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On the algebra of smooth operators

by

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Abstract. Let s be the space of rapidly decreasing sequences. We give the spectral representation of normal elements in the Fréchet algebra $L(s', s)$ of so-called smooth operators. We also characterize closed commutative *-subalgebras of $L(s', s)$ and establish a Hölder continuous functional calculus in this algebra. The key tool is the property (DN) of s.

1. Introduction. The space s of rapidly decreasing sequences plays a significant role in the structure theory of nuclear Fréchet spaces. One of the most explicit examples of this is provided by the Kōmura–Kōmura theorem which implies that a Fréchet space is nuclear if and only if it is isomorphic to some closed subspace of $s^{\mathbb{N}}$ (see e.g. [12, Cor. 29.9]). The space s has also many interesting representations. For instance, it is isomorphic as a Fréchet space to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions, the space $\mathcal{D}(K)$ of test functions with support in a compact set $K \subset \mathbb{R}^n$ such that $\mathrm{int}(K) \neq \emptyset$, the space $C^\infty(M)$ of smooth functions on a compact smooth manifold M, the space $C^\infty[0,1]$ of smooth functions on the interval $[0,1]$. Finally, the space s and all of the spaces above are Fréchet commutative algebras with pointwise multiplication. However, these algebras do not have to be isomorphic as algebras (for instance, s and $C^\infty[0,1]$ with pointwise multiplication are not isomorphic as algebras).

A natural candidate for the "noncommutative s" is the algebra $L(s', s)$ of so-called smooth operators, where multiplication is just the composition of operators (note that $s \subseteq s'$ continuously). It appears in K-theory for Fréchet algebras ([13, Def. 2.1], [1, Ex. 2.12], [6, p. 144], [10]) and in C^*-dynamical systems ([8, Ex. 2.6]). The algebra $L(s', s)$ is also an example of a dense

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smooth subalgebra of a C^*-algebra (namely, it is a dense subalgebra of the C^*-algebra $K(\ell_2)$ of compact operators on ℓ_2) which is especially important in noncommutative geometry (see [1], [2], [4, pp. 23, 183–184]). From the philosophical point of view C^*-algebras just correspond to analogues of topological spaces whereas some of their dense smooth subalgebras play the role of smooth structures.

Representations of s may lead to representations of the algebra $L(s', s)$. Many of them are collected in [7, Th. 2.1]. For example, $L(s', s)$ is isomorphic as a Fréchet *-algebra to the following *-algebras of continuous linear operators with appropriate multiplication and involution: $L(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, $L(\mathcal{E}'(M), C^\infty(M))$, $L(\mathcal{E}'[0,1], C^\infty[0,1])$, where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions, $M \subset \mathbb{R}^n$ is a compact smooth manifold, $\mathcal{E}'(M)$ is the space of distributions on M, and $\mathcal{E}'[0,1]$ is the space of distributions with support in $[0,1]$. Two extra representations of $L(s', s)$ are also worth mentioning: the algebra of rapidly decreasing matrices

$$\mathcal{K} := \Big\{(\xi_{j,k})_{j,k\in\mathbb{N}} : \sup_{j,k\in\mathbb{N}} |\xi_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0\Big\}$$

with matrix multiplication and conjugation of the transpose as involution (see e.g. [4, p. 238], [13, Def. 2.1]), and also the algebra $\mathcal{S}(\mathbb{R}^2)$ equipped with the Volterra convolution $(f \cdot g)(x, y) := \int_{\mathbb{R}} f(x, z) g(z, y)\, dz$ and the involution $f^*(x, y) := \overline{f(y, x)}$ (see e.g. [1, Ex. 2.12]).

The purpose of this paper is to present some spectral, algebra and functional calculus properties of the algebra of smooth operators. The results are derived from the basic theory of nuclear Fréchet spaces and the theory of bounded operators on a separable Hilbert space. The heart of the paper is the theorem on the spectral representation of normal elements in $L(s', s)$ (Theorem 3.1). In the proof we use the fact that the operator norm $\|\cdot\|_{\ell_2\to\ell_2}$ is a dominating norm on $L(s', s)$ (Proposition 3.2). As a by-product we obtain a kind of spectral description of normal elements of $L(s', s)$ among those of $K(\ell_2)$ (Corollary 3.6). Next, we characterize closed commutative *-subalgebras of $L(s', s)$. We prove that every such subalgebra is generated by a single operator and also by its spectral projections (Theorem 4.8), and moreover that it is a Köthe algebra with pointwise multiplication. To do this, we show that every closed commutative *-subalgebra of $L(s', s)$ has a canonical Schauder basis (Lemma 4.4). Finally, we establish a Hölder-continuous functional calculus in $L(s', s)$ (Corollary 5.1) and we prove the functional calculus theorem for normal elements in this algebra (Theorem 5.2).

By a *Fréchet space* we mean a complete metrizable locally convex space. A *Fréchet algebra* is a Fréchet space which is an algebra with continuous multiplication. A *Fréchet *-algebra* is a Fréchet algebra with an involution.

We use the standard notation and terminology. All the notions from functional analysis are explained in [12] and those from topological algebras in [9] or [17].

2. Preliminaries. Throughout the paper, $\mathbb{N}$ will denote the set of natural numbers $\{1,2,\ldots\}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

By *projection* on ℓ_2 we always mean a continuous orthogonal (i.e., self-adjoint) projection.

We define the *space of rapidly decreasing sequences* as the Fréchet space

$$s := \Big\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \forall q\in\mathbb{N}_0\ \ |\xi|_q := \Big(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2q}\Big)^{1/2} < \infty\Big\}$$

with the topology corresponding to the system $(|\cdot|_q)_{q\in\mathbb{N}_0}$ of norms. Its strong dual is isomorphic to the *space of slowly increasing sequences*

$$s' := \Big\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \exists q\in\mathbb{N}_0\ \ |\xi|_q' := \Big(\sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q}\Big)^{1/2} < \infty\Big\}$$

equipped with the inductive limit topology given by the system $(|\cdot|_q')_{q\in\mathbb{N}_0}$ of norms.

Every $\eta\in s'$ corresponds to the continuous functional $\xi\mapsto\langle\xi,\eta\rangle$ on s, where

$$\langle\xi,\eta\rangle := \sum_{j=1}^{\infty} \xi_j\overline{\eta_j}.$$

Furthermore, by the Cauchy–Schwarz inequality we get

$$|\langle\xi,\eta\rangle| \le |\xi|_q|\eta|_q'$$

for all $q\in\mathbb{N}_0$, $\xi\in s$ and $\eta\in s'$ with $|\eta|_q' < \infty$.

For $1\le p<\infty$ and a Köthe matrix $(a_{j,q})_{j\in\mathbb{N},\,q\in\mathbb{N}_0}$ we define the Köthe space

$$\lambda^p(a_{j,q}) := \Big\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \forall q\in\mathbb{N}_0\ \ |\xi|_{p,q} := \Big(\sum_{j=1}^{\infty} |\xi_j a_{j,q}|^p\Big)^{1/p} < \infty\Big\}$$

and for $p=\infty$,

$$\lambda^\infty(a_{j,q}) := \Big\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \forall q\in\mathbb{N}_0\ \ |\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| a_{j,q} < \infty\Big\}$$

with the topology generated by the norms $(|\cdot|_{p,q})_{q\in\mathbb{N}_0}$ (see e.g. [12, Def. p. 326]). Note that these spaces are sometimes Fréchet *-algebras with pointwise multiplication.

Now, s is just the Köthe space $\lambda^2(j^q)$. Moreover, since s is a nuclear Fréchet space, it is isomorphic to any Köthe space $\lambda^p(j^q)$ for $1\le p\le\infty$

(see e.g. [12, Prop. 28.16, Ex. 29.4 (1)]). We use ℓ_2-norms to simplify further computations, for example we have $|\cdot|_0 = |\cdot|_0' = \|\cdot\|_{\ell_2}$.

It is well known that the space $L(s', s)$ of continuous linear operators from s' to s with the fundamental system of norms $(\|\cdot\|_q)_{q\in\mathbb{N}_0}$,

$$\|x\|_q := \sup_{|\xi|_q' \le 1} |x\xi|_q,$$

is isomorphic to s as a Fréchet space. Moreover, $L(s', s)$ is isomorphic to $s \mathbin{\widehat{\otimes}} s$, the completed tensor product of s (see [11, §41.7 (5)]).

The canonical inclusion $j : s \hookrightarrow s'$ is continuous. Hence, for $x, y \in L(s', s)$,

$$x \cdot y := x \circ j \circ y$$

is in $L(s', s)$ as well and with this operation $L(s', s)$ is a Fréchet algebra.

The diagram

$$\ell_2 \hookrightarrow s' \to s \hookrightarrow \ell_2$$

defines the canonical (continuous) embedding of the algebra $L(s', s)$ in the algebra $L(\ell_2)$ of continuous linear operators on the Hilbert space ℓ_2. In fact, this inclusion acts into the space $K(\ell_2)$ of compact operators on ℓ_2, and the sequence of singular numbers of elements in $L(s', s)$ belongs to s (see [7, Prop. 3.1, Cor. 3.2]). Therefore, $L(s', s)$ can be regarded as some class of compact operators on ℓ_2. Clearly, multiplication in $L(s', s)$ coincides with composition in $L(\ell_2)$, and further $L(s', s)$ is invariant under the hilbertian involution $x \mapsto x^*$.

