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O-minimal version of Whitney's extension theorem

by

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Abstract. This is a generalized and improved version of our earlier article [Studia Math. 124 (1997)] on the Whitney extension theorem for subanalytic $\mathcal{C}^p$-Whitney fields (with p finite). In this new version we consider Whitney fields definable in an arbitrary o-minimal structure on any real closed field R and obtain an extension which is a $\mathcal{C}^p$-function definable in the same o-minimal structure. The Whitney fields that we consider are defined on any locally closed definable subset of R^n. In such a way, a local version of the theorem is included.

Introduction. This paper is a generalized and improved version of our [KPaw]. Assume that R is any real closed field, and a fixed o-minimal structure on R is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].) The main theorem of our paper is the following version of the Whitney extension theorem [W]:

Theorem 1. *Let E be a definable closed subset of an open definable subset Ω of R^n, and let p and q be positive integers such that $p \le q$. Let*

$$F(x,X) = \sum_{|\kappa|\le p} \frac{1}{\kappa!} F^\kappa(x) X^\kappa \qquad (X = (X_1,\ldots,X_n))$$

be a definable $\mathcal{C}^p$-Whitney field on E. (Definability of F means that all F^κ are definable functions.) Then there exists a definable $\mathcal{C}^p$-function $f : \Omega \to R$, $\mathcal{C}^q$ on $\Omega \setminus E$, such that $D^\kappa f = F^\kappa$ on E whenever $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$.

The method of proof is by explicit construction as in [KPaw]. It is based on stratifications of definable sets into Λ_p-regular cells which will be presented in Section 2, while Section 1 is devoted to some basic facts on Whitney fields.

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A. Thamrongthanyalak has independently written a paper on Whitney's extension theorem in o-minimal structures (see [Th]). Taking into account some differences both in the approach and the results, we think that our article may still be of interest and therefore worth publishing.

1. $\mathcal{C}^p$-Whitney fields. In Section 1 we principally follow Glaeser [G]. Let $p \in \mathbb{N}$ and let A be any subset of R^n. We denote by $\mathcal{C}(A)$ the algebra of continuous functions on A with values in R. A $\mathcal{C}^p$-*Whitney field* on A is a polynomial

$$F(u,X) = \sum_{|\kappa|\le p} \frac{1}{\kappa!} F^\kappa(u) X^\kappa \in \mathcal{C}(A)[X] = \mathcal{C}(A)[X_1,\ldots,X_n],$$

which fulfills the following condition:

(1.1) for each $c \in A$ and each $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le p$,

$$D_X^\alpha F(a,0) - D_X^\alpha F(b,a-b) = o(|a-b|^{p-|\alpha|}),$$
$$\text{when } A \ni a \to c \text{ and } A \ni b \to c,$$

or equivalently (see [M, Chapter I, Theorem 2.2] or [T, Chapitre IV, Proposition 1.5]),

(1.2) for each $c \in A$,

$$F(a,x-a) - F(b,x-b) = o(|x-a|^p + |x-b|^p),$$

uniformly with respect to $x \in R^n$, when $A \ni a \to c$ and $A \ni b \to c$.

REMARK 1. A $\mathcal{C}^0$-Whitney field is simply a continuous function on A.

REMARK 2. If $p \ge 1$, a polynomial

$$F(u,X) = \sum_{|\kappa|\le p} \frac{1}{\kappa!} F^\kappa(u) X^\kappa \in \mathcal{C}(A)[X]$$

is a $\mathcal{C}^p$-Whitney field on A if and only if

(1.3) for each $c \in A$,

$$F^0(a) - \sum_{|\kappa|\le p} \frac{1}{\kappa!} F^\kappa(b)(a-b)^\kappa = o(|a-b|^p),$$
$$\text{when } A \ni a \to c \text{ and } A \ni b \to c,$$

and

(1.4) for any $i \in \{1,\ldots,n\}$, $\frac{\partial F}{\partial X_i}$ is a $\mathcal{C}^{p-1}$-Whitney field on A.

