $g \in C(K)$, $0 \le g \le 1$.

In particular, the mapping

$$S = \{\nu \in M(K) : \|\nu\| = 1\} \ni \nu \mapsto |\nu| \in P(K)$$

is weak continuous.*

Proof. Let us fix a nonnegative function $g \in C(K)$. For every measure $\mu \in M(K)$ it follows from the definition of $|\mu|$ that

$$(*) \qquad |\mu|(g) = \sup\{|\mu(f)| : f \in C(K) \text{ and } |f| \le g\},$$

and hence $|\mu(f)| \le |\mu|(|f|)$ for every $f \in C(K)$.

Given a net $(\mu_i)_{i \in I}$ in $M_1(K)$ converging to $\mu \in M_1(K)$, if $f \in C(K)$ and $|f| \le g$ then

$$|\mu(f)| = \lim_i |\mu_i(f)| \le \liminf_i |\mu_i|(|f|) \le \liminf_i |\mu_i|(g).$$

Hence $|\mu|(g) \le \liminf_i |\mu_i|(g)$ by $(*)$.

The second statement follows from the first one applied to $1 - g$.

If $|\mu|(K) = 1$ we get $|\mu|(g) \ge \limsup_i |\mu_i|(g)$, so $|\mu_i|(g) \to |\mu|(g)$, and thus the final statement follows. ■

Let us recall that a compact space K is *Corson compact* if, for some cardinal number κ, K is homeomorphic to a subset of the Σ-product of real lines,

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \ne 0\}| \le \omega\}.$$

Concerning Corson compacta and their role in functional analysis we refer the reader to a paper [2] by Argyros, Mercourakis and Negrepontis, and to the extensive surveys of Negrepontis [12] and Kalenda [8].

We now recall several countability-like concepts for arbitrary topological spaces. A topological space K is said to be

(i) *Fréchet* if for every $A \subseteq K$ and $x \in \overline{A}$ there is a sequence $(a_n)_n$ in A converging to x;
(ii) *sequential* if for every nonclosed set $A \subseteq K$ there is a sequence $(a_n)_n$ in A converging to a point $x \in K \setminus A$;
(iii) *sequentially compact* if every sequence in K has a converging subsequence.

The *tightness* of a topological space K, denoted here by $\tau(X)$, is the least cardinal number such that for every $A \subseteq K$ and $x \in \overline{A}$ there is a set $I \subseteq A$ with $|I| \leq \tau(K)$ such that $x \in \overline{I}$.

Every Corson compactum is a Fréchet space and every Fréchet space is clearly sequential; the reader may consult [5, e.g. p. 78 and 3.12.7–3.12.11] for further information.

3. Finite-valued maps. In the following we consider, for a given pair of compact spaces K, L and a fixed natural number p, set-valued mappings $\varphi : L \to [K]^{\leq p}$. Such a mapping φ is said to be

(a) *onto* if $\bigcup_{y\in L} \varphi(y) = K$;
(b) *upper semicontinuous* if the set $\{y \in L : \varphi(y) \subseteq U\}$ is open for every open $U \subseteq K$.

Clearly upper semicontinuity is equivalent to saying that the set

$$\varphi^{-1}[F] := \{y \in L : \varphi(y) \cap F \neq \emptyset\}$$

is closed whenever $F \subseteq K$ is closed. We shall need later a result on preserving compactness by multifunctions. The following proposition is well-known (see e.g. [13, p. 336]); we enclose its proof for completeness.

PROPOSITION 3.1. *If φ is an upper semicontinuous multifunction from a compact space L into the family of compact subsets of a topological space X then $G = \bigcup\{\varphi(y) : y \in L\}$ is compact.*

Proof. Let $\mathcal{U}$ be a family of open subsets of X such that $\bigcup\mathcal{U} \supseteq G$. Assuming that $\mathcal{U}$ is closed under taking finite unions we shall check that $G \subseteq U$ for some $U \in \mathcal{U}$.

Given $y \in L$, $\varphi(y)$ is covered by a finite subfamily of $\mathcal{U}$ so $\varphi(y) \subseteq U_y$ for some $U_y \in \mathcal{U}$. By upper semicontinuity, $\varphi(z) \subseteq U_y$ for all z from some open set V_y containing y. Take a finite set $Y_0 \subseteq Y$ such that $\bigcup_{y\in Y_0} V_y = L$. Then $U = \bigcup_{y\in Y_0} U_y$ contains G. ∎

Semicontinuous mappings with finite values have been considered by Okunev [13] in connection with some problems in $C_p(X)$ theory. The first assertion of the following auxiliary result is a particular case of [13, Proposition 1.2]; we give here a different self-contained argument.