To see this, consider the Fréchet *-algebra of rapidly decreasing matrices

$$\mathcal{K} := \Big\{ \varXi = (\xi_{j,k})_{j,k\in\mathbb{N}} : |||\varXi|||_q := \sup_{j,k\in\mathbb{N}} |\xi_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0 \Big\}$$

with matrix multiplication, with involution defined by $((\xi_{j,k})_{j,k\in\mathbb{N}})^* := (\overline{\xi_{k,j}})_{j,k\in\mathbb{N}}$ and with $(|||\cdot|||_q)_{q\in\mathbb{N}_0}$ as its fundamental sequence of norms. By [7, Th. 2.1], $\varPhi\colon L(s', s) \to \mathcal{K}$, $\varPhi(x) := (\langle xe_k, e_j\rangle)_{j,k\in\mathbb{N}}$, is an algebra isomorphism and we have

$$\varPhi(x)^* = \big(\overline{\langle xe_j, e_k\rangle}\big)_{j,k\in\mathbb{N}} = (\langle x^*e_k, e_j\rangle)_{j,k\in\mathbb{N}}.$$

Hence, $x^* = \varPhi^{-1}(\varPhi(x)^*) \in L(s', s)$ and $\varPhi$ is even a *-isomorphism. Clearly, for every matrix $\varXi \in \mathcal{K}$ and $q \in \mathbb{N}_0$, $|||\varXi^*|||_q = |||\varXi|||_q$, thus the hilbertian involution is continuous on $L(s', s)$.

The Fréchet algebra $L(s', s)$ with the involution * is called the *algebra of smooth operators*. We will also consider the algebra with unit

$$\widetilde{L(s', s)} := \{x + \lambda\mathbf{1} : x \in L(s', s),\ \lambda \in \mathbb{C}\},$$

where $\mathbf{1}$ is the identity operator on ℓ_2. We endow the algebra $\widetilde{L(s', s)}$ with the product topology.

Now, we shall recall some basic spectral properties of the algebra $L(s',s)$. For the sake of convenience, we state the following definition.

DEFINITION 2.1. We say that a sequence $(\lambda_n)_{n\in\mathbb{N}}\subset\mathbb{C}$ is a *sequence of eigenvalues* of an infinite-dimensional compact operator x on ℓ_2 if it satisfies the following conditions:

(i) $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of eigenvalues of x without zero;
(ii) $|\lambda_1|\geq|\lambda_2|\geq\cdots>0$ and if two eigenvalues have the same absolute value, then we can order them in an arbitrary way;
(iii) the number of occurrences of the eigenvalue λ_n is equal to its geometric multiplicity (i.e., the dimension of the space $\ker(\lambda_n\mathbf{1}-x)$).

Proposition 2.3 below is well known (see e.g. [10]) and it is a simple consequence of Proposition 2.2. However, Propositions 2.2 and 2.3 also follow from [3, Prop. A.2.8]. Straightforward proofs of Propositions 2.2 and 2.4 can be found in [7, Th. 3.3, Cor. 3.5].

PROPOSITION 2.2. *An operator in* $\widetilde{L(s',s)}$ *is invertible if and only if it is invertible in* $L(\ell_2)$.

PROPOSITION 2.3. *The algebra* $\widetilde{L(s',s)}$ *is a Q-algebra, i.e., the set of invertible elements is open.*

PROPOSITION 2.4. *The spectrum of x in $L(s',s)$ equals the spectrum of x in $L(\ell_2)$ and it consists of zero and the set of eigenvalues. If moreover x is infinite-dimensional, then the sequence of eigenvalues of x (see Definition 2.1) belongs to s.*

The first part of the following proposition is also known (see e.g. [13, Lemma 2.2]). We give a simple proof that the norms $\|\cdot\|_q$ are submultiplicative.

PROPOSITION 2.5. *The algebra $L(s',s)$ is locally m-convex, i.e., it has a fundamental system of submultiplicative norms. Moreover, $\|xy\|_q\leq\|x\|_q\|y\|_q$ for every $q\in\mathbb{N}_0$.*

Proof. Let $x,y\in L(s',s)$ and let B_q, B'_q denote the closed unit ball for the norms $|\cdot|_q$, $|\cdot|'_q$, respectively. Clearly, $y(B'_q)\subseteq\|y\|_qB_q$ and $B_q\subseteq B'_q$. Hence

$$\begin{aligned}\|xy\|_q&=\sup_{|\xi|'_q\leq1}|x(y(\xi))|_q=\sup_{\eta\in y(B'_q)}|x(\eta)|_q\leq\sup_{\eta\in\|y\|_qB_q}|x(\eta)|_q\\&=\|y\|_q\sup_{\eta\in B_q}|x(\eta)|_q\leq\|y\|_q\sup_{\eta\in B'_q}|x(\eta)|_q=\|x\|_q\|y\|_q.\ \blacksquare\end{aligned}$$

3. Spectral representation. In this section we prove the following theorem on the spectral representation of normal elements in $L(s',s)$.

Theorem 3.1. *Every infinite-dimensional normal operator x in $L(s', s)$ has a unique spectral representation $x = \sum_{n=1}^{\infty} \lambda_n P_n$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a decreasing (in modulus) sequence in s of nonzero pairwise different elements, $(P_n)_{n\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections belonging to $L(s', s)$ (i.e., the canonical inclusion of P_n into $L(\ell_2)$ is a projection onto ℓ_2) and the series converges absolutely in $L(s', s)$. Moreover, $(|\lambda_n|^{\theta} \|P_n\|_q)_{n\in\mathbb{N}} \in s$ for all $q \in \mathbb{N}_0$ and all $\theta \in (0, 1]$.*

To prove this result, we need some preparation. Recall (see [12, Def. on p. 359 and Lemma 29.10]) that a Fréchet space $(X, (\|\cdot\|_q)_{q\in\mathbb{N}_0})$ has the *property* (DN) if there is a continuous norm $\|\cdot\|$ on X such that for any $q \in \mathbb{N}_0$ and $\theta \in (0, 1)$ there are $r \in \mathbb{N}_0$ and $C > 0$ such that for all $x \in X$,

$$\|x\|_q \le C\|x\|^{1-\theta}\|x\|_r^{\theta}.$$

The norm $\|\cdot\|$ is called a *dominating norm.*

The following result is closely related to the result of K. Piszczek [14, Th. 4]. For convenience, we give a more straightforward proof.

Proposition 3.2. *The norm $\|\cdot\|_{\ell_2\to\ell_2}$ is a dominating norm on $L(s', s)$.*

Proof. Clearly, $\|\cdot\|_{\ell_2\to\ell_2} = \|\cdot\|_0$. By [16, Th. 4.3] (see the proof), the conclusion is equivalent to the condition

$$\forall q, \theta > 0\ \exists r, C > 0\ \forall h > 0 \qquad \|\cdot\|_q \le C\Big(h^{\theta}\|\cdot\|_r + \frac{1}{h}\|\cdot\|_0\Big).$$

From Hölder's inequality, the norm $|\cdot|_0$ is a dominating norm on s. Hence, again by [16, Th. 4.3], we get

$$\forall q, \eta > 0\ \exists r, D_0 > 0\ \forall k > 0 \qquad |\cdot|_q \le D_0\Big(k^{\eta}|\cdot|_r + \frac{1}{k}|\cdot|_0\Big).$$

Now, by the bipolar theorem (see e.g. [12, Th. 22.13]), we obtain (following the proof of [12, Lemma 29.13]) an equivalent condition

$$\forall q, \eta > 0\ \exists r, D > 0\ \forall k > 0 \qquad U_q^{\circ} \subset D\Big(k^{\eta}U_r^{\circ} + \frac{1}{k}U_0^{\circ}\Big), \tag{3.1}$$

where $U_q := \{\xi \in s : |\xi|_q \le 1\}$ and U_q° is its polar. If $\theta > 0$ and $h \in (0, 1]$ are given, we define $\eta := 2\theta + 1$ and $k := \sqrt{h}$. Since $k^{2\eta} \le k^{\eta-1}$, we obtain

$$\begin{aligned} U_q^{\circ} \otimes U_q^{\circ} := \{x \otimes y : x, y \in U_q^{\circ}\} &\subset D\Big(k^{\eta}U_r^{\circ} + \frac{1}{k}U_0^{\circ}\Big) \otimes D\Big(k^{\eta}U_r^{\circ} + \frac{1}{k}U_0^{\circ}\Big) \\ &\subset D^2\Big(k^{2\eta}U_r^{\circ} \otimes U_r^{\circ} + 2k^{\eta-1}U_r^{\circ} \otimes U_r^{\circ} + \frac{1}{k^2}U_0^{\circ} \otimes U_0^{\circ}\Big) \end{aligned}$$

$$\subset 3D^2\Big(k^{\eta-1}U_r^\circ\otimes U_r^\circ+\frac{1}{k^2}U_0^\circ\otimes U_0^\circ\Big)=3D^2\Big(h^\theta U_r^\circ\otimes U_r^\circ+\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big).$$