REMARK 3. If F is a $\mathcal{C}^p$-Whitney field on a subset $A \subset R^n$, then its restriction $F|B$ to a subset $B \subset A$, defined by $F|B(u,X) = F(u,X)$ for any $u \in B$, is a $\mathcal{C}^p$-Whitney field on B.

Let $\pi_p : \mathcal{C}(A)[X] \to \mathcal{C}(A)[X]$ denote the natural projection

$$\pi_p\Big(\sum_{\kappa} \frac{1}{\kappa!} F^\kappa X^\kappa\Big) = \sum_{|\kappa|\le p} \frac{1}{\kappa!} F^\kappa X^\kappa$$

onto the space of polynomials of degree $\le p$.

Remark 4. If F is a $\mathcal{C}^p$-Whitney field on A, where $p \ge 1$, then $\pi_{p-1}(F)$ is a $\mathcal{C}^{p-1}$-Whitney field on A, so any $\mathcal{C}^p$-Whitney field defines in a natural way some $\mathcal{C}^{p-1}$-Whitney field.

Proposition 1. *The set $\mathcal{E}^p(A)$ of all $\mathcal{C}^p$-Whitney fields on a subset $A \subset R^n$ with the natural addition and multiplication defined by $FG := \pi_p(FG)$ is an R-algebra.*

Proof. It is clear that the sum of two $\mathcal{C}^p$-Whitney fields is a $\mathcal{C}^p$-Whitney field. To check this for the product, using induction on p, take any F, G in $\mathcal{E}^p(A)$. Since

$$\begin{aligned}
&F^0(a)G^0(a) - F(b, a-b)G(b, a-b)\\
&\qquad = [F^0(a) - F(b, a-b)]G^0(a) + F(b, a-b)[G^0(a) - G(b, a-b)]\\
&\qquad = o(|a-b|^p)
\end{aligned}$$

when $A \ni a \to c$ and $A \ni b \to c$, condition (1.3) is satisfied. On the other hand, for any $i \in \{1, \dots, n\}$,

$$\frac{\partial(FG)}{\partial X_i} = \frac{\partial F}{\partial X_i} G + F \frac{\partial G}{\partial X_i}$$

is a $\mathcal{C}^{p-1}$-Whitney field by induction hypothesis, which proves condition (1.4).

It is also natural to define the composition of $\mathcal{C}^p$-Whitney fields as follows. Let $F_1, \dots, F_m \in \mathcal{E}^p(A)$, where $A \subset R^n$, and let $H \in \mathcal{E}^p(B)$, where $B \subset R^m$ is such that $(F_1^0, \dots, F_m^0)(A) \subset B$. Put $F = (F_1, \dots, F_m)$ and

$$\begin{aligned}
&(H \circ F)(u, X)\\
&\qquad = \pi_p[H(F_1^0(u), \dots, F_m^0(u), F_1(u, X) - F_1^0(u), \dots, F_m(u, X) - F_m^0(u))],
\end{aligned}$$

for any $u \in A$.

Proposition 2. *The composition of $\mathcal{C}^p$-Whitney fields defined as above is a $\mathcal{C}^p$-Whitney field.*

Proof. By condition (1.2) for H, we have

$$\begin{aligned}
&H^0(F_1^0(a), \dots, F_m^0(a))\\
&\qquad - H(F_1^0(b), \dots, F_m^0(b), F_1(b, a-b) - F_1^0(b), \dots, F_m(b, a-b) - F_m^0(b))\\
&\quad = o(|(F_1(b, a-b) - F_1^0(b), \dots, F_m(b, a-b) - F_m^0(b))|^p) = o(|a-b|^p),
\end{aligned}$$

when $A \ni a \to c$ and $A \ni b \to c$, which gives (1.3). On the other hand,

$$\frac{\partial(H \circ F)}{\partial X_i} = \sum_{j=1}^{m} \Big(\frac{\partial H}{\partial Y_j} \circ F\Big) \frac{\partial F_j}{\partial X_i};$$

hence it is a $\mathcal{C}^{p-1}$-Whitney field, by induction hypothesis and Proposition 1, so condition (1.4) is satisfied.