LEMMA 3.2. *Let K and L be compact spaces and suppose that for some natural number p there is an upper semicontinuous onto mapping $\varphi : L \to [K]^{\leq p}$. Then*

(i) *$\tau(K) \leq \tau(L)$;*
(ii) *if L is a Fréchet space then K is Fréchet;*
(iii) *if L is a sequential space then so is K;*
(iv) *if L is sequentially compact then so is K.*

Proof. To prove (i) fix $A \subseteq K$ and $x \in \overline{A}$. For every $a \in A$ there is $y_a \in L$ such that $a \in \varphi(y_a)$. Denote by $\mathcal{V}(x)$ some local neighbourhood base at $x \in K$; the set

$$F = \bigcap_{V \in \mathcal{V}(x)} \overline{\{y_a : a \in V \cap A\}}$$

is nonempty and for any $y \in F$ we have $x \in \varphi(y)$. Indeed, if $x \notin \varphi(y)$ and $y \in F$ then $\varphi(y) \subseteq K \setminus \overline{V}$ for some $V \in \mathcal{V}(x)$. But then

$$U = \{z \in L : \varphi(z) \subseteq K \setminus \overline{V}\}$$

is a neighbourhood of y disjoint from $\{y_a : a \in V \cap A\}$, a contradiction.

Now fix $y \in F$ and choose $V \in \mathcal{V}(x)$ such that $\varphi(y) \cap \overline{V} = \{x\}$. Then

$$y \in \overline{\{y_a : a \in A \cap V\}},$$

so by the definition of tightness there is $I \subseteq A \cap V$ with $|I| \leq \tau(L)$ such that $y \in \overline{\{y_a : a \in I\}}$. Now it suffices to check that $x \in \overline{I}$.

If we suppose that $x \notin \overline{I}$ then $\varphi(y) \subseteq K \setminus \overline{I}$ (by our choice of V), so

$$W = \{z \in L : \varphi(z) \subseteq K \setminus \overline{I}\}$$

is an open neighbourhood of y not intersecting $\{y_a : a \in I\} = \emptyset$, a contradiction.

Suppose now that L is a Fréchet space. Then we can argue as above but this time we can choose a sequence $a(n) \in A \cap V$ such that $y_{a(n)} \to y$. Then $a(n) \to x$. Indeed, otherwise there is a cluster point $x' \neq x$ of $(a(n))_n$. We have $x' \in \overline{V}$; let V_1 be a neighbourhood of x' such that $x \notin \overline{V_1}$. Then $x \notin \overline{V \cap V_1}$ and

$$W_1 = \{z \in L : \varphi(z) \subseteq K \setminus \overline{V \cap V_1}\}$$

is an open neighbourhood of y such that $W_1 \cap \{y_{a(n)} : a(n) \in V_1\} = \emptyset$, a contradiction since the subsequence $(y_{a(n)})$ indexed by $a(n) \in V_1$ should converge to y.

(iii) can be checked in a similar way (cf. [13, Proposition 1.6]).

To prove (iv), for any sequence $x_n \in K$ choose $y_n \in L$ such that $x_n \in \varphi(y_n)$. Since L is sequentially compact we can assume that y_n converges to some $y \in L$. Then every cluster point of the sequence $(x_n)_n$ must lie in $\varphi(y)$ so $(x_n)_n$ has only a finite number of cluster points and therefore must contain a converging subsequence. ∎

LEMMA 3.3. *Let K and L be compact spaces and let $\varphi : L \to [K]^{\le p}$ be an upper semicontinuous mapping. If $F \subseteq K$ is closed then the mapping $\psi : L \to [F]^{\le p}$ given by $\psi(y) = \varphi(y) \cap F$ is also semicontinuous.*

Proof. If $V \subseteq F$ is open in F then $V = F \cap U$ for some open $U \subseteq K$; note that $\psi(y) \subseteq V$ is equivalent to $\varphi(y) \subseteq U \cup (K \setminus F)$. ∎

4. Positive embeddings. In this section we investigate what can be said about a pair of compact space K and L assuming there is a positive embedding $T : C(K) \to C(L)$. The positivity implies that $T^*\delta_y$ is a nonnegative measure on K for every $y \in L$, which is crucial for the proofs given below. The first lemma will show that, after a suitable reduction, we can in fact assume that every $T^*\delta_y$ is a probability measure on K.

