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Embeddings of Besov–Morrey spaces on bounded domains

by

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Abstract. We study embeddings of spaces of Besov–Morrey type, $\mathrm{id}_\Omega : \mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, and obtain necessary and sufficient conditions for the continuity and compactness of id_Ω. This continues our earlier studies relating to the case of $\mathbb{R}^d$. Moreover, we also characterise embeddings into the scale of L_p spaces or into the space of bounded continuous functions.

1. Introduction. In recent years smoothness spaces related to Morrey spaces, in particular Besov–Morrey and Triebel–Lizorkin–Morrey spaces, attracted some attention. The classical Morrey spaces $\mathcal{M}_{p,u}(\mathbb{R}^d)$, $0 < u \le p < \infty$, were introduced by Ch. B. Morrey [Mo] and are part of the wider class of Morrey–Campanato spaces (cf. [Pe]). They can be considered as a complement to L_p spaces, since $\mathcal{M}_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. However, on the one hand the Morrey spaces with $u < p$ consist of locally u-integrable functions, but on the other hand the spaces scale with d/p instead of d/u, that is,

$$\|f(\lambda\,\cdot) \,|\, \mathcal{M}_{p,u}(\mathbb{R}^d)\| = \lambda^{-d/p}\|f \,|\, \mathcal{M}_{p,u}(\mathbb{R}^d)\|, \qquad \lambda > 0.$$

This property is very useful for some partial differential equations.

Built upon this basic family $\mathcal{M}_{p,u}(\mathbb{R}^d)$, different spaces of Besov–Triebel–Lizorkin type were defined in the last years. H. Kozono and M. Yamazaki [KY] and A. Mazzucato [Ma] introduced the Besov–Morrey $\mathcal{N}^s_{p,u,q}$ spaces and used them in the theory of Navier–Stokes equations. As before, if $u = p$, then these spaces coincide with the classical ones, i.e., $\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$. Some of their properties including wavelet characterisations were proved by Y. Sawano [S1, S4, S3], Y. Sawano and H. Tanaka [ST2, ST1] and L. Tang and J. Xu [TX]. The most systematic and general approach to spaces of this type can be found in the recent book [YSY] of W. Yuan, W. Sickel and D. Yang or in the very recent survey papers by W. Sickel [Si1, Si2]. We recommend the monograph and the survey for further up-to-date references

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on this subject. In contrast to this approach Triebel [T3] followed the original Morrey–Campanato ideas to develop local spaces.

Embeddings of classical Besov–Triebel–Lizorkin spaces of Sobolev type are well understood nowadays. In particular a lot is known about the compactness of these embeddings. This includes the behaviour of analytic and geometric quantities describing their compactness, for example, the relevant approximation and entropy numbers. As an application D. E. Edmunds and H. Triebel [ET] proposed a program of investigating the spectral properties of certain pseudo-differential operators based on the asymptotic behaviour of entropy and approximation numbers, together with Carl's inequality and the Birman–Schwinger principle. In contrast to the classical Besov and Triebel–Lizorkin spaces, little is known about the properties of embeddings as well as about applications of smoothness spaces related to Morrey spaces.

Y. Sawano, S. Sugano, and H. Tanaka found sufficient conditions for the continuity of embeddings in the case of a homogeneous Besov–Morrey space $\dot{\mathcal{N}}^s_{p,u,q}(\mathbb{R}^d)$ (cf. [SST]). The analogue for similar embeddings of nonhomogeneous Triebel–Lizorkin–Morrey spaces can be found in [YSY]. Quite recently we proved sufficient and necessary conditions for the continuity of embeddings of nonhomogeneous Besov–Morrey spaces $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ (cf. [HS]). Embeddings of some weighted spaces are also considered there.

Almost nothing is known about Sobolev embeddings of Besov–Morrey spaces defined on bounded domains. The first approach to the problem is due to G. T. Dzhumakaeva [D1–D3] and Yu. Netrusov [Ne]. They considered spaces of Besov–Morrey type on domains defined in terms of differences as well as Sobolev–Morrey spaces and proved some embedding theorems. Besov–Morrey spaces $\mathcal{N}^s_{p,u,q}(\Omega)$ on bounded smooth domains Ω in $\mathbb{R}^d$ were considered by Y. Sawano [S2]. These spaces are defined by restrictions of elements from $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ to Ω. He proved inter alia the extension property for these spaces.

In this paper we investigate the continuity and compactness of embeddings of $\mathcal{N}^s_{p,u,q}(\Omega)$ spaces. Our first goal is to find necessary and sufficient conditions for the boundedness and compactness of the embeddings

$$\mathrm{id}_\Omega : \mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$$

(cf. Theorems 3.1 and 4.1). In particular, we show that id_Ω is compact if, and only if,

$$\frac{s_1 - s_2}{d} > \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\},$$

where $s_i \in \mathbb{R}$, $0 < q_i \le \infty$, $0 < u_i \le p_i < \infty$, $i = 1, 2$. Special attention is paid to the cases when the source or the target space is a classical Besov space. Moreover we investigate embeddings where the target space is a Lebesgue space or the space of bounded continuous functions, that is,

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow L_r(\Omega) \quad \text{and} \quad \mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow C(\Omega).$$

For the last two embeddings we consider both the situation when Ω is a smooth bounded domain in $\mathbb{R}^d$, and when $\Omega = \mathbb{R}^d$. In particular, we show that the distinction between bounded domains and $\mathbb{R}^d$ is relevant for embeddings into Lebesgue spaces only, unlike the case of the space of continuous functions. We refer the reader to [T3, Sect. 2.1] for embeddings of smoothness spaces built on Morrey–Campanato spaces.

The paper is organised as follows. In Section 2 we collect basic facts about Morrey and Besov–Morrey spaces needed later on. We also recall the wavelet characterisation of Besov–Morrey spaces via compactly supported wavelets and Sawano's extension theorem. Section 3 is devoted to the continuity of the above-described embeddings of Besov–Morrey spaces on bounded domains. Afterwards we prove sufficient and necessary conditions for the compactness of these embeddings. In the last section we deal with embeddings into $L_p(\Omega)$ and $C(\Omega)$.

2. Morrey and Besov–Morrey spaces. First we fix some notation. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$, and by $\mathbb{Z}^d$ the set of all lattice points in $\mathbb{R}^d$ having integer components. The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$, the integer part of $a \in \mathbb{R}$ is $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \le a\}$. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ we write $\alpha_k \sim \beta_k$ when there exist constants $c_1, c_2 > 0$ such that $c_1\alpha_k \le \beta_k \le c_2\alpha_k$ for all $k \in \mathbb{N}$; similarly for positive functions. We denote by $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ the ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$.

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts.

2.1. Function spaces on $\mathbb{R}^d$. We start by recalling the definition of Morrey spaces.

Definition 2.1. Let $0 < u \le p < \infty$. The *Morrey space* $\mathcal{M}_{p,u}(\mathbb{R}^d)$ is the set of all locally u-integrable functions $f \in L^{\mathrm{loc}}_u(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{M}_{p,u}(\mathbb{R}^d)\| = \sup_{x\in\mathbb{R}^d,\, R>0} R^{d/p-d/u}\Big(\int_{B(x,R)} |f(y)|^u\,dy\Big)^{1/u} < \infty.$$

Remark 2.2. The spaces $\mathcal{M}_{p,u}(\mathbb{R}^d)$ are quasi-Banach spaces (Banach spaces for $u \ge 1$). They were introduced by Ch. B. Morrey [Mo] and are part of the wider class of Morrey–Campanato spaces (cf. [Pe]). They can be considered as a complement to L_p spaces. As a matter of fact, $\mathcal{M}_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$, $0 < p < \infty$. To extend this relation we put $\mathcal{M}_{\infty,\infty}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. One can easily see that for $0 < u_2 \le u_1 \le p < \infty$,

$$L_p(\mathbb{R}^d) = \mathcal{M}_{p,p}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{p,u_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{p,u_2}(\mathbb{R}^d).$$

In an analogous way one can define the spaces $\mathcal{M}_{\infty,u}(\mathbb{R}^d)$, $0 < u < \infty$, but using the Lebesgue differentiation theorem one can easily prove that $\mathcal{M}_{\infty,u}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$.

Now we define Besov spaces modelled on $\mathcal{M}_{p,u}(\mathbb{R}^d)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and its dual space $\mathcal{S}'(\mathbb{R}^d)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R}^d : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if } |x| \le 1,$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then the Paley–Wiener–Schwartz theorem implies that $\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^d$.

Definition 2.3. Let $0 < u \le p < \infty$ or $p = u = \infty$. Let $0 < q \le \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. *The Besov–Morrey space* $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \,|\, \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \,|\, \mathcal{M}_{p,u}(\mathbb{R}^d)\|^q\Big)^{1/q} \tag{2.1}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 2.4. The spaces $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $u, q \ge 1$), and $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{p,u,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. Moreover, for $u = p$ we recover the usual Besov spaces,

$$\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d). \tag{2.2}$$

Besov–Morrey spaces were introduced by H. Kozono and M. Yamazaki [KY]. They studied semilinear heat equations and Navier–Stokes equations with initial data belonging to such spaces. The investigations were continued by A. Mazzucato [Ma], who found the atomic decomposition of some spaces. We follow the ideas of L. Tang and J. Xu [TX], where a somewhat different definition is proposed. These ideas were developed by Y. Sawano and H. Tanaka [ST2, ST1, S1, S4].