Since r and D in condition (3.1) can be chosen so that $q\le r$ and $D\ge 1$, we obtain

$$U_q^\circ\otimes U_q^\circ\subset U_r^\circ\otimes U_r^\circ\subset 3D^2\Big(h^\theta U_r^\circ\otimes U_r^\circ+\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big)$$

for $h>1$, whence

$$\forall q,\theta>0\ \exists r,C>0\ \forall h>0\quad U_q^\circ\otimes U_q^\circ\subset C\Big(h^\theta U_r^\circ\otimes U_r^\circ+\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big).$$

Therefore,

$$\begin{aligned}\sup_{z\in U_q^\circ\otimes U_q^\circ}|z(x)|&\le C\sup\Big\{|z(x)|:z\in h^\theta U_r^\circ\otimes U_r^\circ+\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big\}\\&=C\sup\Big\{|(z'+z'')(x)|:z'\in h^\theta U_r^\circ\otimes U_r^\circ,\ z''\in\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big\}\\&\le C\sup\Big\{|z'(x)|+|z''(x)|:z'\in h^\theta U_r^\circ\otimes U_r^\circ,\ z''\in\frac{1}{h}U_0^\circ\otimes U_0^\circ\Big\}\\&=C\Big(h^\theta\sup_{z\in U_r^\circ\otimes U_r^\circ}|z(x)|+\frac{1}{h}\sup_{z\in U_0^\circ\otimes U_0^\circ}|z(x)|\Big)\end{aligned}$$

for all $x:=\sum_{j=1}^n x_j\otimes y_j\in s\otimes s$.

Let $\chi\colon s\otimes s\to L(s',s)$, $\chi(\sum_{j=1}^n x_j\otimes y_j)(z):=\sum_{j=1}^n z(y_j)x_j$. We have, for all $p\in\mathbb{N}_0$,

$$\begin{aligned}&\sup_{z\in U_p^\circ\otimes U_p^\circ}\Big|z\Big(\sum_{j=1}^n x_j\otimes y_j\Big)\Big|=\sup\Big\{\Big|\sum_{j=1}^n z_1(x_j)z_2(y_j)\Big|:z_1,z_2\in U_p^\circ\Big\}\\&=\sup\Big\{\Big|z_1\Big(\sum_{j=1}^n z_2(y_j)x_j\Big)\Big|:z_1,z_2\in U_p^\circ\Big\}=\sup\Big\{\Big|\sum_{j=1}^n z(y_j)x_j\Big|_p:z\in U_p^\circ\Big\}\\&=\sup\Big\{\Big|\chi\Big(\sum_{j=1}^n x_j\otimes y_j\Big)(z)\Big|_p:z\in U_p^\circ\Big\}=\Big\|\chi\Big(\sum_{j=1}^n x_j\otimes y_j\Big)\Big\|_p.\end{aligned}$$

Hence

$$\Big\|\chi\Big(\sum_{j=1}^n x_j\otimes y_j\Big)\Big\|_q\le C\Big(h^\theta\Big\|\chi\Big(\sum_{j=1}^n x_j\otimes y_j\Big)\Big\|_r+\frac{1}{h}\Big\|\chi\Big(\sum_{j=1}^n x_j\otimes y_j\Big)\Big\|_0\Big).$$

Finally, since the set $\{\chi(\sum_{j=1}^n x_j\otimes y_j):x_j,y_j\in s,\ n\in\mathbb{N}\}$ is dense in $L(s',s)$, we obtain

$$\|x\|_q\le C\Big(h^\theta\|x\|_r+\frac{1}{h}\|x\|_0\Big)$$

for all $x\in L(s',s)$. ∎

Lemma 3.3. *Let $(E, (\|\cdot\|_q)_{q\in\mathbb{N}_0})$ be a Fréchet space with the property* (DN) *and let $\|\cdot\|_p$ be a dominating norm. If $(x_n)_{n\in\mathbb{N}} \subset E$ and $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ satisfy the conditions*

(i) $\sup_{n\in\mathbb{N}} \|x_n\|_p < \infty$,
(ii) $\forall q \in \mathbb{N}_0\ \sup_{n\in\mathbb{N}} |\lambda_n|\,\|x_n\|_q < \infty$,

then

$$\forall q \in \mathbb{N}_0\ \forall \theta \in (0,1] \qquad \sup_{n\in\mathbb{N}} |\lambda_n|^\theta \|x_n\|_q < \infty.$$

Moreover, for any other sequence $(y_n)_{n\in\mathbb{N}} \subset E$ satisfying conditions (i) *and* (ii) *we have*

$$\forall q \in \mathbb{N}_0\ \forall q' \in \mathbb{N}_0\ \forall \theta \in (0,1] \qquad \sup_{n\in\mathbb{N}} |\lambda_n|^\theta \|x_n\|_q \|y_n\|_{q'} < \infty.$$

Proof. Fix $q \in \mathbb{N}_0$ and $\theta \in (0,1)$. Since $\|\cdot\|_p$ is a dominating norm on E, we obtain, for some $C > 0$ and $r \in \mathbb{N}_0$,

$$\|x_n\|_q \le C\|x_n\|_p^{1-\theta}\|x_n\|_r^\theta \tag{3.2}$$

for all $n \in \mathbb{N}$. Let $C_1 := \sup_{n\in\mathbb{N}} \|x_n\|_p < \infty$, $C_2 := \sup_{n\in\mathbb{N}} |\lambda_n|\,\|x_n\|_q < \infty$. Then by (3.2),

$$|\lambda_n|^\theta \|x_n\|_q \le C\|x_n\|_p^{1-\theta}(|\lambda_n|\,\|x_n\|_r)^\theta \le CC_1^{1-\theta}C_2^\theta =: C_3,$$

where C_3 does not depend on n.

To prove the second assertion we also fix $q' \in \mathbb{N}_0$ and let $(y_n)_{n\in\mathbb{N}} \subset E$ satisfy conditions (i) and (ii). We have

$$|\lambda_n|^\theta \|x_n\|_q \|y_n\|_{q'} = (|\lambda_n|^{\theta/2}\|x_n\|_q)(|\lambda_n|^{\theta/2}\|y_n\|_{q'})$$

and from the first part of the proof,

$$\sup_{n\in\mathbb{N}} |\lambda_n|^{\theta/2}\|x_n\|_q < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} |\lambda_n|^{\theta/2}\|y_n\|_{q'} < \infty,$$

so we are done. ∎

Proposition 3.4. *Let $\mathcal{N}$ be a finite set or $\mathbb{N}$. If $(P_n)_{n\in\mathcal{N}}$ is a sequence of pairwise orthogonal finite-dimensional projections on ℓ_2, $(\lambda_n)_{n\in\mathcal{N}} \subset \mathbb{C}\setminus\{0\}$ and $x := \sum_{n\in\mathcal{N}} \lambda_n P_n \in L(s', s)$* (*the series converging in the norm $\|\cdot\|_{\ell_2\to\ell_2}$*), *then $(P_n)_{n\in\mathcal{N}} \subset L(s', s)$.*

Proof. Since $P_n = \frac{1}{\lambda_n} x \circ P_n$, it follows that $P_n\colon \ell_2 \to s$. On the other hand, $P_n = P_n \circ \frac{1}{\lambda_n} x$, so P_n extends to $P_n\colon s' \to \ell_2$. Hence $P_n = P_n \circ P_n\colon s' \to s$. ∎

Lemma 3.5. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a decreasing* (*in modulus*) *sequence of nonzero complex numbers and let $(P_n)_{n\in\mathbb{N}}$ be a sequence of nonzero pairwise orthogonal finite-dimensional projections on ℓ_2. Moreover, assume that the series $\sum_{n=1}^\infty \lambda_n P_n$ converges in the norm $\|\cdot\|_{\ell_2\to\ell_2}$ and its limit belongs to $L(s', s)$. Then $(\lambda_n)_{n\in\mathbb{N}} \in s$, $(P_n)_{n\in\mathbb{N}} \subset L(s', s)$ and the series converges* *absolutely in $L(s',s)$. Moreover, $(|\lambda_n|^\theta\|P_n\|_q)_{n\in\mathbb{N}}\in s$ for all $q\in\mathbb{N}_0$ and $\theta\in(0,1]$.*

Proof. By Proposition 2.4, the sequence of eigenvalues of the operator $x:=\sum_{n=1}^\infty\lambda_nP_n$ belongs to s. Clearly, λ_n is an eigenvalue of $\sum_{n=1}^\infty\lambda_nP_n$ and the number of its occurrences is less than or equal to the geometric multiplicity, so $(\lambda_n)_{n\in\mathbb{N}}$ is, likewise, in s.

By Proposition 3.4, $P_n\in L(s',s)$. We will show that $(|\lambda_n|^\theta\|P_n\|_q)_{n\in\mathbb{N}}\in s$ for all $q\in\mathbb{N}_0$ and $\theta\in(0,1]$, which implies that the series $\sum_{n=1}^\infty\lambda_nP_n$ converges absolutely in $L(s',s)$. Consider the operator $T_x\colon L(\ell_2)\to L(s',s)$ which sends $z\in L(\ell_2)$ to the following composition (in $L(s',s)$):

$$s'\xrightarrow{x}s\hookrightarrow\ell_2\xrightarrow{z}\ell_2\hookrightarrow s'\xrightarrow{x}s.$$

By the closed graph theorem for Fréchet spaces (see e.g. [12, Th. 24.31]), T_x is continuous and since the sequence $(P_n)_{n\in\mathbb{N}}$ is bounded in $L(\ell_2)$, the sequence $(\lambda_n^2P_n)_{n\in\mathbb{N}}=(T_xP_n)_{n\in\mathbb{N}}$ is bounded in $L(s',s)$, i.e.,

$$\sup_{n\in\mathbb{N}}|\lambda_n|^2\|P_n\|_q<\infty \tag{3.3}$$

for all $q\in\mathbb{N}_0$.