When restricting to definable functions $f : U \to R$ and definable mappings $g = (g_1, \ldots, g_m) : U \to R^m$, where U is an open subset of R^n, all the notions and basic results of classical differentiable calculus are valid, at least for finite differentiability classes. This is so principally because the Mean Value Theorem holds in this case (see [vdD, Chapter 7]). In particular, we have the well defined notion of $\mathcal{C}^p$-function and the Taylor formula in the following version.

Theorem 2. *If $f : U \to R$ is a definable $\mathcal{C}^p$-function on an open definable subset U of R^n, then the polynomial*

$$Tf(u, X) = T_u^p f(X) := \sum_{|\kappa| \le p} \frac{1}{\kappa!} D^\kappa f(u) X^\kappa,$$

*called the **Taylor field** of f, is a $\mathcal{C}^p$-Whitney field on U.*

A $\mathcal{C}^p$-Whitney field

$$F(u, X) = \sum_{|\kappa| \le p} \frac{1}{\kappa!} F^\kappa(u) X^\kappa$$

on a definable subset A of R^n is called *definable* if all the functions F^κ are definable. It is clear that the sum, product and composition of definable $\mathcal{C}^p$-Whitney fields are definable, and the Taylor field of a definable $\mathcal{C}^p$-function is a definable $\mathcal{C}^p$-Whitney field. We will use the following remark.

Remark 5 (see [G, pp. 87–88]). Let k and n be integers such that $1 \le k \le n$. Let Ω be a definable open subset of R^k treated as a subset of R^n via the injection $\Omega \ni v \mapsto (v, 0) \in R^n$. Then every definable $\mathcal{C}^p$-Whitney field

$$F(v, X) = F(v, V, W) = \sum_{|\alpha| + |\beta| \le p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(v) V^\alpha W^\beta$$

on Ω, where $\alpha \in \mathbb{N}^k$, $\beta \in \mathbb{N}^{n-k}$, $V = (X_1, \ldots, X_k)$ and $W = (X_{k+1}, \ldots, X_n)$, can be identified with the polynomial

$$\tilde{F}(v, W) = \sum_{|\beta| \le p} \frac{1}{\beta!} F^{(0,\beta)}(v) W^\beta,$$

where, for each $\beta \in \mathbb{N}^{n-k}$ with $|\beta| \le p$, $F^{(0,\beta)}$ is a definable $\mathcal{C}^{p-|\beta|}$-function on Ω such that

$$D^{\alpha}F^{(0,\beta)} = F^{(\alpha,\beta)}, \quad \text{for each } \alpha \in \mathbb{N}^k \text{ with } |\alpha| \le p - |\beta|. \tag{1.5}$$

2. Λ_p-regular mappings and Λ_p-regular cells. In the rest of the paper all the subsets of spaces R^n ($n \in \mathbb{N}$) and mappings between such subsets will be assumed definable. Therefore, for simplicity we will often skip the adjective *definable*.

Let $\varphi : Q \to R^l$ be a $\mathcal{C}^p$-mapping defined on an open subset Q of R^k. We say that φ is a *Λ_p-regular mapping* (in Q) if there exists a positive integer C such that

$$|D^{\alpha}\varphi(y)| \le C/\operatorname{dist}(y, \partial Q)^{|\alpha|-1}, \quad \text{for } \alpha \in \mathbb{N}^k \text{ with } 1 \le |\alpha| \le p. \tag{2.1}$$

A subset A of R^k is called *quasi-convex* if there is a positive integer M such that for any two points $a_1, a_2 \in A$ there exists a (definable) continuous arc $\lambda : [0, |a_1 - a_2|] \to A$ such that $\lambda(0) = a_1$, $\lambda(|a_1 - a_2|) = a_2$ and $|\lambda'(t)| \le M$ for any $t \in [0, |a_1 - a_2|]$ such that $\lambda'(t)$ exists. (Then λ is necessarily piecewise $\mathcal{C}^1$.)

By the Mean Value Theorem we immediately obtain the following

REMARK 6. If $\varphi : Q \to R^l$ is a Λ_1-regular mapping and A is any quasi-convex subset of Q, then $\varphi|A$ is a Lipschitz mapping; consequently, it extends in a unique way by continuity to $\overline{A}$.