It has turned out that many of the results from the classical situation have their counterparts, e.g., $\mathcal{N}^{s+\varepsilon}_{p,u,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)$ if $\varepsilon > 0$, $\mathcal{N}^{s}_{p,u,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s}_{p,u,q_2}(\mathbb{R}^d)$ if $q_1 \le q_2$, and $\mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ if $s > d/p$. We recall two properties that will be needed later on.

First, Besov–Morrey spaces have lifting properties similar to the classical Besov spaces. Let $\sigma \in \mathbb{R}$. Then I_σ is the operator defined by

$$I_\sigma f = \mathcal{F}^{-1}(1+|\xi|^2)^{\sigma/2}\mathcal{F}f, \qquad f \in \mathcal{S}'(\mathbb{R}^d). \tag{2.3}$$

It is well-known that I_σ is an isomorphic mapping of $\mathcal{S}'(\mathbb{R}^d)$, as well as an isomorphic mapping of $\mathcal{S}(\mathbb{R}^d)$ onto itself.

THEOREM 2.5 (cf. [TX]). *Let $s, \sigma \in \mathbb{R}$, $0 < q \le \infty$, $0 < u \le p < \infty$. Then I_σ is an isomorphic mapping from $\mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)$ onto $\mathcal{N}^{s+\sigma}_{p,u,q}(\mathbb{R}^d)$.*

Let $C(\mathbb{R}^d)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^d$, equipped with the sup-norm as usual. If $m \in \mathbb{N}$, then $C^m(\mathbb{R}^d) = \{f : \mathrm{D}^\alpha f \in C(\mathbb{R}^d) \text{ for all } |\alpha| \le m\}$.

THEOREM 2.6 (cf. [S2]). *If $m \in \mathbb{N}$ is sufficiently large and $h \in C^m(\mathbb{R}^d)$, then the natural pointwise multiplication mapping*

$$M_h : f \ni \mathcal{S}(\mathbb{R}^d) \mapsto h \cdot f$$

extends to a continuous mapping from $\mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)$ to $\mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)$. Furthermore, there exists a constant $C > 0$ such that

$$\|M_h f \,|\, \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)\| \le C \|h \,|\, C^m(\mathbb{R}^d)\| \, \|f \,|\, \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)\|. \tag{2.4}$$

2.2. Function spaces on bounded domains. Let Ω be a bounded C^∞ domain in $\mathbb{R}^d$. We define the Besov–Morrey spaces on Ω by restrictions. Namely, since we are able to define an extension operator $\mathrm{ext} : C^\infty_0(\Omega) \to \mathcal{S}(\mathbb{R}^d)$, the restriction operator $\mathrm{re} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{D}'(\Omega)$ can be defined naturally as the adjoint operator. We will write $f|_\Omega = \mathrm{re}(f)$ for $f \in \mathcal{S}'(\mathbb{R}^d)$.

DEFINITION 2.7. Let $0 < u \le p < \infty$ or $u = p = \infty$. Let $0 < q \le \infty$ and $s \in \mathbb{R}$. Then the *Besov–Morrey space* $\mathcal{N}^{s}_{p,u,q}(\Omega)$ is defined by

$$\mathcal{N}^{s}_{p,u,q}(\Omega) := \{f \in \mathcal{D}'(\Omega) : f = g|_\Omega \text{ for some } g \in \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)\} \tag{2.5}$$

and

$$\|f \,|\, \mathcal{N}^{s}_{p,u,q}(\Omega)\| = \inf\{\|g \,|\, \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)\| : f = g|_\Omega,\ g \in \mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)\}. \tag{2.6}$$

REMARK 2.8. The spaces $\mathcal{N}^{s}_{p,u,q}(\Omega)$ are quasi-Banach spaces (Banach spaces for $q, u \ge 1$). Moreover, for $u = p$ we recover the usual Besov spaces defined on bounded smooth domains. The properties of the above spaces, in particular, the extension property, were investigated in [S2].

Theorem 2.9 (cf. [S2]). *Let $N \in \mathbb{N}$ and either $N^{-1} \le u \le p < \infty$ or $u = p = \infty$. Let $|s| < N$ and $N^{-1} < q \le \infty$. Then for any N there is a common extension operator*

$$\mathrm{ext}_N : \mathcal{N}^s_{p,u,q}(\Omega) \to \mathcal{N}^s_{p,u,q}(\mathbb{R}^d) \tag{2.7}$$

such that

$$\mathrm{re} \circ \mathrm{ext}_N = \mathrm{id} \quad (\textit{identity in } \mathcal{N}^s_{p,u,q}(\Omega)), \tag{2.8}$$

that is, $\mathrm{ext}_N f|_\Omega = f$ *for* $f \in \mathcal{N}^s_{p,u,q}(\Omega)$.

2.3. Wavelet characterisation. Finally, we briefly describe the wavelet characterisation of Besov–Morrey spaces proved in [S4]. For $m \in \mathbb{Z}^d$ and $\nu \in \mathbb{Z}$ we define a d-dimensional dyadic cube $Q_{\nu,m}$ with sides parallel to coordinate axes by

$$Q_{\nu,m} = \prod_{i=1}^{d} \left[\frac{m_i}{2^\nu}, \frac{m_i+1}{2^\nu}\right), \qquad m = (m_1, \dots, m_d) \in \mathbb{Z}^d,\ \nu \in \mathbb{Z}.$$

For $0 < p < \infty$, $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$ we denote by $\chi^{(p)}_{\nu,m}$ the p-normalised characteristic function of the cube $Q_{\nu,m}$,

$$\chi^{(p)}_{\nu,m}(x) = 2^{\nu d/p} \chi_{\nu,m}(x) = \begin{cases} 2^{\nu d/p} & \text{for } x \in Q_{\nu,m}, \\ 0 & \text{for } x \notin Q_{\nu,m}, \end{cases} \tag{2.9}$$

hence $\|\chi^{(p)}_{\nu,m} \,|\, L_p(\mathbb{R}^d)\| = 1$ and $\|\chi^{(p)}_{\nu,m} \,|\, \mathcal{M}_{p,u}(\mathbb{R}^d)\| = 1$.

Let $\widetilde{\phi}$ be a scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\widetilde{\psi}$ be the associated wavelet. Then the tensor-product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \dots, \psi_{2^d-1}$, all defined now on $\mathbb{R}^d$. We suppose $\widetilde{\phi} \in C^{N_1}(\mathbb{R})$ and $\operatorname{supp} \widetilde{\phi} \subset [-N_2, N_2]$ for certain natural numbers N_1 and N_2. This implies

$$\phi, \psi_i \in C^{N_1}(\mathbb{R}^d) \quad \text{and} \quad \operatorname{supp} \phi, \operatorname{supp} \psi_i \subset [-N_3, N_3]^d \tag{2.10}$$

for $i = 1, \dots, 2^d - 1$. We use the standard abbreviations

$$\phi_{\nu,m}(x) = 2^{\nu d/2} \phi(2^\nu x - m) \quad \text{and} \quad \psi_{i,\nu,m}(x) = 2^{\nu d/2} \psi_i(2^\nu x - m). \tag{2.11}$$

To formulate the result we introduce some sequence spaces. For $0 < u \le p < \infty$, $0 < q \le \infty$ and $\sigma \in \mathbb{R}$, let

$$\begin{aligned} n^\sigma_{p,u,q} := \Big\{ &\lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \lambda_{\nu,m} \in \mathbb{C}, \\ &\|\lambda \,|\, n^\sigma_{p,u,q}\| = \bigg\| \bigg\{ 2^{\nu(\sigma - d/p)} \bigg\| \sum_{m \in \mathbb{Z}^d} \lambda_{\nu,m} \chi^{(p)}_{\nu,m} \,\bigg|\, \mathcal{M}_{p,u}(\mathbb{R}^d) \bigg\| \bigg\}_{\nu \in \mathbb{N}_0} \,\bigg|\, \ell_q \bigg\| < \infty \Big\}. \end{aligned} \tag{2.12}$$

The following theorem was proved in [Ro, Thm. 4.5, Cor. 4.17] (see also [S4]).

Theorem 2.10. *Let $0 < u \le p < \infty$ or $u = p = \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let ϕ be a scaling function and let ψ_i, $i = 1, \ldots, 2^d - 1$, be the corresponding wavelets satisfying* (2.10). *Assume that*

$$\max\{(1 + \lfloor s \rfloor)_+, \lfloor d(1/u - 1)_+ - s \rfloor\} \le N_1.$$

Then a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ if, and only if,

$$\|f \,|\, \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)\|^\star = \|\{\langle f, \phi_{0,m}\rangle\}_{m\in\mathbb{Z}^d} \,|\, \ell_p\| + \sum_{i=1}^{2^d-1} \|\{\langle f, \psi_{i,\nu,m}\rangle\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^d} \,|\, n^\sigma_{p,u,q}\|$$

is finite, where $\sigma = s + d/2$. Furthermore, $\|f \,|\, \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)\|^\star$ may be used as an equivalent (*quasi-*) *norm in $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$.*

Remark 2.11. It follows from Theorem 2.10 that the mapping

$$T : f \mapsto \big(\{\langle f, \phi_{0,m}\rangle\}_{m\in\mathbb{Z}^d}, \{\langle f, \psi_{i,\nu,m}\rangle\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^d,\, i=1,\ldots,2^d-1}\big) \tag{2.13}$$

is an isomorphism of $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ onto $\ell_p \oplus \bigoplus_{i=1}^{2^d-1} n^\sigma_{p,u,q}$, $\sigma = s + d/2$ (cf. [Ro, Thm. 4.5, Cor. 4.17] and [S4]).