Let $(e_n)_{n\in\mathbb{N}}$ be the canonical orthonormal basis in ℓ_2 and let $E_n\colon s'\to s$,

$$E_n\xi:=\xi_ne_n,$$

for $\xi=(\xi_n)_{n\in\mathbb{N}}\in s'$ and $n\in\mathbb{N}$. Clearly, each E_n is a projection in $L(s',s)$. Moreover,

$$\|E_n\|_q=\sup_{|\xi|'_q\le1}|E_n\xi|_q=\sup_{|\xi|'_q\le1}|\xi_ne_n|_q=\sup_{|\xi|'_q\le1}|\xi_n|\cdot|e_n|_q=n^q\cdot n^q=n^{2q}. \tag{3.4}$$

Since $(\lambda_n)_{n\in\mathbb{N}}\in s$, we have

$$\sup_{n\in\mathbb{N}}|\lambda_n|^2\|E_n\|_q<\infty \tag{3.5}$$

for $q\in\mathbb{N}_0$.

By Proposition 3.2, $\|\cdot\|_{\ell_2\to\ell_2}$ is a dominating norm on $L(s',s)$, and of course $\|P_n\|_{\ell_2\to\ell_2}=\|E_n\|_{\ell_2\to\ell_2}=1$ for $n\in\mathbb{N}$. Thus, from (3.3)–(3.5) and Lemma 3.3 (applied to the sequences $(\lambda_n^2)_{n\in\mathbb{N}}$, $(P_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$) we get

$$\sup_{n\in\mathbb{N}}|\lambda_n|^{2\theta}\|P_n\|_qn^{2q'}=\sup_{n\in\mathbb{N}}|\lambda_n|^{2\theta}\|P_n\|_q\|E_n\|_{q'}<\infty$$

for all $\theta\in(0,1]$ and $q,q'\in\mathbb{N}_0$. Hence, $(|\lambda_n|^\theta\|P_n\|_q)\in s$ for all $q\in\mathbb{N}_0$ and $\theta\in(0,1]$. ∎

Now, it is not hard to prove the main theorem of this section.

Proof of Theorem 3.1. Let x be a normal infinite-dimensional operator in $L(s',s)$. The operator x (as an operator on ℓ_2) is compact (see [7, Prop. 3.1]), thus by the spectral theorem for normal compact operators (see e.g.

[5, Th. 7.6]), $x=\sum_{n=1}^{\infty}\lambda_n P_n$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a decreasing null sequence of nonzero pairwise different elements, $(P_n)_{n\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections and the series converges in the norm $\|\cdot\|_{\ell_2\to\ell_2}$. Now, the conclusion follows by Lemma 3.5. ∎

As a corollary, we get a characterization of normal operators in $L(s',s)$ among compact operators on ℓ_2.

Corollary 3.6. *Let x be a compact infinite-dimensional normal operator on ℓ_2 with spectral representation $x=\sum_{n=1}^{\infty}\lambda_n P_n$ (the series converges in norm $\|\cdot\|_{\ell_2\to\ell_2}$). Then the following assertions are equivalent:*

(i) $x\in L(s',s)$ *(as an operator on ℓ_2);*
(ii) $P_n\in L(s',s)$ *for* $n\in\mathbb{N}$ *and* $(|\lambda_n|^{\theta}\|P_n\|_q)_{n\in\mathbb{N}}\in s$ *for all* $q\in\mathbb{N}_0$ *and every* $\theta\in(0,1]$*;*
(iii) $P_n\in L(s',s)$ *for* $n\in\mathbb{N}$, $(\lambda_n)_{n\in\mathbb{N}}\in s$ *and* $\sup_{n\in\mathbb{N}}|\lambda_n|\,\|P_n\|_q<\infty$ *for all* $q\in\mathbb{N}_0$*;*
(iv) $P_n\in L(s',s)$ *for* $n\in\mathbb{N}$ *and* $\sum_{n=1}^{\infty}|\lambda_n|\,\|P_n\|_q<\infty$ *for all* $q\in\mathbb{N}_0$.

Moreover, if $x=\sum_{n=1}^{N}\lambda_n P_n$ is a finite-dimensional operator on ℓ_2, then $x\in L(s',s)$ if and only if $P_n\in L(s',s)$ for $n=1,\dots,N$.

Proof. The implication (i)⇒(ii) follows directly from Theorem 3.1. The implications (ii)⇒(iii) and (iv)⇒(i) are obvious.

(iii)⇒(iv). We have

$$\sum_{n=1}^{\infty}|\lambda_n|\,\|P_n\|_q\le\sup_{n\in\mathbb{N}}|\lambda_n|^{1/2}\|P_n\|_q\cdot\sum_{n=1}^{\infty}|\lambda_n|^{1/2}<\infty,$$

because, by Lemma 3.3, $\sup_{n\in\mathbb{N}}|\lambda_n|^{1/2}\|P_n\|_q<\infty$ and $s\subset\bigcap_{p>0}\ell_p$.

The finite case is an immediate consequence of Proposition 3.4. ∎

4. Closed commutative *-subalgebras. The aim of this section is to describe all closed commutative *-subalgebras of $L(s',s)$ (see Theorem 4.8) and to identify maximal ones among them (see Theorem 4.10).

We will need the following lemma.

Lemma 4.1. *Let A be a subalgebra of the algebra $\widetilde{A}$ over $\mathbb{C}$. Let $N\in\mathbb{N}$, $a_1,\dots,a_N\in\widetilde{A}$, $\lambda_1,\dots,\lambda_N\in\mathbb{C}$, $a_j\neq0$, $a_j^2=a_j$, $a_ja_k=0$ for $j\neq k$, $\lambda_j\neq0$ and $\lambda_j\neq\lambda_k$ for $j\neq k$. Then $a_1,\dots,a_N\in A$ whenever $\lambda_1a_1+\dots+\lambda_Na_N\in A$.*

Proof. We use induction on N. The case $N=1$ is trivial.

Assume that the conclusion holds for all $M<N$. Let $a:=\lambda_1a_1+\dots+\lambda_Na_N\in A$. We have

$$\lambda_1^2a_1+\dots+\lambda_N^2a_N=a^2\in A,$$

and, on the other hand,

$$\lambda_N\lambda_1 a_1 + \cdots + \lambda_N^2 a_N = \lambda_N a \in A,$$

so

$$(\lambda_1^2 - \lambda_N\lambda_1)a_1 + \cdots + (\lambda_{N-1}^2 - \lambda_N\lambda_{N-1})a_{N-1} = a^2 - \lambda_N a \in A.$$

Since $\lambda_j \neq 0$ and $\lambda_j \neq \lambda_N$ for $j \in \{1, \ldots, N-1\}$, we have $\lambda_j^2 - \lambda_N\lambda_j = \lambda_j(\lambda_j - \lambda_N) \neq 0$ for $j \in \{1, \ldots, N-1\}$. If $\lambda_j^2 - \lambda_N\lambda_j$ are pairwise different then, from the inductive assumption, $a_1, \ldots, a_{N-1} \in A$ so $a_N \in A$ as well.

Assume that these numbers are not pairwise different. We define an equivalence relation $\mathcal{R}$ on the set $\{1, \ldots, N-1\}$ in the following way:

$$j\mathcal{R}k \;\Leftrightarrow\; \lambda_j(\lambda_j - \lambda_N) = \lambda_k(\lambda_k - \lambda_N).$$

Let $I_1, \ldots, I_{N_1}$ denote the equivalence classes which contain not less than two elements and let $I_0 := \{i_1, \ldots, i_{N_0}\}$ be the remaining indices. From our assumption, $I_1 \neq \emptyset$. For $j \in \{1, \ldots, N_1\}$ and $k \in I_j$ let

$$\lambda_j' := \lambda_k(\lambda_k - \lambda_N),$$

and let

$$a_j' := \sum_{n \in I_j} a_n.$$

We also define

$$\lambda'_{N_1+1} := \lambda_{i_1}(\lambda_{i_1} - \lambda_N), \lambda'_{N_1+2} := \lambda_{i_2}(\lambda_{i_2} - \lambda_N), \ldots, \lambda'_{N_1+N_0} := \lambda_{i_{N_0}}(\lambda_{i_{N_0}} - \lambda_N)$$

and

$$a'_{N_1+1} := a_{i_1}, \quad a'_{N_1+2} := a_{i_2}, \; \ldots, \; a'_{N_1+N_0} := a_{i_{N_0}}.$$