We say that two closed subsets K and L of R^m are *simply separated* if either $K \cap L = \emptyset$ or there is a positive integer N such that $\operatorname{dist}(u, K \cap L) \le N \operatorname{dist}(u, L)$ for each $u \in K$.

The following proposition motivates our interest in Λ_p-regular mappings.

PROPOSITION 3 (see [KPaw, Proposition 3 and Remark 3]). *Let $\Phi : \Omega \to R^n$ be a Λ_p-regular mapping defined on an open subset $\Omega \subset R^m$ and let A be a closed quasi-convex subset of Ω such that $\overline{A}$ and $\partial\Omega$ are simply separated. Let B be a subset of R^n such that $\Phi(A) \subset B$ and let $r : B \to [0, +\infty)$ be a function such that*

$$r(x) \le C' \operatorname{dist}(x, \Phi(\overline{A} \setminus A)), \quad \textit{for any } x \in B,$$

where C' is a positive constant. Let

$$F(x, X) = \sum_{|\kappa| \le p} \frac{1}{\kappa!} F^{\kappa}(x) X^{\kappa}$$

be a $\mathcal{C}^p$-Whitney field on B such that, for each $b \in \Phi(\overline{A} \setminus A)$, $F^{\kappa}(x) = o(r(x)^{p-|\kappa|})$, when $B \ni x \to b$ and $|\kappa| \le p$. Put

$$G(y, Y) := F \circ T\Phi(y, Y) = \sum_{|\sigma| \le p} \frac{1}{\sigma!} G^{\sigma}(y) Y^{\sigma}, \quad \textit{where } Y = (Y_1, \ldots, Y_m).$$

Then, for each $a \in \overline{A} \setminus A$, $G^\sigma(y) = o(r(\Phi(x))^{p-|\sigma|})$, when $A \ni x \to a$ and $|\sigma| \le p$.

Proof. It suffices to repeat the proof of Proposition 3 in [KPaw] (see also Remark 3 there).

REMARK 7. If Ω is quasi-convex, we can take $A = \Omega$ in Proposition 3.

Now we recall after [KPaw] (see also [Paw2]) the definition of Λ_p-regular cells. We say that S is an *open Λ_p-regular cell* in R^n if

(2.2) S is any open interval in R, when $n = 1$;

(2.3) $S = \{(x', x_n) : x' \in T, \psi_1(x') < x_n < \psi_2(x')\}$, where $x' = (x_1, \ldots, x_{n-1})$, T is an open Λ_p-regular cell in R^{n-1} and every ψ_i $(i = 1, 2)$ is either a Λ_p-regular function on T with values in R, or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for each $x' \in T$, when $n > 1$.

REMARK 8. It follows from Remark 6, by induction on n, that such a cell S is quasi-convex and if ψ_i is finite, then it is Lipschitz on T, thus it admits a continuous extension $\overline{\psi}_i$ to $\overline{T}$ (cf. [Paw1, Proposition 1]).

For any open Λ_p-regular cell in R^n, one defines, by induction on n, a sequence $\rho_j : \overline{S} \to [0, +\infty]$ $(j = 1, \ldots, 2n)$ of *functions associated with the cell S*:

(1) When $n = 1$ and $S = (a_1, a_2)$, we set

$$\rho_1(x) = \begin{cases} x - a_1 & \text{if } a_1 \in R, \\ +\infty & \text{if } a_1 = -\infty, \end{cases} \qquad \rho_2(x) = \begin{cases} a_2 - x & \text{if } a_2 \in R, \\ +\infty & \text{if } a_2 = +\infty. \end{cases}$$

(2) When $n > 1$ and $S = \{(x', x_n) : x' \in T, \psi_1(x') < x_n < \psi_2(x')\}$, let σ_j $(j = 1, \ldots, 2n-2)$ be the functions associated with T. We set, for any $x = (x', x_n) \in \overline{S}$, $\rho_j(x) = \sigma_j(x')$, for $j = 1, \ldots, 2n-2$, and

$$\rho_{2n-1}(x) = \begin{cases} x_n - \overline{\psi}_1(x') & \text{if } \psi_1 : T \to R, \\ +\infty & \text{if } \psi_1 \equiv -\infty, \end{cases}$$
$$\rho_{2n}(x) = \begin{cases} \overline{\psi}_2(x') - x_n & \text{if } \psi_2 : T \to R, \\ +\infty & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