The theorem covers the characterisation of Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ by Daubechies wavelets (cf. [T2, pp. 15–16] and the references given there).

We define an equivalent norm in the sequence spaces $n^s_{p,u,q}$ that is more convenient for our purposes. Let $s \in \mathbb{R}$, $0 < q \le \infty$, $0 < u \le p < \infty$ or $u = p = \infty$. For a sequence $\{\lambda_{j,m}\}_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^d$, we define a quasi-norm

$$\|\lambda \,|\, n^s_{p,u,q}\|^* = \Big(\sum_{j=0}^{\infty} 2^{qj(s-\frac{d}{p})} \sup_{\substack{\nu:\, \nu\le j\\ k\in\mathbb{Z}^d}} 2^{qd(j-\nu)(\frac{1}{p}-\frac{1}{u})} \Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^u\Big)^{\frac{q}{u}}\Big)^{\frac{1}{q}}, \tag{2.14}$$

with the usual modification if $q = \infty$ or $u = p = \infty$.

Lemma 2.12 (cf. [HS]). *Let $s \in \mathbb{R}$, $0 < q \le \infty$, $0 < u \le p < \infty$. Then*

$$n^s_{p,u,q} = \big\{\lambda = \{\lambda_{j,m}\}_{j,m} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^d \text{ and } \|\lambda \,|\, n^s_{p,u,q}\|^* < \infty\big\}. \tag{2.15}$$

Remark 2.13. If $0 < u = p \le \infty$, then

$$\|\lambda \,|\, n^s_{p,p,q}\|^* = \Big(\sum_{j=0}^{\infty} 2^{qj(s-\frac{d}{p})} \Big(\sum_{m\in\mathbb{Z}^d} |\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}, \tag{2.16}$$

with the usual modification if $q = \infty$ or $p = \infty$.

3. Continuous embeddings. First we consider sufficient and necessary conditions for boundedness of embeddings.

THEOREM 3.1. *Let* $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < u_2 \le p_2 < \infty$. *Then the embedding*

$$\mathrm{id}_\Omega : \mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega) \tag{3.1}$$

holds if either

$$\frac{s_1 - s_2}{d} > \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\}, \tag{3.2}$$

or

$$\frac{s_1 - s_2}{d} = \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\}, \tag{3.3}$$

with

$$q_1 \le \min\left(1, \max\left(1, \frac{p_2}{p_1}\right)\frac{u_1}{u_2}\right) q_2. \tag{3.4}$$

Conversely, if there is a continuous embedding (3.1), *then either the parameters satisfy* (3.2), *or* (3.3) *holds and* $q_1 \le q_2$.

REMARK 3.2. Note that the above theorem implies that in the case of

$$\frac{u_1}{u_2} \ge \min\left(1, \frac{p_1}{p_2}\right) \tag{3.5}$$

the embedding (3.1) is continuous if, and only if, either (3.2) is satisfied, or (3.3) holds and $q_1 \le q_2$. In the classical case, i.e., when $p_i = u_i$, $i = 1, 2$, condition (3.5) is obviously true and the well-known embedding result for Besov spaces on bounded domains is recovered. In all other cases apart from (3.5) there is clearly a gap between necessary and sufficient conditions in the limiting case (3.3) in view of the fine indices q_i.

Proof. STEP 1 (Sufficiency)

SUBSTEP 1.1. Let $\Omega_t = \{x \in \mathbb{R}^d : \mathrm{dist}(x, \Omega) < t\}$ for some $t > 0$. We choose a dyadic cube Q (applying some appropriate dilations or translations first, if necessary) such that

$$\mathrm{supp}\,\psi_{i,\nu,m} \subset Q \quad \text{if} \quad \mathrm{supp}\,\psi_{i,\nu,m} \cap \Omega_t \ne \emptyset$$

and

$$\mathrm{supp}\,\phi_{0,m} \subset Q \quad \text{if} \quad \mathrm{supp}\,\phi_{0,m} \cap \Omega_t \ne \emptyset.$$

We define a sequence space by

$$\widetilde{n}^{\sigma}_{p,u,q}(Q) := \Big\{\lambda = \{\lambda_{j,m}\}_{j,m} : \lambda_{j,m} \in \mathbb{C},\ j \in \mathbb{N}_0,\ Q_{j,m} \subset Q, \tag{3.6}$$
$$\|\lambda \,|\, n^{\sigma}_{p,u,q}\| = \Big(\sum_{j=0}^{\infty} 2^{jq(\sigma-\frac{d}{p})} \sup_{\substack{\nu:\, \nu\le j\\ k:\, Q_{\nu,k}\subset Q}} 2^{qd(j-\nu)(\frac{1}{p}-\frac{1}{u})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^u\Big)^{\frac{q}{u}}\Big)^{\frac{1}{q}} < \infty\Big\}.$$

Let $h \in C_0^\infty(\mathbb{R}^d)$ be a test function such that $\operatorname{supp} h \subset \Omega_t$ and $h(x) = 1$ for any $x \in \Omega_{t/2}$. We choose $N \in \mathbb{N}$ such that $N^{-1} \le u_i \le p_i$, $N^{-1} < q_i$ and $|s_i| < N$, $i = 1, 2$. Then there exists an extension operator ext_N, common for both spaces $\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega)$ and $\mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$ (cf. Theorem 2.9). Note that $\mathrm{re} \circ M_h = \mathrm{re}$, that is, $\mathrm{re} \circ M_h \circ \mathrm{ext}_N = \mathrm{id}$ on $\mathcal{N}^{s_i}_{p_i,u_i,q_i}(\Omega)$ (cf. (2.8) and Theorem 2.6). Now using the wavelet characterisation of $\mathcal{N}^{s}_{p,u,q}(\mathbb{R}^d)$ by Daubechies wavelets (cf. Theorem 2.10), in particular the isomorphism T defined by (2.13), one can factorise the embedding (3.1) through the embedding of the above sequence spaces. Namely the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) & \xrightarrow{\ \mathrm{id}\ } & \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega) \\
{\scriptstyle T\circ M_h \circ \mathrm{ext}_N}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{re}\circ T^{-1}} \\
\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q) & \xrightarrow{\ \mathrm{id}\ } & \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)
\end{array}$$

with $\sigma_i = s_i + d/2$.

Thus it is enough to find sufficient conditions for the embedding

$$\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q) \hookrightarrow \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)$$

to hold. It should be clear that we can assume that $Q = Q_{\nu_0,0}$ for some $\nu_0 < 0$.

Moreover, the diagonal operator $(\lambda_{j,m}) \mapsto (2^{-jt}\lambda_{j,m})$, $t \in \mathbb{R}$, is an isometry of $\widetilde{n}^{\sigma}_{p,u,q}(Q)$ onto $\widetilde{n}^{\sigma+t}_{p,u,q}(Q)$. Thus we can assume that $\sigma_1 = d/p_1$.

Substep 1.2. If $\frac{u_2}{p_2} \le \frac{u_1}{p_1}$ and $p_1 \le p_2$ we can apply the method used in the proof of [HS, Theorem 3.2]. This shows that the inequality

$$\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}\| \le C\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}\| \tag{3.7}$$

holds if either

$$\frac{s_1 - s_2}{d} > \frac{1}{p_1} - \frac{1}{p_2}, \quad \text{or} \quad \frac{s_1 - s_2}{d} = \frac{1}{p_1} - \frac{1}{p_2} \quad \text{and} \quad q_1 \le q_2,$$

which coincides with (3.4) in this case.

Substep 1.3. Let $u_2 \le u_1$. The case $p_1 \le p_2$ is covered by Substep 1.2 since $u_2 \le u_1 \le \frac{p_2}{p_1}u_1$. So we can assume $p_2 < p_1$. By the Hölder inequality we get

$$\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \le C\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1}\Big)^{\frac{1}{u_1}} 2^{d(j-\nu)(\frac{1}{u_2}-\frac{1}{u_1})}. \tag{3.8}$$

Consequently,

$$\begin{aligned}\sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} & 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \\ &\le \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{p_1})}\, 2^{d(j-\nu)(\frac{1}{p_1}-\frac{1}{u_1})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1}\Big)^{\frac{1}{u_1}}.\end{aligned} \tag{3.9}$$

But $\frac{1}{p_2}-\frac{1}{p_1}>0$, so that

$$2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{p_1})} \le 2^{d(j-\nu_0)(\frac{1}{p_2}-\frac{1}{p_1})} \le C 2^{jd(\frac{1}{p_2}-\frac{1}{p_1})}$$

and $\sigma_2-\frac{d}{p_2}+\frac{d}{p_2}-\frac{d}{p_1}=\sigma_2-\frac{d}{p_1}=s_2-s_1$. Thus

$$\begin{aligned} 2^{j(\sigma_2-\frac{d}{p_2})} & \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \\ &\le C 2^{-j(s_1-s_2)} \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_1}-\frac{1}{u_1})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1}\Big)^{\frac{1}{u_1}}.\end{aligned} \tag{3.10}$$

If $s_1-s_2>0$, or $s_1-s_2=0$ and $q_1\le q_2$ (according to (3.4) in this case), then (3.7) holds.