LEMMA 4.1. *Suppose that $C(K)$ can be embedded into $C(L)$ by a positive operator T of norm one. Then there is a compact subspace $L_0 \subseteq L$ and a positive embedding $S : C(K) \to C(L_0)$ such that $S1_K = 1_{L_0}$ and $\|S^{-1}\| \le \|T^{-1}\|$.*

Proof. Since $\|T\| = 1$, there is a positive constant $m > 0$ such that $m \le \|Tg\| \le 1$ whenever $\|g\| = 1$.

By a standard application of Zorn's lemma there is a minimal element L_0 in the family of those compact subsets $L' \subseteq L$ such that

$$\|Tg\| = \|(Tg)\restriction L'\| \quad \text{for every } g \in C(K).$$

Letting $h = (T1_K)\restriction L_0$ we have $h \ge m$; indeed, otherwise $U = \{y \in L_0 : h(y) < m\}$ is a nonempty open subset of L_0, so $L_1 = L_0 \setminus U$ is a proper closed subspace of L_0. On the other hand, if $g \in C(K)$ and $\|g\| = 1$, then $\|Tg\| = |Tg(y_1)| \ge m$ for some $y_1 \in L_0$; since $\pm Tg \le T1_K = h$ we get $y_1 \in L_1$. This shows that L_0 is not minimal, a contradiction.

We can now define $S : C(K) \to C(L_0)$ letting Sg be the function $(1/h)(Tg)$ restricted to L_0. Then $S1_K = 1_{L_0}$; as S is positive this implies $\|S\| = 1$. Moreover, $\|Sg\| \ge \|Tg\|$ by the choice of L_0. ∎

PROPOSITION 4.2. *Suppose that $T : C(K) \to C(L)$ is a positive embedding such that $T1_K = 1_L$ and $\|T^{-1}\| = 1/m$. For $r \in (0, 1]$ and $y \in L$ set*

$$\varphi_r(y) = \{x \in K : \nu_y(\{x\}) \ge r\},$$

where $\nu_y = T^*\delta_y$. Then for every $r \in (0, 1]$:

(i) *φ_r takes its values in $[K]^{\le p}$, where p is the integer part of $1/r$, and φ_r is upper semicontinuous;*
(ii) *$\bigcup_{y \in L} \varphi_r(y)$ is closed in K;*
(iii) *φ_m is onto K.*

Proof. Note first that since $\nu_y(g) = Tg(y) \geq 0$ for every $g \geq 0$, we have $\nu_y \in M^+(K)$. Moreover, $\nu_y(K) = \nu_y(1_K) = T1_K(y) = 1$, so every ν_y is in fact a probability measure on K.

It is clear that φ_r has at most $1/r$ elements, so to verify (i) we shall check upper semicontinuity.

Take an open set $U \subseteq K$ and $y \in L$ such that $\varphi_r(y) \subseteq U$. Writing $F = K \setminus U$, for every $x \in F$ we have $\nu_y(\{x\}) < r$ so there is an open set $U_x \ni x$ such that $\nu_y(\overline{U_x}) < r$. This defines an open cover U_x of a closed set F so we have $F \subseteq \bigcup_{x \in I} U_x$ for some finite set $I \subseteq F$. Now

$$V = \{z \in L : \nu_z(\overline{U_x}) < r \text{ for every } x \in I\}$$

is an open neighbourhood of y and $\varphi_r(z) \subseteq U$ for all $z \in V$.

Part (ii) follows directly from (i) and Proposition 3.1.

To prove (iii), we will check that for every $x \in K$ there is $y \in L$ such that $\nu_y(\{x\}) \geq m$.

Take any open neighbourhood U of x, and a continuous function $g_U : K \to [0,1]$ that vanishes outside U and $g_U(x) = 1$. Then $\|g_U\| = 1$, so by Lemma 2.1 there is $y(U) \in L$ such that $\nu_{y(U)}(g_U) \geq m$. Let y be a cluster point of the net $y(U)$, indexed by all open $U \ni x$ (ordered by inverse inclusion). Then $\nu_y(\{x\}) \geq m$.