Clearly, $1 \leq N' := N_1 + N_0 < N$, $a_j' \neq 0$, $a_j'^2 = a_j'$, $a_j' a_k' = 0$, $\lambda_j' \neq 0$, $\lambda_j' \neq \lambda_k'$ for $j, k \in \{1, \ldots, N'\}$, $j \neq k$, and

$$\lambda_1' a_1' + \cdots + \lambda_{N'}' a_{N'}' = a^2 - \lambda_N a \in A.$$

From the inductive assumption, $a_1' \in A$, hence

$$\sum_{n \in I_1} \lambda_n a_n = \sum_{n \in I_1} a_n \cdot \sum_{n=1}^{N} \lambda_n a_n = a_1' a \in A.$$

Again, from the inductive assumption, $a_n \in A$ for $n \in I_1$, and therefore $\sum_{n \in \{1,\ldots,N\} \setminus I_1} \lambda_n a_n \in A$. Once again, from the inductive assumption, $a_n \in A$ for $n \in \{1, \ldots, N\} \setminus I_1$. Thus $a_1, \ldots, a_N \in A$, which completes the proof. ∎

Proposition 4.2. *Let A be a closed *-subalgebra of $L(s', s)$ (not necessary commutative) and let x be an infinite-dimensional normal operator in $L(s', s)$ with spectral representation $x = \sum_{n=1}^{\infty} \lambda_n P_n$. Then $x \in A$ if and only if $P_n \in A$ for all $n \in \mathbb{N}$.*

Proof. Let $N_0 := 0$, $N_1 := \sup\{n \in \mathbb{N} : |\lambda_n| = |\lambda_1|\}$ and for $j = 2, 3, \ldots$ let $N_j := \sup\{n \in \mathbb{N} : |\lambda_n| = |\lambda_{N_{j-1}+1}|\}$. Since $(|\lambda_n|)_{n\in\mathbb{N}}$ is a null sequence, we have $N_j < \infty$.

By Theorem 3.1, if $P_n \in A$ for all $n \in \mathbb{N}$ then $x \in A$. To prove the converse assume that $x \in A$. Then $x^* = \sum_{n=1}^{\infty} \overline{\lambda_n} P_n \in A$ so $xx^* = \sum_{n=1}^{\infty} |\lambda_n|^2 P_n \in A$, whence

$$y_k := \sum_{n=1}^{\infty} \left(\frac{|\lambda_n|}{|\lambda_1|}\right)^{2k} P_n = \left(\frac{xx^*}{|\lambda_1|^2}\right)^k \in A$$

for all $k \in \mathbb{N}$. Hence for q and k arbitrary we get

$$\begin{aligned}\|y_k - (P_1 + \cdots + P_{N_1})\|_q &= \left\| \sum_{n=1}^{\infty} \left(\frac{|\lambda_n|}{|\lambda_1|}\right)^{2k} P_n - (P_1 + \cdots + P_{N_1}) \right\|_q \\ &= \left\| \sum_{n=N_1+1}^{\infty} \left(\frac{|\lambda_n|}{|\lambda_1|}\right)^{2k} P_n \right\|_q \le \sum_{n=N_1+1}^{\infty} \left(\frac{|\lambda_n|}{|\lambda_1|}\right)^{2k} \|P_n\|_q \\ &\le \frac{1}{|\lambda_1|} \left(\frac{|\lambda_{N_1+1}|}{|\lambda_1|}\right)^{2k-1} \sum_{n=N_1+1}^{\infty} |\lambda_n| \, \|P_n\|_q.\end{aligned}$$

By Theorem 3.1, $\sum_{n=N_1+1}^{\infty} |\lambda_n| \, \|P_n\|_q < \infty$, and moreover $|\lambda_{N_1+1}|/|\lambda_1| < 1$. Thus

$$\|y_k - (P_1 + \cdots + P_{N_1})\|_q \to 0$$

as $k \to \infty$. Therefore, since A is closed, we conclude that $P_1 + \cdots + P_{N_1} \in A$. Consequently,

$$\sum_{n=N_1+1}^{\infty} |\lambda_n|^2 P_n = xx^* - |\lambda_1|^2 (P_1 + \cdots + P_{N_1}) \in A;$$

hence, proceeding by induction, $P_{N_j+1} + \cdots + P_{N_{j+1}} \in A$ for $j \in \mathbb{N}_0$, so

$$\sum_{n=N_j+1}^{N_{j+1}} \lambda_n P_n = (P_{N_j+1} + \cdots + P_{N_{j+1}}) x \in A.$$

Finally, by Lemma 4.1, $P_n \in A$ for $n \in \mathbb{N}$. ∎

Proposition 4.3. *For every othonormal system $(e_n)_{n\in\mathbb{N}}$ in ℓ_2 and sequence $(\lambda_n)_{n\in\mathbb{N}} \in c_0$, the series $\sum_{n=1}^{\infty} \lambda_n \langle \cdot, e_n \rangle e_n$ converges in the norm $\| \cdot \|_{\ell_2 \to \ell_2}$.*

Proof. This is a simple consequence of the Pythagorean theorem and the Bessel inequality. ∎

Lemma 4.4. *Let A be a commutative subalgebra of $L(s', s)$. Let $\mathcal{P}$ denote the set of nonzero projections belonging to A and let $\mathcal{M}$ be the set of minimal* *elements in $\mathcal{P}$ with respect to the order relation*

$$\forall P, Q \in \mathcal{P} \quad P \preceq Q \ \Leftrightarrow\ PQ = QP = P.$$

Then

(i) *$\mathcal{M}$ is an at most countable family of pairwise orthogonal projections belonging to $L(s', s)$ such that*

$$\forall P \in \mathcal{P}\ \exists P'_1, \ldots, P'_m \in \mathcal{M} \quad P = P'_1 + \cdots + P'_m.$$

(ii) *If A is also a closed *-subalgebra of $L(s', s)$, then $\mathcal{M}$ is a Schauder basis in A.*

Proof. (i) By the definition

$$\mathcal{M} = \{P \in \mathcal{P} : \forall Q \in \mathcal{P}\ \ (Q \preceq P \Rightarrow Q = P)\}.$$

Firstly, we will show that

$$\forall P \in \mathcal{P}\ \exists P'_1, \ldots, P'_m \in \mathcal{M} \quad P = P'_1 + \cdots + P'_m. \tag{4.1}$$

Take $P \in \mathcal{P}$. If $P \in \mathcal{M}$, then we are done. Otherwise, there is $Q \in \mathcal{P}$ such that $Q \preceq P$, $Q \neq P$. Of course, $P - Q \in \mathcal{P}$. If $Q, P - Q \in \mathcal{M}$, then $P = Q + (P - Q)$ is the desired decomposition. Otherwise, we decompose Q or $P - Q$ into smaller projections as was done above for P. Since P is finite-dimensional, after finitely many steps we finish our procedure.

Next, we shall prove that projections in $\mathcal{M}$ are pairwise orthogonal. Let $P, Q \in \mathcal{M}$, $P \neq Q$, and suppose, to derive a contradiction, that $PQ \neq 0$. Since A is commutative,

$$(PQ)^2 = P^2Q^2 = PQ$$

and thus $PQ \in \mathcal{P}$. Moreover,

$$P(PQ) = P^2Q = PQ$$

so $PQ \preceq P$. Now, $PQ \neq P$ implies that $P \notin \mathcal{M}$ and if $PQ = P$ then $Q \notin \mathcal{M}$, which is a contradiction.

Finally, since projections in $\mathcal{M}$ are pairwise orthogonal (as projections on ℓ_2), $\mathcal{M}$ is at most countable.

(ii) Let $x \in A$. If x is finite-dimensional and $\sum_{n=1}^{N} \mu_n Q_n$ is its spectral decomposition, then from (i) and Lemma 4.1, x is a linear combination of projections in $\mathcal{M}$.

Assume that x is infinite-dimensional and let $x = \sum_{n=1}^{\infty} \mu_n Q_n$ (the spectral representation of x). Since A is a closed commutative *-subalgebra of $L(s', s)$, by Proposition 4.2, $Q_n \in A$ for $n \in \mathbb{N}$. Next, from (i),

$$\forall n \in \mathbb{N}\ \exists Q_1^{(n)}, \ldots, Q_{l_n}^{(n)} \in \mathcal{M} \quad Q_n = \sum_{j=1}^{l_n} Q_j^{(n)}.$$

Hence

$$x = \sum_{n=1}^{\infty} \sum_{j=1}^{l_n} \mu_n Q_j^{(n)}.$$

For $l_0 = 0$, $j = l_0 + l_1 + \cdots + l_{n-1} + k$, $1 \le k \le l_n$ let $P_j := Q_k^{(n)}$ and let $\lambda_j := \mu_n$. Consider the series $\sum_{n=1}^{\infty} \lambda_n P_n$. Clearly, if the series converges in $L(s', s)$ then its limit is x. To prove this we shall first show that the series converges in the norm $\|\cdot\|_{\ell_2 \to \ell_2}$.