SUBSTEP 1.4. We assume that $u_1<u_2$. The case $p_1\le p_2$ and $u_1<u_2\le\frac{p_2}{p_1}u_1$ is covered by Substep 1.2. Thus—also in view of (3.4)—we are left with two cases:

(i) $p_2<p_1$, $u_1<u_2$, and $q_1\le\frac{u_1}{u_2}q_2$ if (3.3) holds,
(ii) $p_1\le p_2$, $u_1\le\frac{p_2}{p_1}u_1<u_2$, and $q_1\le\frac{u_1p_2}{u_2p_1}q_2$ if (3.3) holds.

In both cases we have

$$\max\Big\{0,\frac{1}{p_1}-\frac{1}{p_2},\frac{u_1}{p_1}\Big(\frac{1}{u_1}-\frac{1}{u_2}\Big)\Big\}=\frac{u_1}{p_1}\Big(\frac{1}{u_1}-\frac{1}{u_2}\Big)\ge\max\Big\{0,\frac{1}{p_1}-\frac{1}{p_2}\Big\}.$$

We take a sequence λ such that

$$\|\lambda\,|\,\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}\|=1. \tag{3.11}$$

Then the terms $\lambda_{j,m}$ of the sequence λ are at most 1 in absolute value, since

$\sigma_1 = d/p_1$. So the inequality $u_1 < u_2$ implies

$$\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2} \le \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1}. \tag{3.12}$$

Thus by (3.11) we get

$$2^{d(j-\nu)(\gamma u_2 -1)} \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2} \le 2^{d(j-\nu)(\frac{u_1}{p_1}-1)} \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1} \le 1 \tag{3.13}$$

if $\gamma u_2 \le \frac{u_1}{p_1}$ and $j-\nu \ge 0$. First we deal with (3.2) and assume that

$$\frac{s_1 - s_2}{d} = \frac{\frac{d}{p_1} - \sigma_2}{d} > \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right). \tag{3.14}$$

It follows from (3.14) that there exists some $\varepsilon > 0$ such that

$$\frac{d}{p_2} - \sigma_2 - \varepsilon > \frac{d}{u_2}\left(\frac{u_2}{p_2} - \frac{u_1}{p_1}\right) > 0. \tag{3.15}$$

Then $2^{-\nu(\frac{d}{p_2}-\sigma_2-\varepsilon)} \le 2^{-\nu_0(\frac{d}{p_2}-\sigma_2-\varepsilon)}$ if $Q_{\nu,k}\subset Q$. Hence

$$\begin{aligned} &2^{j(\sigma_2-\frac{d}{p_2})} \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})}\Bigg(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Bigg)^{\frac{1}{u_2}} \\ &\le 2^{\nu_0(\sigma_2-\frac{d}{p_2}-\varepsilon)} 2^{-j\varepsilon} \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{\sigma_2+\varepsilon}{d}-\frac{1}{u_2})}\Bigg(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Bigg)^{\frac{1}{u_2}}. \end{aligned} \tag{3.16}$$

Since (3.15) implies that $\frac{\sigma_2+\varepsilon}{d}u_2 < \frac{u_1}{p_1}$ we can take $\gamma = \frac{\sigma_2+\varepsilon}{d}$ in (3.13). Now (3.13) and (3.16) yield

$$2^{j(\sigma_2-\frac{d}{p_2})} \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})}\Bigg(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Bigg)^{\frac{1}{u_2}} \le C2^{-j\varepsilon},$$

which leads to

$$\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}\| \le C \tag{3.17}$$

for any $0 < q_2 \le \infty$.

Substep 1.5. Now we deal with (3.3), that is,

$$\frac{s_1 - s_2}{d} = \frac{\frac{d}{p_1} - \sigma_2}{d} = \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right) > 0. \tag{3.18}$$

We shall first verify the continuity of (3.1) in both cases (i) and (ii) above when $q_1 \le \frac{u_1}{u_2}q_2$. Afterwards, in Substep 1.6, we finally prove the extension to values $\frac{u_2 p_1}{u_1 p_2}q_1 \le q_2 \le \frac{u_2}{u_1}q_1$ in case (ii).

We put

$$a_j = \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{u_1}{p_1}-1)} \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1}. \tag{3.19}$$

Then (3.11) implies

$$\sum_{j=0}^{\infty} a_j^{q_1/u_1} = 1 \tag{3.20}$$

(recall $\sigma_1 = d/p_1$). As in the previous substep, we get

$$\begin{aligned} 2^{j(\sigma_2-\frac{d}{p_2})} &\sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \\ &\le C \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{\sigma_2}{d}-\frac{1}{u_2})}\Big(\sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}}. \end{aligned} \tag{3.21}$$

We now choose $\gamma = \sigma_2/d$ in (3.13) to obtain

$$\begin{aligned} \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} &2^{d(j-\nu)(\frac{\sigma_2}{d}u_2-1)} \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2} \\ &\le \sup_{\substack{\nu:\,\nu\le j\\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{u_1}{p_1}-1)} \sum_{\substack{m:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1} = a_j. \end{aligned}$$

Now by (3.20), (3.21) and the monotonicity of the ℓ_q scale we have

$$\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}\| \le C\Big(\sum_{j=0}^{\infty} a_j^{q_2/u_2}\Big)^{1/q_2} \le C \tag{3.22}$$

if $q_2/u_2 \ge q_1/u_1$.

Substep 1.6. It remains to consider case (ii) above, that is, where (3.18) is satisfied with $p_1 \le p_2$, $u_1 \le \frac{p_2}{p_1}u_1 < u_2$, and $\frac{u_2 p_1}{u_1 p_2}q_1 \le q_2 \le \frac{u_2}{u_1}q_1$. Let

$$u = u_1\frac{p_2}{p_1} \quad \text{and} \quad \sigma = s_1 - d\frac{u_1}{p_1}\Big(\frac{1}{u_1}-\frac{1}{u}\Big).$$

Then $u_1 < u < u_2$, $\frac{s_1-\sigma}{d} = \frac{u_1}{p_1}\big(\frac{1}{u_1}-\frac{1}{u}\big) = \frac{1}{p_1}-\frac{1}{p_2} \ge 0$ and according to Substep 1.2 we obtain

$$\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{\sigma}_{p_2,u,q_1}(\Omega). \tag{3.23}$$

On the other hand,

$$\frac{\sigma - s_2}{d} = \frac{s_1-s_2}{d} - \frac{u_1}{p_1}\Big(\frac{1}{u_1}-\frac{1}{u}\Big) = \frac{u_1}{p_1}\Big(\frac{1}{u}-\frac{1}{u_2}\Big),$$

so that Substep 1.5 implies

$$\mathcal{N}^{\sigma}_{p_2,u,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega) \tag{3.24}$$

if $q_1 \le \frac{u}{u_2} q_2 = \frac{u_1 p_2}{u_2 p_1} q_2$. Thus (3.23), (3.24) yield (3.1) as desired and complete the proof of the sufficiency part.

Step 2 (Necessity). To prove necessity we construct some counterexamples. It is sufficient to consider the following cases:

- $\frac{s_1-s_2}{d} = \max\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}(\frac{1}{u_1} - \frac{1}{u_2})\} = \frac{1}{p_1} - \frac{1}{p_2} > 0$ and $q_1 > q_2$,
- $\frac{s_1-s_2}{d} = \max\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}(\frac{1}{u_1} - \frac{1}{u_2})\} = 0$ and $q_1 > q_2$,
- $\frac{s_1-s_2}{d} = \max\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}(\frac{1}{u_1} - \frac{1}{u_2})\} = \frac{u_1}{p_1}(\frac{1}{u_1} - \frac{1}{u_2}) > \max\{0, \frac{1}{p_1} - \frac{1}{p_2}\}$ and $q_1 > q_2$.

The rest follows by monotonicity of the function spaces with respect to the smoothness parameter s.

Substep 2.1. By the diffeomorphic properties of Besov–Morrey spaces, using translations and dilations if necessary we can assume that the domain Ω satisfies the following conditions:

$$Q_{\nu_0,0} \subset \Omega, \tag{3.25}$$

$$\text{if } Q_{\nu,m} \subset Q_{\nu_0,0},\ \nu \ge 0, \text{ then } \operatorname{supp} \psi_{i,\nu,m} \subset \Omega, \tag{3.26}$$

$$\text{if } Q_{0,m} \subset Q_{\nu_0,0}, \text{ then } \operatorname{supp} \phi_{0,m} \subset \Omega \tag{3.27}$$

(cf. [S2]).

Let $\widetilde{T}$ denote the restriction of the isomorphism T^{-1} to the subspaces $n^s_{p,u,q}(Q_{\nu_0,0}) \subset n^s_{p,u,q}(\mathbb{R}^d)$ (cf. (2.13)). Then

$$\widetilde{T}(\lambda) \in n^s_{p,u,q}(\mathbb{R}^d), \quad \operatorname{supp} \widetilde{T}(\lambda) \subset \Omega \quad \text{for any } \lambda \in n^s_{p,u,q}(Q_{\nu_0,0}). \tag{3.28}$$

Moreover there exists a positive $\varepsilon > 0$ such that for any $\lambda \in n^s_{p,u,q}(Q_{\nu_0,0})$, $\operatorname{dist}(\operatorname{supp} \widetilde{T}(\lambda), \mathbb{R}^d \setminus \Omega) > \varepsilon$. But then

$$\|\widetilde{T}(\lambda) \,|\, \mathcal{N}^s_{p,u,q}(\Omega)\| \sim \|\widetilde{T}(\lambda) \,|\, \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)\| \sim \|\lambda \,|\, n^s_{p,u,q}(Q_{\nu_0,0})\| \tag{3.29}$$

for any $\lambda \in n^s_{p,u,q}(Q_{\nu_0,0})$. We can take the same system of wavelets, and hence the same operator $\widetilde{T}$, for $\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\mathbb{R}^d)$ and $\mathcal{N}^{s_2}_{p_2,u_2,q_2}(\mathbb{R}^d)$. So, if we find a counterexample showing that the inequality

$$\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0})\| \le C \|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q_{\nu_0,0})\|, \qquad \lambda \in \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q_{\nu_0,0}), \tag{3.30}$$

does not hold for any $C > 0$ we immediately get a counterexample showing that

$$\|f \,|\, \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)\| \le C \|f \,|\, \mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega)\|, \qquad f \in \mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega), \tag{3.31}$$

does not hold for any $C > 0$.