Indeed, suppose that $\nu_y(\{x\}) < m$. Then there is an open set $U \ni x$ such that $\nu_y(\overline{U}) < m$. Since ν_y is a cluster point of $\nu_{y(V)}$, there must be V such that $x \in V \subseteq U$ and $\nu_{y(V)}(\overline{U}) < m$. But

$$\nu_{y(V)}(\overline{U}) \geq \nu_{y(V)}(U) \geq \nu_{y(V)}(V) \geq \nu_{y(V)}(g_V) \geq m,$$

a contradiction. ∎

THEOREM 4.3. *If K and L are compact spaces such that there is a positive isomorphic embedding $C(K) \to C(L)$ then $\tau(K) \leq \tau(L)$. Moreover, if L is a Fréchet* (*sequential, sequentially compact*) *space then so is K.*

Proof. Note that if $\tau(L) \leq \kappa$ and $L_0 \subseteq L$ is closed then $\tau(L_0) \leq \kappa$; likewise, the Fréchet property and sequentiality are inherited by subspaces. Therefore by Lemma 4.1 we can assume that there is a positive embedding $T : C(K) \to C(L)$ with $T1_K = 1_L$, and the theorem follows from Lemma 3.2 and Proposition 4.2(iii). ∎

For the sake of the next result let us write $ci(L)$ for the class of compact spaces that are continuous images of closed subspaces of a given compact space L.

THEOREM 4.4. *Let K and L be compact spaces such that there is a positive isomorphic embedding $C(K) \to C(L)$. Let p be the least integer such that $2^p > \|T\| \cdot \|T^{-1}\|$. Then there is a sequence*

$$K_1 = K \supseteq \cdots \supseteq K_p$$

of closed subspaces of K such that

(a) $K_p \in ci(L)$;

(b) *for every $i \le p-1$ and every $x \in K_i \setminus K_{i+1}$ there is an open set U containing x such that $\overline{U} \cap K_i \in ci(L)$.*

Proof. Take a positive embedding $T : C(K) \to C(L)$ of norm one and write $m = 1/\|T^{-1}\|$. As in the previous proof we can assume that $T1_K = 1_L$. Following Proposition 4.2 we write $\nu_y = T^*\delta_y$ and $\varphi_r(y) = \{x \in K : \nu_y(\{x\}) \ge r\}$ for $y \in L$ and $r > 0$.

Let $m_i = 2^{i-1}m$ for $i = 1, \ldots, p$ and put

$$K_i = \bigcup_{y \in L} \varphi_{m_i}(y).$$

Then every K_i is closed by Proposition 4.2(ii) and $K_1 = K$ by 4.2(iii).

To prove (a) note that $m_p > 1/2$ by our choice of p, so every $\varphi_{m_p}(y)$ has at most one element. Let $\psi(y) = \varphi_{m_p} \cap K_p$ for $y \in L$. Then $\psi : L \to [K_p]^{\le 1}$ is upper semicontinuous by Lemma 3.3 and onto by the definition of K_p. If $L_0 = \{y \in L : \psi(y) \ne \emptyset\}$ then L_0 is closed and clearly ψ defines a continuous surjection from L_0 onto K_p.

Let $i \le p-1$ and fix $x \in K_i \setminus K_{i+1}$. Consider the mapping ψ defined as $\psi(y) = \varphi_{m_i}(y) \cap K_i$ for $y \in L$; again ψ is onto and upper semicontinuous.

We claim that x has a neighbourhood H in K_i such that $|\psi(y) \cap H| \le 1$ for $y \in L$.

Suppose otherwise; then for every set H which is open in K_i and contains x there is $y_H \in L$ and distinct $u_H, w_H \in H$ such that $u_H, w_H \in \psi(y_H)$. Passing to a subnet we can assume that $y_H \to y \in L$. Then $\nu_y(\{x\}) < m_{i+1}$ since $x \notin K_{i+1}$, so there is an open U such that $\nu_y(\overline{U}) < m_{i+1}$. Then $\nu_{y_H}(\overline{U}) < m_{i+1}$ should hold eventually but

$$\nu_{y_H}(U) \ge \nu_{y_H}(\{u_h, w_h\}) \ge 2m_i = m_{i+1},$$

a contradiction.