Since P_n is an (orthogonal) projection of finite dimension d_n, we have $P_n = \sum_{j=1}^{d_n} \langle \cdot, e_j^{(n)} \rangle e_j^{(n)}$ for every orthonormal basis $(e_j^{(n)})_{j=1}^{d_n}$ of the image of P_n. For $d_0 = 0$, $j = d_0 + d_1 + \cdots + d_{n-1} + k$, $1 \le k \le d_n$ let $e_j := e_k^{(n)}$ and let $\lambda'_j := \lambda_n$. By Proposition 4.3, the series $\sum_{j=1}^{\infty} \lambda'_j \langle \cdot, e_j \rangle e_j$ converges in the norm $\|\cdot\|_{\ell_2 \to \ell_2}$. Hence $\sum_{n=1}^{\infty} \lambda_n P_n$ converges in the norm $\|\cdot\|_{\ell_2 \to \ell_2}$ because $(\sum_{n=1}^{N} \lambda_n P_n)_{N \in \mathbb{N}}$ is a subsequence of the sequence of partial sums of the series $\sum_{j=1}^{\infty} \lambda'_j \langle \cdot, e_j \rangle e_j$.

Now, by Lemma 3.5, $x = \sum_{n=1}^{\infty} \lambda_n P_n$ and the series converges absolutely in $L(s', s)$. This shows that every operator in A is represented by an absolutely convergent series $\sum_{n=1}^{\infty} \lambda''_n P''_n$ with $P''_n \in \mathcal{M}$. To prove the uniqueness of this representation assume that $\sum_{n=1}^{\infty} \lambda''_n P''_n = 0$. Then

$$\lambda''_m P''_m = P''_m \sum_{n=1}^{\infty} \lambda''_n P''_n = 0,$$

so $\lambda''_m = 0$ for $m \in \mathbb{N}$. This shows that the sequence of coefficients is unique, hence $\mathcal{M}$ is a Schauder basis in A. ∎

For a closed commutative *-subalgebra A of $L(s', s)$ the Schauder basis $\mathcal{M}$ from the preceding lemma will be called the *canonical Schauder basis* (of A).

For a subset Z of $L(s', s)$ we will denote by $\operatorname{alg}(Z)$ the closed *-subalgebra of $L(s', s)$ generated by Z and by $\overline{\operatorname{lin}}(Z)$ the closure (in $L(s', s)$) of the linear span of Z. If A is a closed *-subalgebra of $L(s', s)$, then $\widehat{A}$ denotes the set of nonzero *-multiplicative functionals on A.

COROLLARY 4.5. *The set $\widehat{A}$ of nonzero *-multiplicative functionals on a closed commutative *-subalgebra A of $L(s', s)$ is exactly the set of coefficient functionals with respect to the canonical Schauder basis of A.*

Proof. Clearly, every coefficient functional is *-multiplicative. Conversely, if φ is a nonzero *-multiplicative functional on A and $\{P_n\}_{n \in \mathbb{N}}$ is the canonical Schauder basis then $\varphi(P_n) = \varphi(P_n^2) = (\varphi(P_n))^2$, thus $\varphi(P_n) = 0$

or $\varphi(P_n) = 1$. Suppose that $\varphi(P_n) = \varphi(P_m) = 1$ for $n \neq m$. Then

$$\begin{aligned}2 &= \varphi(P_n) + \varphi(P_m) = \varphi(P_n + P_m) = \varphi((P_n + P_m)^2) = (\varphi(P_n + P_m))^2 \\ &= (\varphi(P_n) + \varphi(P_m))^2 = 4,\end{aligned}$$

a contradiction. Hence, there is at most one $n \in \mathbb{N}$ such that $\varphi(P_n) = 1$. If $\varphi(P_n) = 0$ for all $n \in \mathbb{N}$ then, since $\{P_n\}_{n\in\mathbb{N}}$ is a basis, $\varphi = 0$, a contradiction. Thus, there is exactly one $n \in \mathbb{N}$ such that $\varphi(P_n) = 1$, and $\varphi(P_m) = 0$ for $m \neq n$, i.e., φ is a coefficient functional. ■

PROPOSITION 4.6. *If $\{P_n\}_{n\in\mathcal{N}}$ is a family of pairwise orthogonal projections belonging to $L(s', s)$, then*

$$\operatorname{alg}(\{P_n\}_{n\in\mathcal{N}}) = \overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}})$$

and it is a commutative **-algebra.*

Proof. Clearly, $\overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}}) \subseteq \operatorname{alg}(\{P_n\}_{n\in\mathcal{N}})$ and $\operatorname{lin}(\{P_n\}_{n\in\mathcal{N}})$ is a commutative *-algebra. By the continuity of multiplication and involution, $\overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}})$ is a commutative *-algebra as well. Hence, $\overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}}) = \operatorname{alg}(\{P_n\}_{n\in\mathcal{N}})$. ■

PROPOSITION 4.7. *Every sequence $\{P_n\}_{n\in\mathcal{N}} \subset L(s', s)$ of nonzero pairwise orthogonal projections is a basic sequence in $L(s', s)$, i.e., it is a* (*canonical*) *Schauder basis of the Fréchet space* (**-algebra*) $\overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}})$.

Proof. Let $\mathcal{M}$ be the canonical Schauder basis of $A := \operatorname{alg}(\{P_n\}_{n\in\mathcal{N}})$ which consists of all projections which are minimal with respect to the order relation described in Lemma 4.4. If we show that $\{P_n\}_{n\in\mathcal{N}} = \mathcal{M}$, then, by Proposition 4.6, we get the conclusion.

Fix $n \in \mathcal{N}$ and assume that $Q \preceq P_n$ for some nonzero projection $Q \in A$, i.e., $QP_n = Q$. Since $A = \overline{\operatorname{lin}}(\{P_n\}_{n\in\mathcal{N}})$, we have

$$Q = \lim_{j\to\infty} \sum_{k=1}^{M_j} \lambda_k^{(j)} P_k$$

for some $M_j \in \mathbb{N}$ and $\lambda_k^{(j)} \in \mathbb{C}$. From the continuity of algebra multiplication and scalar multiplication, we get

$$\begin{aligned}Q = QP_n &= \Big(\lim_{j\to\infty} \sum_{k=1}^{M_j} \lambda_k^{(j)} P_k\Big) P_n = \lim_{j\to\infty} \Big(\sum_{k=1}^{M_j} \lambda_k^{(j)} P_k P_n\Big) = \lim_{j\to\infty} \lambda_n^{(j)} P_n \\ &= \Big(\lim_{j\to\infty} \lambda_n^{(j)}\Big) P_n = \lambda_n P_n,\end{aligned}$$

where $\lambda_n := \lim_{j\to\infty} \lambda_n^{(j)} \in \mathbb{C}$. Since Q is a nonzero projection, we deduce that $\lambda_n = 1$ and $Q = P_n$. Hence $\{P_n\}_{n\in\mathcal{N}} \subseteq \mathcal{M}$.

Now, suppose that there is a projection Q in $\mathcal{M} \setminus \{P_n\}_{n\in\mathcal{N}}$. We have already proved that $\{P_n\}_{n\in\mathcal{N}} \subseteq \mathcal{M}$, hence by Lemma 4.4(i), $Qx = 0$ for all $x \in \mathrm{lin}(\{P_n\}_{n\in\mathcal{N}})$. By continuity of multiplication, $Qx = 0$ for every $x \in \overline{\mathrm{lin}}(\{P_n\}_{n\in\mathcal{N}}) = A$. In particular, $Q = Q^2 = 0$, a contradiction. Hence, $\{P_n\}_{n\in\mathcal{N}} = \mathcal{M}$. ∎

Closed commutative *-subalgebras of $L(s', s)$ are, in some sense, quite simple: each of them is generated by a single operator and also by its spectral projections. From nuclearity we also get the following sequence space representations.

THEOREM 4.8. *Every closed commutative infinite-dimensional *-subalgebra A of $L(s', s)$ has a (canonical) Schauder basis $\{P_n\}_{n\in\mathbb{N}}$ consisting of pairwise othogonal finite-dimensional minimal projections (see Lemma 4.4) such that*

$$A = \mathrm{alg}(\{P_n\}_{n\in\mathbb{N}}) \cong \lambda^1(\|P_n\|_q) = \lambda^\infty(\|P_n\|_q)$$

as Fréchet *-algebras. Moreover, there is an operator $x \in A$ with spectral representation $x = \sum_{n=1}^\infty \lambda_n P_n$ such that $A = \mathrm{alg}(x)$.

Proof. By Lemma 4.4, A has a Schauder basis with the desired properties. By Proposition 4.6, $A = \overline{\mathrm{lin}}(\{P_n\}_{n\in\mathbb{N}}) = \mathrm{alg}(\{P_n\}_{n\in\mathbb{N}})$ and since A is a nuclear Fréchet space with Schauder basis $\{P_n\}_{n\in\mathbb{N}}$, we deduce that

$$A \cong \lambda^1(\|P_n\|_q) = \lambda^\infty(\|P_n\|_q)$$

as Fréchet spaces (see e.g. [12, Cor. 28.13, Prop. 28.16]). Since on the linear span of $\{P_n\}_{n\in\mathbb{N}}$ multiplication (resp. involution) corresponds to pointwise multiplication (resp. conjugation) in $\lambda^1(\|P_n\|_q)$, the isomorphism is also a *-algebra isomorphism where the Köthe space is equipped with pointwise multiplication.