Thus we are left with necessary conditions for the bounded embeddings

$$\mathrm{id} : \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q_{\nu_0,0}) \to \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0}) \tag{3.32}$$

with $\nu_0 < 0$. For convenience, let $Q = Q_{\nu_0,0}$ again.

We assume that

$$\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \le C\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\|. \tag{3.33}$$

Since we have $q_1 > q_2$ in all cases, we can choose a sequence $(\gamma_j)_j$ of positive numbers such that $\gamma = (\gamma_j)_j \in \ell_{q_1}(\mathbb{N}_0) \setminus \ell_{q_2}(\mathbb{N}_0)$.

Substep 2.2. If $\frac{s_1-s_2}{d} = \frac{1}{p_1} - \frac{1}{p_2} > 0$ and $q_1 > q_2$, then we define the sequence $\lambda = (\lambda_{j,m})_{j,m}$ by

$$\lambda_{j,m} = \begin{cases} \gamma_j & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.34}$$

Then (3.1) together with (3.33) implies

$$\Big(\sum_{j=0}^{\infty} \gamma_j^{q_2}\Big)^{\frac{1}{q_2}} = \|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \le C\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| = C\Big(\sum_{j=0}^{\infty} \gamma_j^{q_1}\Big)^{\frac{1}{q_1}} < \infty,$$

which contradicts $\gamma \notin \ell_{q_2}$.

Substep 2.3. If $\frac{s_1-s_2}{d} = 0$ and $q_1 > q_2$, then we take

$$\lambda_{j,m} = \begin{cases} \gamma_j 2^{-jd/p_1} & \text{if } Q_{j,m} \subset Q, \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

Then

$$\sup_{\substack{\nu:\,\nu\le j \\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_2}-\frac{1}{u_2})} \Big(\sum_{\substack{m: \\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2} \Big)^{\frac{1}{u_2}} = \gamma_j 2^{dj(\frac{1}{p_2}-\frac{1}{p_1})} 2^{d|\nu_0|\frac{1}{p_2}}$$

and

$$\sup_{\substack{\nu:\,\nu\le j \\ k:\,Q_{\nu,k}\subset Q}} 2^{d(j-\nu)(\frac{1}{p_1}-\frac{1}{u_1})} \Big(\sum_{\substack{m: \\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_1} \Big)^{\frac{1}{u_1}} = \gamma_j 2^{d|\nu_0|\frac{1}{p_1}}.$$

Therefore

$$\begin{aligned} \Big(\sum_{j=0}^{\infty} \gamma_j^{q_2}\Big)^{\frac{1}{q_2}} &\le C\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \\ &\le C'\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| \le C\Big(\sum_{j=0}^{\infty} \gamma_j^{q_1}\Big)^{\frac{1}{q_1}} < \infty, \end{aligned} \tag{3.36}$$

which again contradicts $\gamma \notin \ell_{q_2}$.

SUBSTEP 2.4. Let

$$\frac{s_1 - s_2}{d} = \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\} = \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)$$
$$> \max\left(0, \frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{and} \quad q_1 > q_2.$$

It follows that $u_1/p_1 < u_2/p_2$, in particular, $u_1/p_1 < 1$. We adopt the example constructed in Substep 2.4 of the proof of Theorem 3.2 in [HS].

For any $0 > \nu \geq \nu_0$ we put

$$k_\nu = \lfloor 2^{d|\nu|(1-u_1/p_1)} \rfloor$$

(recall $\lfloor x \rfloor = \max\{l \in \mathbb{Z} : l \leq x\}$). Then $1 \leq k_\nu < 2^{d|\nu|}$ and

$$k_\nu \leq \ c_{p_1 u_1}\ 2^{d(\mu-\nu)} k_\mu \quad \text{if } \nu \leq \mu < 0. \tag{3.37}$$

For convenience let us assume that $c_{p_1 u_1} = 1$ (otherwise the argument below has to be modified in an obvious way). For any ν we defined in [HS] a sequence $\lambda^{(\nu)} = \{\lambda^{(\nu)}_{j,m}\}_{j,m}$ in the following way: We assume that k_ν elements of the sequence equal 1 and the others are 0. If $j \neq 0$ or $Q_{0,m} \not\subseteq Q_{\nu,0}$, then we put $\lambda^{(\nu)}_{j,m} = 0$. Moreover, because of the inequality (3.37), we can choose the elements equal to 1 in such a way that:

$$\text{if} \quad Q_{\mu,\ell} \subseteq Q_{\nu,0} \quad \text{and} \quad Q_{\mu,\ell} = \bigcup_{i=1}^{2^{-d\mu}} Q_{0,m_i},$$
$$\text{then} \quad \text{at most } k_\mu \text{ elements } \lambda^{(\nu)}_{0,m_i} \text{ equal 1.}$$

Now we define a new sequence $\lambda = (\lambda_{j,k}) \in \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}$ by

$$\lambda_{j,m} = \gamma_j \lambda^{(\nu)}_{0,m} \quad \text{if} \quad j = \nu - \nu_0 \quad \text{and} \quad Q_{0,m} \subset Q_{\nu,0}. \tag{3.38}$$

If $Q_{\mu,l} \subset Q$, then for fixed $j \geq \mu$ there are at most $k_{\mu-j}$ nonzero elements $\lambda_{j,m}$ such that the corresponding cube $Q_{j,m}$ is a subset of $Q_{\mu,l}$. Thus we have

$$\sum_{m:\, Q_{j,m} \subset Q_{\mu,\ell}} |\lambda_{j,m}|^{u_1} \leq \gamma_j^{u_1} k_{\mu-j} \leq \gamma_j^{u_1} 2^{d(j-\mu)(1-\frac{u_1}{p_1})} \tag{3.39}$$

and the last sum is equal to $k_{\mu-j}\gamma_j^{u_1}$ if $\mu = \nu_0$. Thus

$$\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| \leq c\Big(\sum_{j=0}^{\infty} \gamma_j^{q_1}\Big)^{\frac{1}{q_1}} < \infty \tag{3.40}$$

(recall $\sigma_1 = d/p_1$). In a similar way we get

$$2^{d(j-\mu)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{m:\, Q_{j,m} \subset Q_{\mu,\ell}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \leq \gamma_j 2^{d(j-\mu)(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})}. \tag{3.41}$$

Moreover, if $\mu = \nu_0$, then

$$2^{d(j-\nu_0)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{m\colon Q_{j,m}\subset Q_{\nu_0,0}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \ge c\gamma_j 2^{d(j-\nu_0)(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})}, \tag{3.42}$$

for some constant c independent of γ. Recall that we are now dealing with the case when $\delta = \sigma_1 - \sigma_2 - \frac{d}{p_1} + \frac{d}{p_2} = d\big(\frac{1}{p_2} - \frac{u_1}{p_1 u_2}\big) > 0$. So (3.42) implies

$$\begin{aligned} c2^{-d\nu_0(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})}\gamma_j &= c2^{-jd(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})}\gamma_j 2^{d(j-\nu_0)(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})} \\ &\le 2^{-jd(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})} 2^{d(j-\nu_0)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{m\colon Q_{j,m}\subset Q_{\nu_0,0}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}} \\ &\le 2^{-j\delta} \sup_{\substack{\mu\colon \mu\le j \\ k\colon Q_{\mu,k}\subset Q}} 2^{d(j-\mu)(\frac{1}{p_2}-\frac{1}{u_2})}\Big(\sum_{m\colon Q_{j,m}\subset Q_{\mu,k}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{1}{u_2}}. \end{aligned}$$

This yields

$$\|\gamma \,|\, \ell_{q_2}\| \le C\|\lambda \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \le C\|\lambda \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| \le C\|\gamma \,|\, \ell_{q_1}\| < \infty,$$

which contradicts $\gamma \notin \ell_{q_2}$. ■

We list a number of immediate consequences of Theorem 3.1.

Corollary 3.3. *Let $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < u_2 \le p_2 < \infty$. The following conditions are equivalent:*

(a) $\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) = \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$,
(b) $(s_1, p_1, u_1, q_1) = (s_2, p_2, u_2, q_2)$.

Now we focus on some limiting cases; recall also Remark 3.2 above.