REMARK 9 ([KPaw, Lemma 3]). There exists a positive integer K such that

$$\frac{1}{K} \min_j \rho_j(x) \le \operatorname{dist}(x, \partial S) \le \min_j \rho_j(x), \qquad \text{for each } x \in \overline{S}.$$

(We adopt the convention $\operatorname{dist}(x, \emptyset) = +\infty$.)

REMARK 10 ([KPaw, Lemma 4]). Each of the functions ρ_j which is finite is Λ_p-regular on S and Lipschitz on $\overline{S}$ (with an integral constant).

Lemma 1 ([KPaw, Lemma 5]). *If $\rho_j \neq \pm\infty$, there exists a positive integer $\tilde{C}$ such that $|D^\alpha(1/\rho_j)(x)| \le \tilde{C}\operatorname{dist}(x, \partial S)^{-|\alpha|-1}$, whenever $x \in S$ and $|\alpha| \le p$.*

Proof. Put $r := \rho_j$. If $\alpha \in \mathbb{N}^n \setminus \{0\}$ and $|\alpha| \le p$, we have

$$D^\alpha(1/r) = \sum_{\nu=1}^{|\alpha|}\Big(\sum_{\substack{\lambda_1+\dots+\lambda_\nu=\alpha\\ \lambda_1\neq 0,\dots,\lambda_\nu\neq 0}} a^\alpha_{\lambda_1\dots\lambda_\nu}(D^{\lambda_1}r)\dots(D^{\lambda_\nu}r)\Big)\cdot r^{-1-\nu}, \tag{2.4}$$

where $a^\alpha_{\lambda_1\dots\lambda_\nu}$ is an integer depending only on $\alpha, \lambda_1, \dots, \lambda_\nu$. The lemma follows from (2.1) and Remark 9.

Extending the definition of an open Λ_p-regular cell, we call a subset S of R^n an *m-dimensional Λ_p-regular cell* in R^n, where $m \in \{0, \dots, n-1\}$, if $S = \{(u, w) : u \in T,\ w = \Phi(u)\}$, where $u = (x_1, \dots, x_m)$, $w = (x_{m+1}, \dots, x_n)$, T is an open Λ_p-regular cell in R^m and $\Phi : T \to R^{n-m}$ is a Λ_p-regular mapping.

Let us recall that a (*definable*) *$\mathcal{C}^p$-stratification* of a (definable) subset E of R^n is a finite decomposition $\mathcal{S}$ of E into (definable) connected $\mathcal{C}^p$-submanifolds of R^n, called *strata*, such that, for each $S \in \mathcal{S}$, its *boundary* in E, i.e. $\partial_E S := (\overline{S} \setminus S) \cap E$, is the union of some strata of dimensions $< \dim S$. If $A_1, \dots, A_k$, where $k \in \mathbb{N}$, are subsets of E, we call a stratification $\mathcal{S}$ *compatible with the subsets* $A_1, \dots, A_k$ if each A_j is a union of some strata.

This is the fundamental theorem on Λ_p-regular stratifications:

Theorem 3 ([KPaw, Proposition 4]). *Given any finite number $A_1, \dots, A_k$ of definable subsets of a definable subset E of R^n, there exists a $\mathcal{C}^p$-stratification $\mathcal{S}$ of E, compatible with the subsets $A_1, \dots, A_k$ and such that every stratum $S \in \mathcal{S}$ is a Λ_p-regular cell in R^n in some linear coordinate system.*

Proof. The proof is based on the main result of [K] (formulated there for subanalytic sets, but automatically generalizable to arbitrary o-minimal structures), which gives the theorem for the case $p = 1$, i.e. for Λ_1-stratifications. Then we have the following.