Now, we shall show that there is a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive numbers such that the series $\sum_{n=1}^\infty \lambda_n P_n$ is absolutely convergent in $L(s', s)$. To do so, choose a sequence $(C_q)_{q\in\mathbb{N}}$ such that $C_q \ge \max_{1\le n\le q} \|P_n\|_q$. Clearly, $C_q/\|P_n\|_q \ge 1$ for $q \ge n$, so

$$\inf_{q\in\mathbb{N}} \frac{C_q}{\|P_n\|_q} \ge \min\left\{\frac{C_1}{\|P_n\|_1}, \ldots, \frac{C_{n-1}}{\|P_n\|_{n-1}}, 1\right\} > 0$$

for $n \in \mathbb{N}$. Let $\lambda_1 := 1$ and let

$$\lambda_n := \min\left\{\frac{1}{n^2} \inf_{q\in\mathbb{N}} \frac{C_q}{\|P_n\|_q}, \frac{\lambda_{n-1}}{2}\right\}.$$

Then $\lambda_n > 0$, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is strictly decreasing and

$$\sum_{n=1}^\infty \lambda_n \|P_n\|_q \le \sum_{n=1}^\infty \frac{1}{n^2} \inf_{r\in\mathbb{N}} \frac{C_r}{\|P_n\|_r} \|P_n\|_q \le C_q \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

Consequently, $x := \sum_{n=1}^\infty \lambda_n P_n \in L(s', s)$ and this series is the spectral representation of x. Moreover, since $P_n \in A$ for $n \in \mathbb{N}$ and A is closed, we have $x \in A$. Finally, the equality $\mathrm{alg}(x) = \mathrm{alg}(\{P_n\}_{n\in\mathbb{N}})$ is a consequence of Proposition 4.2. ∎

A commutative closed *-subalgebra A of $L(s', s)$ is said to be *maximal commutative* if it is not properly contained in any larger closed commutative *-subalgebra of $L(s', s)$. We say that a sequence $\{P_n\}_{n\in\mathbb{N}}$ of nonzero pairwise orthogonal projections in $L(s', s)$ is *complete* if there is no nonzero projection P in $L(s', s)$ such that $P_n P = 0$ for every $n \in \mathbb{N}$. For a subset Z of $L(s', s)$, the set $Z' := \{x \in L(s', s) : xy = yx \text{ for all } y \in Z\}$ is called the *commutant* of Z.

Proposition 4.9. *For every self-adjoint subset Z of $L(s', s)$, the commutant Z' is a closed *-subalgebra of $L(s', s)$.*

Proof. Clearly, if x, y commute with every $z \in Z$ then λx, $x + y$, xy and x^* commute as well. Hence, from the continuity of the algebra operations and the involution, Z' is a closed *-subalgebra of $L(s', s)$. ∎

Theorem 4.10. *For a closed commutative *-subalgebra A of $L(s', s)$ the following assertions are equivalent:*

(i) *A is maximal commutative;*
(ii) *the canonical Schauder basis $\{P_n\}_{n\in\mathbb{N}}$ of A is a complete sequence of pairwise orthogonal one-dimensional projections belonging to $L(s', s)$;*
(iii) *$A = A'$.*

Proof. (i)⇒(ii). Suppose that for some $m \in \mathbb{N}$ the projection P_m is not one-dimensional. Then there are two nonzero pairwise orthogonal projections $Q_1, Q_2 \in L(s', s)$ such that $P_m = Q_1 + Q_2$. By Proposition 4.6, $\overline{\mathrm{lin}}(\{P_n : n \neq m\} \cup \{Q_1, Q_2\})$ is a closed commutative *-subalgebra of $L(s', s)$, and clearly

$$A = \overline{\mathrm{lin}}(\{P_n\}_{n\in\mathbb{N}}) \subseteq \overline{\mathrm{lin}}(\{P_n : n \neq m\} \cup \{Q_1, Q_2\}).$$

By Proposition 4.7, $\{P_n\}_{n\in\mathbb{N}}$ is the canonical Schauder basis of A, and $\{P_n : n \neq m\} \cup \{Q_1, Q_2\}$ is the canonical Schauder basis of $\overline{\mathrm{lin}}(\{P_n : n \neq m\} \cup \{Q_1, Q_2\})$, so

$$A \neq \overline{\mathrm{lin}}(\{P_n : n \neq m\} \cup \{Q_1, Q_2\}).$$

Thus, A is not maximal, a contradiction.

If $P \in L(s', s)$ is a nonzero projection orthogonal to all P_n, then, using similar arguments, we find that $\overline{\mathrm{lin}}(\{P_n\}_{n\in\mathbb{N}} \cup \{P\})$ is a closed commutative *-subalgebra of $L(s', s)$ properly containing A, a contradiction.

(ii)⇒(iii). Since A is commutative, we have $A \subset A'$. Now, suppose that there is $x \in A' \setminus A$. By Proposition 4.9, $x^* \in A'$ so $x + x^*, x - x^* \in A'$, and moreover $x^* \notin A$. Since $x = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*)$, we have $x + x^* \notin A$ or $x - x^* \notin A$. Without loss of generality assume that $x + x^* \notin A$. The operator

$x+x^*$ is self-adjoint, hence it has a spectral representation $\sum_{m=1}^{\infty}\mu_m Q_m$. Then, by Propositions 4.2 and 4.9, $Q_m \in A'$ for all $m \in \mathbb{N}$ and there exists m_0 for which $Q_{m_0} \notin A$ (otherwise $x+x^* \in A$). Let $J := \{n : P_n \preceq Q_{m_0}\}$ (see the definition of $\preceq$ in Lemma 4.4). Since Q_{m_0} is finite-dimensional, J is finite. One can easily check that $Q_{m_0} - \sum_{j\in J} P_j$ is a projection (if $J = \emptyset$, then $\sum_{j\in J} P_j := 0$). Moreover,

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big)P_k = 0 \tag{4.2}$$

for all $k \in \mathbb{N}$. Indeed, if $k \in J$, then from the definition of $\preceq$, $Q_{m_0}P_k = P_k$, so

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big)P_k = Q_{m_0}P_k - P_k = 0.$$

Let $k \notin J$. We have $Q_{m_0}P_k = P_kQ_{m_0}$ because $Q_{m_0} \in A'$. This implies that $Q_{m_0}P_k$ is a projection and $\operatorname{im} Q_{m_0}P_k = \operatorname{im} Q_{m_0} \cap \operatorname{im} P_k$. Therefore, since the P_k are one-dimensional, we have $Q_{m_0}P_k = P_k$ or $Q_{m_0}P_k = 0$. But, by our assumption, $Q_{m_0}P_k \neq P_k$, so $Q_{m_0}P_k = 0$. Now,

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big)P_k = Q_{m_0}P_k = 0.$$

Since the sequence $(P_n)_{n\in\mathbb{N}}$ is complete, (4.2) implies that

$$Q_{m_0} - \sum_{j\in J} P_j = 0.$$

Hence $Q_{m_0} \in A$, a contradiction.

(iii)⇒(i). Follows directly from the definition of the commutant of A. ∎

Remark 4.11. (i) Since $(P_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal one-dimensional projections, we have $P_n = \langle\cdot, e_n\rangle e_n$, where $(e_n)_{n\in\mathbb{N}} \subset s$ is an orthonormal system in ℓ_2. Then $\lambda^\infty(\|P_n\|_q) = \lambda^\infty(|e_n|_q)$ as Fréchet *-algebras. Indeed, from the Hölder inequality, if $\xi \in s$ and $q \in \mathbb{N}_0$, then

$$|\xi|_q^2 \le |\xi|_{\ell_2}|\xi|_{2q}.$$

Hence

$$1 \le |e_n|_q \le |e_n|_q^2 = \|P_n\|_q = |e_n|_q^2 \le |e_n|_{2q}.$$

This implies that $\lambda^\infty(\|P_n\|_q) = \lambda^\infty(|e_n|_q)$ as Fréchet spaces, and, since the algebra operations are the same in both algebras, $\lambda^\infty(\|P_n\|_q) = \lambda^\infty(|e_n|_q)$ as Fréchet *-algebras.

(ii) The sequence $(e_n)_{n\in\mathbb{N}}$ from the previous item need not be an orthonormal basis of ℓ_2. Indeed, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of ℓ_2 such that $e_n \in s$ for $n \in \mathbb{N}\setminus\{1\}$ and $e_1 \notin s$. Clearly, $(e_n)_{n\in\mathbb{N}\setminus\{1\}}$ is not an orthonormal basis of ℓ_2 and $(\langle\cdot, e_n\rangle e_n)_{n\in\mathbb{N}\setminus\{1\}}$ is a complete sequence in $L(s', s)$.

(iii) Applying the Kuratowski–Zorn lemma, one can easily prove that every closed commutative *-subalgebra of $L(s', s)$ is contained in some maximal commutative *-subalgebra of $L(s', s)$. If $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal finite-dimensional projections, then, by Proposition 4.6, $\mathrm{alg}(\{P_n\}_{n\in\mathbb{N}})$ is a closed commutative *-subalgebra of $L(s', s)$ so it is contained in some maximal commutative *-subalgebra $\mathrm{alg}(\{Q_n\}_{n\in\mathbb{N}})$ of $L(s', s)$, where $\{Q_n\}_{n\in\mathbb{N}}$ is a complete sequence of one-dimensional projections in $L(s', s)$ (see Theorems 4.8 and 4.10). Now, applying Lemma 4.4(i), it is easy to show that the sequence $\{P_n\}_{n\in\mathbb{N}}$ can be extended to some complete sequence of projections belonging to $L(s', s)$.