Corollary 3.4. *Let $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < u_2 \le p_2 < \infty$.*

(a) *If $s_1 = s_2$, then the embedding* (3.1) *holds if, and only if, $p_1 \ge p_2$, $u_1 \ge u_2$ and $q_1 \le q_2$.*
(b) *If $u_1 = u_2$, then the embedding* (3.1) *holds if, and only if, either*

$$\frac{s_1 - s_2}{d} > \max\Big\{0, \frac{1}{p_1} - \frac{1}{p_2}\Big\} \tag{3.43}$$

or

$$\frac{s_1 - s_2}{d} = \max\Big\{0, \frac{1}{p_1} - \frac{1}{p_2}\Big\} \quad \text{and} \quad q_1 \le q_2. \tag{3.44}$$

(c) *If $p_1 = p_2 = p$, then the embedding* (3.1) *holds if one of the following conditions is satisfied:*
 (i) $s_1 - s_2 > \frac{d}{p}\big(1 - \frac{u_1}{u_2}\big)_+$,
 (ii) $s_1 = s_2$, $u_1 \ge u_2$ *and* $q_1 \le q_2$,

(iii) $s_1 - s_2 = \frac{d}{p}\big(1 - \frac{u_1}{u_2}\big)$, $u_1 < u_2$ *and* $q_1 \le \frac{u_1}{u_2} q_2$.

Conversely, embedding (3.1) *implies either* (i) *or* $s_1 - s_2 = \frac{d}{p}\big(1 - \frac{u_1}{u_2}\big)_+$ *with* $q_1 \le q_2$.

Finally we consider the special cases when the source or target space is of Besov type, that is, when $p_1 = u_1$ or $p_2 = u_2$.

COROLLARY 3.5. *Let* $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < p_2 < \infty$. *Then the embedding*

$$\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega) \tag{3.45}$$

holds if one of the following conditions holds:

(a) $s_1 - s_2 > \frac{d}{p_1}\big(1 - \frac{u_1}{p_2}\big)_+$,
(b) $s_1 - s_2 = 0$, $p_2 \le u_1$ *and* $q_1 \le q_2$,
(c) $s_1 - s_2 = \frac{d}{p_1}\big(1 - \frac{u_1}{p_2}\big) > 0$, $p_2 > u_1$ *and* $q_1 \le \frac{u_1}{\min(p_1,p_2)} q_2$.

Conversely, embedding (3.45) *implies either* (a) *or* $s_1 - s_2 = \frac{d}{p_1}\big(1 - \frac{u_1}{p_2}\big)_+$ *with* $q_1 \le q_2$.

REMARK 3.6. Note that the above result essentially depends on Ω being bounded since in [HS, Rem. 3.8] we found for $\Omega = \mathbb{R}^d$ that whenever $u_1 < p_1$, then there is never an embedding of type (3.45) for any choice of parameters s_2, p_2, q_2.

COROLLARY 3.7. *Let* $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < p_1 < \infty$, $0 < u_2 \le p_2 < \infty$. *Then the following conditions are equivalent:*

(a) $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$
(b) $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$
(c) *either* $s_1 - s_2 > \big(\frac{d}{p_1} - \frac{d}{p_2}\big)_+$, *or* $s_1 - s_2 = \big(\frac{d}{p_1} - \frac{d}{p_2}\big)_+$ *and* $q_1 \le q_2$.

REMARK 3.8. In contrast to Remark 3.6 the above agrees with the parallel result for $\Omega = \mathbb{R}^d$ (cf. [HS, Cor. 3.6]).

4. Compact embeddings. Now we prove sufficient and necessary conditions for the compactness of such embeddings.

THEOREM 4.1. *Let* $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < u_2 \le p_2 < \infty$. *Then the embedding*

$$\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega) \tag{4.1}$$

is compact if, and only if,

$$\frac{s_1 - s_2}{d} > \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\}. \tag{4.2}$$

Proof. Once more it is sufficient to consider the compactness of the embedding

$$\mathrm{id} : \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q_{\nu_0,0}) \to \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0}) \tag{4.3}$$

where $\sigma_1 = d/p_1$ and $\sigma_2 = s_2 - s_1 + d/p_1$.

STEP 1 (Sufficiency). The proof is standard. Let G be a bounded set in $\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q_{\nu_0,0})$. We show that for any $\varepsilon > 0$ there is an ε-net of G in $\widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0})$. We take $\gamma > 0$ such that

$$\frac{s_1 - s_2 - \gamma}{d} > \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\}, \tag{4.4}$$

and $j_0 \in \mathbb{N}$ such that $2^{-j_0\gamma} < \varepsilon/2$. Then it follows from Theorem 3.1 and (4.2) that the set $\widetilde{G} = \{2^{j\gamma}\lambda : \lambda \in G\}$ is bounded in $\widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0})$. Thus if $\lambda \in G$, then

$$\Bigg(\sum_{j=j_0}^{\infty} 2^{jq_2(\sigma_2 - \frac{d}{p_2})} \sup_{\substack{\nu:\, \nu \le j\\ k:\, Q_{\nu,k} \subset Q}} 2^{q_2 d(j-\nu)(\frac{1}{p_2} - \frac{1}{u_2})} \Big(\sum_{\substack{m:\\ Q_{j,m} \subset Q_{\nu,k}}} |\lambda_{j,m}|^{u_2}\Big)^{\frac{q_2}{u_2}}\Bigg)^{\frac{1}{q_2}} < C 2^{-j_0\gamma} < C\varepsilon/2. \tag{4.5}$$

On the other hand $\{\lambda = (\lambda_{j,m}) \in G : \lambda_{j,m} = 0 \text{ if } j \ge j_0\}$ is relatively compact in $\widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0})$ since it is a bounded subset of a finite-dimensional subspace. This and (4.5) imply that there exists an ε-net for G.

STEP 2 (Necessity). We assume that

$$\frac{s_1 - s_2}{d} = \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}, \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\right\}. \tag{4.6}$$

SUBSTEP 2.1. We assume that $(s_1 - s_2)/d = 0$. We define the sequence $\lambda^{(n)} = (\lambda^{(n)}_{j,m})_{j,m}$, $n \in \mathbb{N}_0$, by

$$\lambda^{(n)}_{j,m} = \begin{cases} 2^{-n\frac{d}{p_1}} & \text{if } j = n \text{ and } Q_{n,m} \subset Q, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Then

$$\|\lambda^{(n)} | \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| = \sup_{\substack{\nu:\, \nu \le n\\ k:\, Q_{\nu,k} \subset Q}} 2^{d(n-\nu)(\frac{1}{p_1} - \frac{1}{u_1})} \Big(\sum_{\substack{m:\\ Q_{n,m} \subset Q_{\nu,k}}} |\lambda^{(n)}_{n,m}|^{u_1}\Big)^{\frac{1}{u_1}} = 2^{d|\nu_0|\frac{1}{p_1}}$$

and

$$\sup_{\substack{\nu:\, \nu \le n\\ k:\, Q_{\nu,k} \subset Q}} 2^{d(n-\nu)(\frac{1}{p_2} - \frac{1}{u_2})} \Big(\sum_{\substack{m:\\ Q_{n,m} \subset Q_{\nu,k}}} |\lambda^{(n)}_{n,m}|^{u_2}\Big)^{\frac{1}{u_2}} = 2^{dn(\frac{1}{p_2} - \frac{1}{p_1})} 2^{d|\nu_0|\frac{1}{p_2}}.$$

Therefore the sequence $\lambda^{(n)}$ is bounded in $\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)$, but

$$\|\lambda^{(n)} - \lambda^{(\ell)} \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \geq 2^{d|\nu_0|\frac{1}{p_2}} \tag{4.8}$$

if $n \neq \ell$ so it does not contain a convergent subsequence.

Substep 2.2. We assume that $s_1 - s_2 = d/p_1 - d/p_2 = \sigma_1 - \sigma_2 > 0$. Then $\sigma_1 = d/p_1$ implies $\sigma_2 = d/p_2$. If we take

$$\lambda^{(n)}_{j,m} = \begin{cases} 1 & \text{if } j = n \text{ and } m = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.9}$$

then $\|\lambda^{(n)} \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}(Q)\| = 1$ and $\|\lambda^{(n)} - \lambda^{(\ell)} \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q)\| \geq 1$ if $n \neq \ell$.

Substep 2.3. We assume that

$$\frac{s_1 - s_2}{d} = \frac{u_1}{p_1}\left(\frac{1}{u_1} - \frac{1}{u_2}\right) > \max\left\{0, \frac{1}{p_1} - \frac{1}{p_2}\right\}.$$

Let $\nu \geq \nu_0$ and let $\lambda^{(\nu)} = \{\lambda^{(\nu)}_{j,m}\}_{j,m}$ be the sequence used in Substep 2.4 of the proof of Theorem 3.1. Now we define a new sequence $\widetilde{\lambda}^{(\nu)} = (\widetilde{\lambda}^{(\nu)}_{j,m}) \in \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}$ by

$$\widetilde{\lambda}^{(\nu)}_{j,m} = \begin{cases} \lambda^{(\nu)}_{0,m} & \text{if } j = \nu - \nu_0 \text{ and } Q_{0,m} \subset Q_{\nu,0}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

If $Q_{\mu,\ell} \subset Q_{\nu_0,0}$, then for fixed $j \geq \mu$ there are at most $k_{\mu-j}$ nonzero elements $\widetilde{\lambda}^{(\nu)}_{j,m}$ such that the corresponding cube $Q_{j,m}$ is a subset of $Q_{\mu,\ell}$.