Take an open set $U \subseteq K$ containing x such that $\overline{U} \cap K_i \subseteq H$; it follows that $\overline{U} \cap K_i$ is in $ci(L)$, as required. ∎

Let us recall that a family $\mathcal{V}$ of nonempty open subsets of a topological space X is called a π-*base* if every nonempty open set $U \subseteq X$ contains some $V \in \mathcal{V}$.

COROLLARY 4.5. *If $T : C(K) \to C(L)$ is a positive embedding then the family of those open $U \subseteq K$ such that $\overline{U} \in ci(L)$ is a π-base of K.*

Proof. Let $W \subseteq K$ be a nonempty open set. Taking K_i as in Theorem 4.4 we infer that there is $i \le p$ such that $W \cap (K_i \setminus K_{i+1})$ (we set $K_{p+1} = \emptyset$) has nonempty interior; we conclude applying 4.4. ∎

COROLLARY 4.6. *If $T : C([0,1]^\kappa) \to C(L)$ is a positive embedding then L can be continuously mapped onto $[0,1]^\kappa$.*

Proof. By the previous corollary there is a basic neighbourhood U in $[0,1]^\kappa$ of the form $U = p_I^{-1}[(a_1,b_1)\times\cdots\times(a_n,b_n)]$, where $p_I : [0,1]^\kappa \to [0,1]^I$ is a projection, such that $\overline{U} \in ci(L)$. Clearly, $\overline{U}$ is homeomorphic to $[0,1]^\kappa$ and we conclude that there is a continuous surjection $h : L_0 \to [0,1]^\kappa$ from a closed subspace $L_0 \subseteq L$; we can then extend h to a continuous function $L \to [0,1]^\kappa$ (applying Tietze's extension theorem coordinatewise). ■

Recall that a space K is called *homogeneous* if for each pair $x_1, x_2 \in K$ there is a homeomorphism $\theta : K \to K$ such that $\theta(x_1) = x_2$.

COROLLARY 4.7. *If $T : C(K) \to C(L)$ is a positive embedding and L is Corson compact then K has a π-base of sets having Corson compact closures.*

If, moreover, K is homogeneous then K is Corson compact itself.

Proof. The first part is a consequence of Corollary 4.5 and the fact that Corson compacta are stable under taking continuous images.

If K is homogeneous then for every $x \in K$ there is an open set $x \in U$ such that $\overline{U}$ is Corson compact. Hence K can be covered by a finite family of Corson compacta and this easily implies that K is Corson compact too (see e.g. [17, Corollary 6.4]). ■

5. Remarks and questions. In connection with Theorem 4.3 it is worth recalling the following result due to Okunev [14]: There is a Fréchet compactum L and a compact space K which is not Fréchet and such that $C(K)$ is isomorphic to $C(L)$. In fact [13] gave the proof that there is a linear isomorphism $T : C_p(K) \to C_p(L)$ between the underlying spaces equipped with the topology of pointwise convergence. It is well-known, however, that such a T is automatically an isomorphism of the Banach spaces $C(K)$ and $C(L)$.

Note that Theorem 4.4 implies the following.

COROLLARY 5.1. *Suppose that $T : C(K) \to C(L)$ is a positive embedding such that $\|T\| \cdot \|T^{-1}\| < 2$. Then there is a compact subspace $L_1 \subseteq L$ which can be continuously mapped onto K.*

This is, however, a particular case of a result due to Jarosz [7], who showed that one may drop the assumption of positivity. It is worth recalling that Jarosz's result was motivated by the following theorem due to Amir [1] and Cambern [3].

THEOREM 5.2. *Suppose that there an isomorphism T from $C(K)$* **onto** *$C(L)$ such that $\|T\| \cdot \|T^{-1}\| < 2$. Then K is homeomorphic to L, and consequently $C(K)$ and $C(L)$ are isometric.*

Our results from the previous sections do not require the condition $\|T\| \cdot \|T^{-1}\| < 2$ at the price of assuming that the operator in question is positive. The following elementary example shows that in the setting of Theorem 4.4 the whole space K need not be an image of a subspace of L.