Corollary 4.12. *Let A be one of the following Fréchet *-algebras with pointwise multiplication:*

(i) *the algebra $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions;*
(ii) *the algebra $\mathcal{D}(K)$ of test functions with support in a compact set $K \subset \mathbb{R}^n$ such that $\mathrm{int}(K) \neq \emptyset$;*
(iii) *the algebra $C_a^\infty(M)$ of smooth functions on a compact smooth manifold M vanishing at $a \in M$;*
(iv) *the algebra $C_a^\infty(\overline{\Omega})$ of smooth functions on $\overline{\Omega}$ vanishing at $a \in \Omega$, where $\Omega \neq \emptyset$ is an open bounded subset of $\mathbb{R}^n$ with C^1-boundary;*
(v) *the algebra $\mathcal{E}_a(K)$ of Whitney jets on a compact set $K \subset \mathbb{R}^n$ with the extension property, flat at $a \in K$ and such that $\mathrm{int}(K) \neq \emptyset$.*

Then A is isomorphic to s as a Fréchet space but it is not isomorphic to any closed commutative *-subalgebra of $L(s', s)$ as a Fréchet *-algebra.

Proof. It is well known that the spaces from items (i)–(v) are isomorphic to s as Fréchet spaces (see e.g. [12, Ch. 31], [15, Satz 4.1]).

To prove the second assertion let us compare the relevant sets of *-multiplicative functionals. If A is one of the spaces from items (i)–(v), then every point evaluation functional on A is *-multiplicative and since the underlying space has the cardinality $\mathfrak{c}$ of the continuum, the cardinality of the set of *-multiplicative functionals on A is not less than $\mathfrak{c}$. On the other hand, by Corollary 4.5, the set of *-multiplicative functionals on any infinite-dimensional closed commutative *-subalgebra of $L(s', s)$ is at most countable, hence none of the spaces from items (i)–(v) is isomorphic to A. ∎

It is clear that the algebra s with pointwise multiplication is a *-subalgebra of $L(s', s)$ (consider, for example, diagonal operators). The previous corollary shows that this is not the case for many other interesting Fréchet *-algebras isomorphic to s (as Fréchet spaces). This leads to the following:

Open Problem 4.13. *Is every closed commutative *-subalgebra of $L(s', s)$ *-isomorphic to some closed *-subalgebra of s with pointwise multiplication?*

By Theorem 4.10 and Remark 4.11, this problem is equivalent to the following:

Open Problem 4.14. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system in ℓ_2 such that $(e_n)_{n\in\mathbb{N}} \subset s$ and $(\langle\cdot, e_n\rangle e_n)_{n\in\mathbb{N}}$ is a complete sequence in $L(s', s)$. Is the algebra $\lambda^\infty(|e_n|_q)$ isomorphic to some closed *-subalgebra of s?*

5. Functional calculus. If x is a normal operator in $L(s', s) \subset K(\ell_2)$ and f is a continuous function on the spectrum $\sigma(x)$ of x vanishing at zero, then the continuous functional calculus for normal operators provides a uniquely determined operator $f(x) \in K(\ell_2)$ (see e.g. [12, Prop. 17.20]). In this section, we want to describe those functions f for which $f(x)$ is again in $L(s', s)$.

From the general theory of Fréchet locally m-convex algebras we get the holomorphic functional calculus on $L(s', s)$ (see Prop. 2.5 and [13, Lemma 1.3], [17, Th. 12.16]). More precisely, if x is an arbitrary operator in $L(s', s)$ and f is a holomorphic function on an open neighborhood U of $\sigma(x)$ with $f(0) = 0$, then $f(x) \in L(s', s)$, and moreover the map $\Phi\colon H_0(U) \to L(s', s)$, $f \mapsto f(x)$, is a continuous homomorphism ($H_0(U)$ stands for the space of holomorphic functions vanishing at zero).

Recall that a function $f\colon X \to \mathbb{C}$ ($X \subset \mathbb{C}$, $0 \in X$) is *Hölder continuous at zero* if there are $\theta \in (0, 1]$ and $C > 0$ such that $|f(t) - f(0)| \le C|t|^\theta$ for all t in a neighborhood of 0. As a consequence of Theorem 3.1 we get the following Hölder continuous functional calculus for normal operators in $L(s', s)$.

Corollary 5.1. *If $x \in L(s', s) \subset K(\ell_2)$ is normal, then for every function $f\colon \sigma(x) \to \mathbb{C}$ Hölder continuous at zero with $f(0) = 0$, we have $f(x) \in L(s', s)$ as well. In particular, for every normal operator $x \in L(s', s)$ with $\sigma(x) \subset [0,\infty)$ and $\theta \in (0,\infty)$, we have $x^\theta \in L(s', s)$.*

Proof. Let $x = \sum_{n\in\mathcal{N}} \lambda_n P_n$ be a normal operator in $L(s', s)$ with non-negative spectrum and let $\theta \in (0,\infty)$. If $\theta \in (0, 1]$, then, by Theorem 3.1, $x^\theta = \sum_{n=1}^\infty \lambda_n^\theta P_n \in L(s', s)$, and for $\theta \in (1,\infty)$ we have $x^\theta = x^{\lfloor\theta\rfloor} \cdot x^{\theta-\lfloor\theta\rfloor} \in L(s', s)$, where $\lfloor\theta\rfloor$ is the floor of θ.

Now, let $x = \sum_{n\in\mathcal{N}} \lambda_n P_n \in L(s', s)$ be normal and let $f\colon \sigma(x) \to \mathbb{C}$ be Hölder continuous at zero with $f(0) = 0$. Then $|f| \le C|\cdot|^\theta$ for some $C > 0$ and $\theta \in (0, 1]$. Hence $\sum_{n\in\mathcal{N}} \|f(\lambda_n)P_n\|_q \le C\sum_{n\in\mathcal{N}} |\lambda_n|^\theta\|P_n\|_q < \infty$ so, by Corollary 3.6, $f(x) \in L(s', s)$. ∎

For a normal operator x in $L(s', s)$ with spectral representation $x = \sum_{n=1}^\infty \lambda_n P_n$, we define the function space

$$C_s(\sigma(x)) := \{f : \sigma(x) \to \mathbb{C} : f(0) = 0,\ (f(\lambda_n))_{n\in\mathbb{N}} \in \lambda^\infty(\|P_n\|_q)\}.$$

It is easy to show that the space $C_s(\sigma(x))$ with the system $(c_q)_{q\in\mathbb{N}_0}$, $c_q(f) := \sup_{n\in\mathbb{N}} |f(\lambda_n)|\,\|P_n\|_q$, of seminorms, pointwise multiplication and conjugation is a Fréchet *-algebra.

THEOREM 5.2. *If* $x = \sum_{n=1}^{\infty} \lambda_n P_n$ *is an infinite-dimensional normal operator in* $L(s', s)$*, then the map*

$$\Phi : C_s(\sigma(x)) \to \mathrm{alg}(x), \quad \Phi(f) := f(x) := \sum_{n=1}^{\infty} f(\lambda_n) P_n,$$

is a Fréchet algebra isomorphism such that $\Phi(\mathrm{id}) = x$ *and* $\Phi(\overline{f}) = \Phi(f)^*$.

Proof. By Theorem 4.8, Φ is well defined, and of course $\Phi(\mathrm{id}) = x$ and $\Phi(\overline{f}) = \Phi(f)^*$. The space $\mathrm{alg}(x)$ is a nuclear Fréchet space (as a closed subspace of the nuclear Fréchet space $L(s', s)$) so $\lambda^\infty(\|P_n\|_q) \cong \mathrm{alg}(x)$ (see Theorem 4.8) is a nuclear Fréchet space as well. Thus, by the Grothendieck–Pietsch theorem (see e.g. [12, Th. 28.15]), for given $q \in \mathbb{N}_0$ one can find $r \in \mathbb{N}_0$ such that $C := \sum_{n=1}^{\infty} \|P_n\|_q / \|P_n\|_r < \infty$. Hence

$$\begin{aligned} \|\Phi(f)\|_q &\le \sum_{n=1}^{\infty} |f(\lambda_n)|\,\|P_n\|_q = \sum_{n=1}^{\infty} |f(\lambda_n)|\,\|P_n\|_r \frac{\|P_n\|_q}{\|P_n\|_r} \\ &\le \sup_{n\in\mathbb{N}} |f(\lambda_n)|\,\|P_n\|_r \cdot \sum_{n=1}^{\infty} \frac{\|P_n\|_q}{\|P_n\|_r} = C c_r(f) \end{aligned}$$

and thus Φ is continuous.

Clearly, Φ is injective. To prove that it is also surjective, take $y \in \mathrm{alg}(x)$. By Theorem 4.8, $(P_n)_{n\in\mathbb{N}}$ is a Schauder basis, so there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $y = \sum_{n=1}^{\infty} \mu_n P_n$. Let $g(\lambda_n) := \mu_n$ for $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}} |g(\lambda_n)|\,\|P_n\|_q = \sup_{n\in\mathbb{N}} |\mu_n|\,\|P_n\|_q < \infty,$$

hence $g \in C_s(\sigma(x))$, and of course, $\Phi(g) = y$. ∎

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