Hence, if $j = \nu - \nu_0$, then

$$\sum_{m\colon Q_{j,m} \subset Q_{\mu,\ell}} |\widetilde{\lambda}^{(\nu)}_{j,m}|^{u_1} \leq k_{\mu-j} \leq 2^{d(j-\mu)(1-\frac{u_1}{p_1})} \tag{4.11}$$

and the last sum is equal to $k_{\mu-j}$ if $\mu = \nu_0$. Moreover,

$$\sum_{m\colon Q_{j,m} \subset Q_{\mu,\ell}} |\widetilde{\lambda}^{(\nu)}_{j,m}|^{u_1} = 0 \quad \text{if } j \neq \nu - \nu_0. \tag{4.12}$$

Thus

$$\|\widetilde{\lambda}^{(\nu)} \,|\, \widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}\| \leq C < \infty. \tag{4.13}$$

In a similar way we get, for $\mu = \nu_0$,

$$\begin{aligned} 2^{d(\nu-\nu_0)(\frac{1}{p_2}-\frac{1}{u_2})}\Bigg(\sum_{m\colon Q_{\nu,m} \subset Q_{\nu_0,0}} |\widetilde{\lambda}^{(\nu)}_{\nu-\nu_0,m}|^{u_2}\Bigg)^{1/u_2} &\geq c 2^{d(\nu-\nu_0)(\frac{1}{p_2}-\frac{u_1}{u_2 p_1})} \\ &= c 2^{\nu(\frac{d}{p_2}-\sigma_2)}, \end{aligned} \tag{4.14}$$

for some constant c independent of $\lambda^{(\nu)}$.

Consequently, if $\nu \neq \ell$, then

$$\|\widetilde{\lambda}^{(\nu)} - \widetilde{\lambda}^{(\ell)} \,|\, \widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}(Q_{\nu_0,0})\| \geq c. \tag{4.15}$$

Thus we get a bounded sequence in $\widetilde{n}^{\sigma_1}_{p_1,u_1,q_1}$ that has no convergent subsequence in $\widetilde{n}^{\sigma_2}_{p_2,u_2,q_2}$. ∎

Corollary 4.2. *Let* $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < u_1 \le p_1 < \infty$, $0 < u_2 \le p_2 < \infty$.

(a) *The embedding* $\mathcal{N}^{s_1}_{p_1,u_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ *is compact if, and only if,* $s_1 - s_2 > \frac{d}{p_1}\big(1 - \frac{u_1}{p_2}\big)_+$.

(b) *The embedding* $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{p_2,u_2,q_2}(\Omega)$ *is compact if, and only if,* $s_1 - s_2 > \frac{d}{p_1}\big(1 - \frac{p_1}{p_2}\big)_+ = d(\frac{1}{p_1} - \frac{1}{p_2})_+$, *that is, if, and only if,* $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ *is compact.*

5. Embeddings into L_p and C. Here we collect a few results on embeddings of spaces of Besov–Morrey type on $\mathbb{R}^d$, $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$, into 'classical' spaces such as $C(\mathbb{R}^d)$, the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^d$ with the sup-norm, and $L_p(\mathbb{R}^d), 0 < p \le \infty$. The analogous question will be studied for spaces on bounded domains. We start with a simple observation from our recent paper [HS].

Proposition 5.1. *Let* $0 < u < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Then* $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ *is never embedded in any* $L_r(\mathbb{R}^d)$ *for* $1 \le r < \infty$.

Proof. Assume that $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d)$ for some $1 \le r < \infty$. Then in view of the classical case,

$$B^0_{r,1}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d) \hookrightarrow B^0_{r,\infty}(\mathbb{R}^d), \quad 1 \le r \le \infty \tag{5.1}$$

(cf. [T1, Prop. 2.5.7]), this implies a continuous embedding $\mathcal{N}^s_{p,u,q}(\mathbb{R}^d) \hookrightarrow B^0_{r,\infty}(\mathbb{R}^d)$, which contradicts our result in [HS], in particular [HS, Rem. 3.8]. ∎

Remark 5.2. As is well-known, in the classical case $p = u$ one has (5.1) (with r replaced by p). According to Proposition 5.1, this feature is essentially different for the Besov–Morrey spaces when $u < p$. Moreover, it also depends on the situation of $\mathbb{R}^d$, as the next observation shows; recall also our Remark 3.6.

Proposition 5.3. *Let* $0 < u < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $1 \le r < \infty$. *Then there is a continuous embedding*

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow L_r(\Omega) \tag{5.2}$$

if

$$s > \frac{d}{p}\left(1 - \frac{u}{r}\right)_+ \tag{5.3}$$

or

$$s = \frac{d}{p}\left(1 - \frac{u}{r}\right)_+ \quad \textit{and} \quad 0 < q \le \min(\max(u, r), 2). \tag{5.4}$$

Conversely, (5.2) *implies*

$$s \ge \frac{d}{p}\left(1-\frac{u}{r}\right)_+ . \tag{5.5}$$

The embedding (5.2) *is compact if, and only if,* (5.3) *holds.*

Proof. We begin with the necessity part of the continuity and combine (the Ω-counterpart of) (5.1) with (5.2). Thus we obtain

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow B^0_{r,\infty}(\Omega)$$

and Corollary 3.5 gives (5.5). Conversely, we decompose (5.2) as

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow B^0_{r,1}(\Omega) \hookrightarrow L_r(\Omega). \tag{5.6}$$

Assuming either (5.3) or

$$\begin{cases} s = \frac{d}{p}\left(1-\frac{u}{r}\right)_+ = 0, \quad u \ge r, \quad \text{and} \quad 0 < q \le 1, \\ s = \frac{d}{p}\left(1-\frac{u}{r}\right)_+ > 0, \quad u < r, \quad \text{and} \quad 0 < q \le \frac{u}{\min(p,r)}, \end{cases} \tag{5.7}$$

Corollary 3.5 implies the continuity of the first embedding in (5.6), whereas the second is covered by (the Ω-counterpart of) (5.1). It remains to extend (5.7) to (5.4).

We first consider the case $r \le u$, that is, $s = 0$. Obviously the case $u = 1$ is already covered (note that our assumption $r \ge 1$ excludes $0 < u < 1$ in this case), so we may assume $u > 1$. Here we use some results of Sawano [S2, Props. 1.9, 1.10], and deduce for $1 < u \le p < \infty$, $0 < q \le \min(u, 2)$ that

$$\mathcal{N}^0_{p,u,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{p,u}(\mathbb{R}^d), \tag{5.8}$$

so that for bounded domains we can further conclude that

$$\mathcal{N}^0_{p,u,q}(\Omega) \hookrightarrow L_u(\Omega) \hookrightarrow L_r(\Omega) \quad \text{if } u \ge r \text{ and } 0 < q \le \min(u,2). \tag{5.9}$$

Now we consider the case $r > u$, so that $s = \frac{d}{p}\left(1-\frac{u}{r}\right)_+ > 0$. We refine our above argument and decompose (5.2) as

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow \mathcal{N}^0_{rp/u,r,\min(r,2)}(\Omega) \hookrightarrow L_r(\Omega). \tag{5.10}$$

The second embedding is covered by what we just proved since $r\frac{p}{u} > r$. As for the first one, we apply Theorem 3.1(b) with $\frac{s}{d} = \frac{1}{p}\left(1-\frac{u}{r}\right)$ and thus have to assume by (3.4) that

$$q \le \min\left(1, \max\left(1, \frac{r}{u}\right)\frac{u}{r}\right) \min(r,2) = \min(r,2).$$

This completes the sufficiency part for the embedding result (5.2).

As for the compactness, it is an immediate consequence of Corollary 4.2 and (the Ω-counterpart of) (5.1): the compactness of (5.2) implies the compactness of $\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow B^0_{r,\infty}(\Omega)$, which leads to (5.3) in view of Corollary 4.2. Conversely, the same argument implies the compactness of $\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow B^0_{r,1}(\Omega)$ when (5.3) is satisfied, which together with (5.1) ensures the compactness of (5.2). ∎

Remark 5.4. Obviously, there remains a gap between necessary and sufficient conditions for (5.2) in the above statement concerning the limiting case $s = \frac{d}{p}\big(1-\frac{u}{r}\big)_+$. In the classical situation $p = u$ the continuity of $\mathcal{N}^s_{p,p,q}(\Omega) = B^s_{p,q}(\Omega) \hookrightarrow L_r(\Omega)$ requires either (5.3) or, in the limiting case $s = d\big(\frac{1}{p}-\frac{1}{r}\big)_+$, $0 < q \le \min(p,2)$ if $s = 0$, i.e., $p \ge r$, or $0 < q \le r$ if $s = d\big(\frac{1}{p}-\frac{1}{r}\big) > 0$, that is, when $p < r$. This indicates that (5.4) might be not optimal, at least when $u < r$ and thus $s > 0$. So, finally, it seems not so clear at the moment what appropriate condition on q should be expected in the limiting case when $s = \frac{d}{p}\big(1-\frac{u}{r}\big)_+$.

But unlike in the case of $\Omega = \mathbb{R}^d$, there is always a continuous embedding (5.2) into the L_r scale for $1 \le r < \infty$, if s is chosen sufficiently large.

The situation changes completely when $r = \infty$. Here the counterpart of (5.1) reads as

$$B^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^d) \tag{5.11}$$

(cf. [T1, Prop. 2.5.7]), where $C(\mathbb{R}^d)$ can be replaced by $L_\infty(\mathbb{R}^d)$. Recall that in the classical case we have, for $0 < p \le \infty$,

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \quad \text{if, and only if,} \quad \begin{cases} 0 < q \le \infty & \text{if } s > d/p, \\ 0 < q \le 1 & \text{if } s = d/p, \end{cases}$$

where again $C(\mathbb{R}^d)$ can be replaced by $L_\infty(\mathbb{R}^d)$. For a proof we refer the reader to [ET, 2.3.3(iii)]. It turns out that the above embedding survives Morreyfication without any change.