EXAMPLE 5.3. Let

$$K = \{x_n : n \geq 1\} \cup \{y_n : n \geq n\} \cup \{x, y\}$$

consist of two disjoint converging sequences $x_n \to x$, $y_n \to y$ (where $x \neq y$); let $L = \{z_n : n \geq 0\} \cup \{z\}$, where $z_n \to z$. Define $T : C(K) \to C(L)$ by

$$Tf(z_0) = f(y),$$
$$Tf(z_{2n-1}) = \frac{f(x_n) + f(y)}{2}, \quad Tf(z_{2n}) = \frac{f(x) + f(y_n)}{2} \quad \text{for } n \geq 1.$$

Then T is a positive operator with $\|T\| = 1$; moreover, T is an isomorphism onto $C(K)$. Indeed, the inverse $S = T^{-1} : C(L) \to C(K)$ is given by

$$Sh(y) = h(z_0), \quad Sh(x) = 2h(z) - h(z_0),$$
$$Sh(x_n) = 2h(z_{2n-1}) - h(z_0), \quad Sh(y_n) = 2h(z_{2n}) - 2h(z) + h(z_0)$$

so $\|T^{-1}\| \leq 5$.

Note that, on the other hand, K is not a continuous image of any subspace of L.

It should be stressed that we have not been able to find a complete solution to Problem 1.1 even for positive embeddings.

One reason is that Corsonnes is not preserved by taking images under finite-valued upper semicontinuous functions: for instance, if we let K be the split interval (which is not Corson compact because every separable Corson compact space is metrizable) then there is an upper semicontinuous onto mapping $\varphi : [0, 1] \to [K]^{\leq 2}$.

The second reason is that facts like Theorem 4.4 are not strong enough: Suppose for instance that $T : C(K) \to C(L)$ is a positive embedding with $\|T\| \cdot \|T^{-1}\| < 4$ and L is Corson compact. Then we conclude from 4.4 that there is a Corson compact $K_2 \subseteq K$ such that every $x \in K \setminus K_2$ has a neighbourhood with a Corson compact closure. But a space K with this property need not be Corson itself—take for instance $K = [0, \omega_1]$ (the space of ordinals $\alpha \leq \omega_1$ with the order topology) and let $K_2 = \{\omega_1\}$.

6. Envelopes of operators between $C(K)$ spaces. Let $T : C(K) \to C(L)$ be a bounded operator; we can consider the following function $e^T : L \to \mathbb{R}$ associated to T:

$$e^T(y) = \sup\{Tg(y) : g \in C(K),\ \|g\| \leq 1\}.$$

We shall call e^T the *envelope* of T.

Lemma 6.1. *The envelope e^T of a bounded operator $T : C(K) \to C(L)$ is a lower semicontinuous function with values in $[0, \|T\|]$. If T is an isomorphism onto $C(L)$ then $1/\|T^{-1}\| \leq e^T(y) \leq \|T\|$ for every $y \in L$.*

Proof. If $e^T(y_0) > r$ then there is norm-one function $g \in C(K)$ such that $Tg(y_0) > r$ and then $Tg(z) > r$ for all z from some neighbourhood V of y_0 so $V \subseteq \{e^T > r\}$. This means that the set $\{e^T > r\}$ is open.

If T is onto then there is $g \in C(K)$ such that $Tg = 1_L$; writing $m = 1/\|T^{-1}\|$ we have $\|mg\| \leq 1$ so $e^T(y) \geq m$ for every $y \in L$. ∎

Lemma 6.2. *For every bounded operator $T : C(K) \to C(L)$ we have $e^T(y) = \|T^*\delta_y\|$ for every $y \in L$.*

Proof. This follows from

$$\|T^*\delta_y\| = \sup_{\|g\|\leq 1} T^*\delta_y(g) = \sup_{\|g\|\leq 1} Tg(y) = e^T(y). \blacksquare$$

The idea of considering envelopes is based on the observation that the assumption of positivity of an embedding $T : C(K) \to C(L)$ might be dropped once we know that the mapping $y \mapsto \|T^*\delta_y\|$ is continuous. In fact the following result explains that an embedding having a continuous envelope can be, in a sense, reduced to a positive embedding.