Proposition 5.5. *Let $0 < u \le p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Then the following three conditions are equivalent:*

$$\text{(i)} \quad \mathcal{N}^s_{p,u,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d), \tag{5.12}$$

$$\text{(ii)} \quad B^s_{p,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d), \tag{5.13}$$

$$\text{(iii)} \quad \begin{cases} 0 < q \le \infty & \text{if } s > d/p, \\ 0 < q \le 1 & \text{if } s = d/p, \end{cases} \tag{5.14}$$

where $C(\mathbb{R}^d)$ can be replaced by $L_\infty(\mathbb{R}^d)$ in (5.12) *and* (5.13). *The embeddings* (5.12) *and* (5.13) *are never compact.*

Proof. As mentioned above, we take the equivalence of (ii) and (iii) for granted, including the possible substitution of $C(\mathbb{R}^d)$ by $L_\infty(\mathbb{R}^d)$ in (5.13). Moreover, since obviously $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$, we have the implication from (i) to (ii) and are thus left to prove that (iii) implies (i).

Let $f \in \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$. We may assume that $s = d/p$ and $q \le 1$, since the rest follows by elementary embeddings (see [HS, Thm. 3.3]). We have to show that $f \in C(\mathbb{R}^d)$. Let

$$f = \sum_{m\in\mathbb{Z}^d} \lambda_{0,m}\varphi_{0,m} + \sum_{i=1}^{2^d-1}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^d} \lambda_{i,\nu,m}\psi_{i,\nu,m},$$

with $(\lambda_{0,m})_m \in \ell_p$ and $(\lambda_{i,\nu,m})_{i,\nu,m} \in \bigoplus_{i=1}^{2^d-1} n^\sigma_{p,u,q}$ be the wavelet decomposition of f, $\sigma = s + d/2$ (cf. [S4] or [HS, Thm. 2.8]). For any fixed $\nu \in \mathbb{N}_0$ and $i = 1,\dots,2^d-1$, the functions

$$\eta_0(y) = \sum_{m\in\mathbb{Z}^d} \lambda_{0,m}\varphi_{0,m}(y), \qquad \eta_{i,\nu}(y) = \sum_{m\in\mathbb{Z}^d} \lambda_{i,\nu,m}\psi_{i,\nu,m}(y) \tag{5.15}$$

are continuous since the sums are locally finite. Moreover they are bounded since the wavelets $\psi_{i,\nu,m}$ are uniformly bounded by $c2^{\nu d/2}$ and the sequence $(\lambda_{i,\nu,m})_m$ is bounded by $c2^{-\nu d/2}$ if $(\lambda_{i,\nu,m})_m \in n^\sigma_{p,u,q}$ with $s = d/p$. Note that the constant c is independent of i and ν.

Let $y \in \mathbb{R}^d$ and let $Q^{(y)}_{j_0,k}$, $j_0 < 0$, be a dyadic cube such that $\operatorname{supp}\psi_{i,0,m} \subset Q^{(y)}_{j_0,k}$ if $y \in \operatorname{supp}\psi_{i,0,m}$. It should be clear by the construction of Daubechies wavelets that one can choose j_0 independent of y. By scaling properties of the multiresolution analysis one can find for any $\nu > 0$ a dyadic cube $Q^{(y)}_{j_\nu,\ell}$ such that

$$j_\nu = j_0 + \nu \quad\text{and}\quad \operatorname{supp}\psi_{i,\nu,m} \subset Q^{(y)}_{j_\nu,\ell} \quad \text{if } y \in \operatorname{supp}\psi_{i,\nu,m}. \tag{5.16}$$

Again we can choose the same j_ν for all $y \in \mathbb{R}^d$. Hence the number of dyadic unit cubes contained in $Q^{(y)}_{j_0,k}$ is the same as the number of dyadic cubes of size $2^{-\nu}$ contained in $Q^{(y)}_{j_\nu,\ell}$ and this number is independent of $y \in \mathbb{R}^d$.

We put $C_0 = c2^{dj_0(\frac{1}{p}-\frac{1}{q}+(1-\frac{1}{q})_+)}$. Then

$$\begin{aligned} |\eta_{i,\nu}(y)| &\le c2^{\nu d/2} \sum_{m:\, Q_{\nu,m}\subset Q^{(y)}_{j_\nu,\ell}} |\lambda_{i,\nu,m}| \\ &\le C_0 2^{\nu d/2} \sup_{j:\, j\le\nu;\,\ell} 2^{d(j-\nu)(\frac{1}{p}-\frac{1}{u})} \Big(\sum_{m:\, Q_{j,m}\subset Q_{\nu,\ell}} |\lambda_{i,\nu,m}|^u\Big)^{\frac{1}{u}}. \end{aligned} \tag{5.17}$$

Thus for any $n_1, n_2 \in \mathbb{N}$, $n_1 < n_2$ we have

(5.18)
$$\Big|\sum_{\nu=n_1}^{n_2} \eta_{i,\nu}(y)\Big| \le C_0 \sum_{\nu=n_1}^{n_2} 2^{\nu d/2} \sup_{j:\, j\le\nu;\,\ell} 2^{d(j-\nu)(\frac{1}{p}-\frac{1}{u})} \Big(\sum_{m:\, Q_{j,m}\subset Q_{\nu,\ell}} |\lambda_{i,\nu,m}|^u\Big)^{\frac{1}{u}}.$$

But $\lambda \in n^\sigma_{p,u,q}$ with $\sigma = d/p + d/2$ and $q \le 1$. Thus for any $\varepsilon > 0$ we get

$$\sum_{\nu=n_1}^{n_2} 2^{\nu d/2} \sup_{j\colon j\le\nu;\ell} 2^{d(j-\nu)(\frac{1}{p}-\frac{1}{u})} \Big(\sum_{m\colon Q_{j,m}\subset Q_{\nu,\ell}} |\lambda_{i,\nu,m}|^u \Big)^{\frac{1}{u}} \le \varepsilon \tag{5.19}$$

for n_1 and n_2 sufficiently large. This implies the uniform convergence of the series $\sum_{\nu=0}^{\infty} \eta_{i,\nu}$ since the estimates are uniform in $y \in \mathbb{R}^d$. The starting term η_0 is treated analogously. This leads to the continuity of f, and the inequality (5.17) implies

$$\|f \,|\, C(\mathbb{R}^d)\| \le c\|f \,|\, \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)\|.$$

So we arrive at (5.12), where the counterpart for $L_\infty(\mathbb{R}^d)$ is immediate.

As for the noncompactness, it is well-known in the case of (5.13) (recall also (5.11)). In view of the elementary embedding $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{p,u,q}(\mathbb{R}^d)$ this immediately implies the noncompactness of (5.12). ∎

Remark 5.6. A partial forerunner of Proposition 5.5 can be found in [S2, Prop. 1.11] dealing with the super-limiting case: it is shown that the first line in (5.14) implies (5.12). We can also refer to [KY, Thm. 2.5].

Corollary 5.7. *Let $0 < u \le p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Then the following three conditions are equivalent:*

(i) $\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow C(\Omega)$, (5.20)

(ii) $B^s_{p,q}(\Omega) \hookrightarrow C(\Omega)$, (5.21)

(iii) $\begin{cases} 0 < q \le \infty & \text{if } s > d/p, \\ 0 < q \le 1 & \text{if } s = d/p, \end{cases}$

where $C(\Omega)$ can be replaced by $L_\infty(\Omega)$ in (5.20) *and* (5.21)*. The embeddings* (5.20) *and* (5.21) *are compact if, and only if, $s > d/p$.*

Proof. The result about the continuity of (5.20) and (5.21) is an immediate consequence of the definition of the spaces by restriction together with Proposition 5.5. Again, the equivalence of (ii) and (iii) was already known, as was the compactness of (5.21) if, and only if, $s > d/p$ (see [ET, Sects. 3.3, 3.4]). Moreover, in view of $B^s_{p,q}(\Omega) \hookrightarrow \mathcal{N}^s_{p,u,q}(\Omega)$ this implies the necessity of that condition for the compactness of (5.20). It remains to deal with the sufficiency of the compactness in the case of $u < p$. Let $s > d/p$ and choose consecutively parameters σ, ϱ and r such that

$$0 < \frac{d}{\varrho} < \min\left(\frac{s-d/p}{1-u/p}, \frac{d}{u}\right), \quad \sigma = s - \frac{d}{p}\left(1 - \frac{u}{\varrho}\right), \quad r \ge \frac{q}{u}\min(p,\varrho).$$

Then by Corollary 3.5(c),

$$\mathcal{N}^s_{p,u,q}(\Omega) \hookrightarrow B^\sigma_{\varrho,r}(\Omega) \hookrightarrow C(\Omega),$$

and the latter embedding is compact since $\sigma > d/\varrho$. ∎

REMARK 5.8. Note that Corollaries 3.5 and 4.2 did not cover the above result immediately since we deal with the case $p_2 = \infty$ now. The only related result known to us can be found in the papers [D2, D3] dealing with Sobolev–Morrey spaces and some special set Ω. As in its counterpart for (usual) Sobolev spaces on domains, there is no embedding of type (5.20) in the limiting case of (5.14), that is, when $s = d/p$ recall $p > 1$ now (unless $d = 1$). According to [D2, D3] there is a continuous embedding in the sense of (5.20) (with Besov spaces replaced by spaces of Sobolev–Morrey type) if, and only if, $s > d/p$.

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