Theorem 6.3. *Suppose that $T : C(K) \to C(L)$ is an isomorphic embedding such that e^T is continuous and positive everywhere on L. Then there exists a positive embedding of $C(K)$ into $C(L \times 2)$.*

Proof. W can assume that $\|T\| = 1$; we have $e^T \in C(L)$ and also $h = 1/e^T \in C(L)$. If we define $T' : C(K) \to C(L)$ by $T'g = hT(g)$ then T' is also an isomorphic embedding. If $g \in C(K)$ with $\|g\| = 1$ then $|T'g(y)| = |h(y)Tg(y)| \leq 1$, so again $\|T'\| = 1$. Moreover, it is easy to check that $e^{T'} \equiv 1$.

By the above remark we can assume that $T : C(K) \to C(L)$ is an embedding such that $\|T\| = 1$ and e^T is equal to 1 on L. As before, for every $y \in L$ write $\nu_y = T^*\delta_y \in M(K)$. By Lemma 6.2, $\|\nu_y\| = 1$ for every $y \in L$.

Every ν_y can be written as $\nu_y = \nu_y^+ - \nu_y^-$, where $\nu_y^+, \nu_y^- \in M^+(K)$. Since the mapping $L \ni y \mapsto \nu_y$ is continuous, it follows from the final assertion of Lemma 2.2 that the mapping $L \ni y \mapsto |\nu_y|$ is also continuous, and consequently so are the mappings $L \ni y \mapsto \nu_y^+$ and $L \ni y \mapsto \nu_y^-$.

We now define

$$\theta : L \times 2 \to M^+(K), \quad \theta(y, 0) = \nu_y^+, \quad \theta(y, 1) = \nu_y^-;$$

by the above remarks, θ is continuous and $L' = \theta[L \times 2]$ is a compact subset of $M^+(K)$.

Let us consider an operator $S : C(K) \to C(L')$ where $Sg(\mu) = \mu(g)$ for $\mu \in L'$. Clearly S is bounded, in fact $\|S\| \le 1$. Let $m = 1/\|T^{-1}\|$, i.e. $\|Tg\| \ge m\|g\|$ for $g \in C(K)$. If $g \in C(K)$ and $\|g\| = 1$ then $|Tg(y)| \ge m$ for some $y \in L$, so $|\nu_y(g)| \ge m$, which gives that either $|\nu_y^+(g)| \ge m/2$ or $|\nu_y^-(g)| \ge m/2$. It follows that $\|Sg\| \ge m/2$, so S is a positive embedding. Finally, $C(L')$ can be embedded into $C(L\times 2)$ by a positive operator sending $g \in C(L')$ to $g \circ \theta \in C(L \times 2)$; the proof is complete. ∎

We can combine Theorem 6.3 with the result of the previous section. We denote by $K+1$ the space K with one isolated point added. Clearly, $C(K+1)$ is isomorphic to $C(K) \times \mathbb{R}$; recall that except for some peculiar spaces K, $C(K+1)$ is isomorphic to $C(K)$ (see [18], Koszmider [9], Plebanek [16]).

Lemma 6.4. *Let $T : C(K) \to C(L)$ be an isomorphic embedding and suppose that $e^T(y_0) = 0$ for some $y_0 \in L$. Then there is an isomorphic embedding $S : C(K+1) \to C(L)$ with $e^S = e^T + 1 \ge 1$.*

Proof. Let T satisfy $m\|g\| \le \|Tg\| \le \|g\|$ for $g \in C(K)$. Put $K+1 = K \cup \{z\}$ and define $S : C(K+1) \to C(L)$ by

$$Sf = T(f{\upharpoonright}K) + f(z)1_L.$$

Clearly $\|T\| \le 2$; if $f \in C(K+1)$ is a norm-one function then either $|f(z)| \ge m/2$, which gives $|Sf(y_0)| \ge m/2$, or $|f(z)| < m/2$, which also implies $\|Sf\| \ge m/2$. Therefore S is an embedding too. Clearly $e^S = e^T + 1 \ge 1$. ∎

Corollary 6.5. *If K and L are compact spaces and there is an isomorphic embedding $T : C(K) \to C(L)$ with a continuous envelope then $\tau(K) \le \tau(L)$. Moreover, the Fréchet property* (*sequentiality, sequential compactness*) *of L implies that K is Fréchet* (*sequential, sequentially compact*).

Proof. If e^T is positive on L then by Theorem 6.3 and Proposition 4.2, K is the image of $L \times 2$ under a finite-valued upper semicontinuous set-function; we conclude applying Lemma 3.2.

If e^T is zero at some point then we first use Lemma 6.4 and continue with $K+1$ replacing K. ∎

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