Proposition 4 ([KPaw, Proposition 1]). *Let $\Phi : \Omega \to R$ be a definable function on an open subset Ω of R^m. Let $p \in \mathbb{N}$ and $p \ge 1$. Then there exists a closed nowhere dense definable subset Z of Ω such that Φ is $\mathcal{C}^p$ on $\Omega \setminus Z$ and for every open ball $K = K(u, r) \subset \Omega \setminus Z$, with center u and radius r, we have $|D^\alpha\Phi(u)| \le C\sup_K|\Phi|/r^{|\alpha|}$, for each $\alpha \in \mathbb{N}^m$ with $|\alpha| \le p$, where C is a positive integer depending only on m and p.*

For the proof of Theorem 3 we need the following immediate corollary to Proposition 4.

Corollary ([KPaw, Proposition 2]). *Let $\Phi : \Omega \to R$ be a definable $\mathcal{C}^1$-function defined on an open subset Ω of R^m and such that $|\partial\Phi/\partial x_j| \le M$*

on Ω, for $j = 1, \ldots, m$. Let $p \in \mathbb{N}$ and $p \geq 1$. Then there exists a closed, definable nowhere dense subset of Ω such that Φ is of class $\mathcal{C}^p$ on $\Omega \setminus Z$ and

$$|D^\alpha \Phi(u)| \leq C(m,p) M \operatorname{dist}(u, Z \cup \partial\Omega)^{1-|\alpha|}$$

whenever $u \in \Omega \setminus Z$, $\alpha \in \mathbb{N}^m$, $1 \leq |\alpha| \leq p$, and where $C(m,p)$ is a positive integer depending only on m and p.

The proof of Proposition 4 is the same as that of [KPaw, Proposition 1]. Now, to finish the proof of Theorem 3, we use the Corollary to Proposition 4 to refine the Λ_1-stratification obtained at the beginning of the proof to a Λ_p-stratification, arguing by induction on $\dim E$. For more details, see [KPaw, proof of Proposition 4].

3. Two lemmas on $\mathcal{C}^p$-functions

LEMMA 3 ([KPaw, Lemma 6]). *Let Γ be an open subset of R^n, $a \in \overline{\Gamma}$ and $r : \Gamma \to R$. Let $g, h : \Gamma \to R$ be two $\mathcal{C}^p$-functions such that $D^\kappa g(x) = o(r(x)^{p-|\kappa|})$ and $D^\kappa h(x) = O(r(x)^{-|\kappa|})$, when $x \to a$, for any $\kappa \in \mathbb{N}^n$ with $|\kappa| \leq p$. Then $D^\kappa(gh)(x) = o(r(x)^{p-|\kappa|})$, when $x \to a$, for any $\kappa \in \mathbb{N}^n$ with $|\kappa| \leq p$.*

Proof. Immediate by Leibniz's formula.

LEMMA 4 ([KPaw, Lemma 7]). *Let $\chi : Q \to R$ be a $\mathcal{C}^p$-function on an open subset Q of R^m $(m < n)$ and $r : Q \to (0, +\infty)$. Let $c \in \overline{Q}$. Assume that $D^\alpha \chi(u) = O(r(u)^{-|\alpha|-1})$, when $u \to c$, for any $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq p$. Let $\psi : R \to R$ be any $\mathcal{C}^p$-function. Let Γ be an open subset of $R^m \times R^{n-m} = R^n$ contained in*

$$\{(u,w) \in Q \times R^{n-m} : w = (w_1, \ldots, w_{n-m}),\ |w_i| \leq Cr(u)\ (i = 1, \ldots, n-m)\},$$

where C is a positive constant. Define $g : \Gamma \to R$ by

$$g(u,w) = \psi(\chi(u)w_1) \cdot \ldots \cdot \psi(\chi(u)w_{n-m}).$$

Then $D^{(\alpha,\beta)} g(u,w) = O(r(u)^{-|\alpha|-|\beta|})$, when $(u,w) \to (c,0)$, for any $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^{n-m}$ such that $|\alpha| + |\beta| \leq p$.

Proof. This a straightforward calculation. For details, see [KPaw, proof of Lemma 7].

4. Proof of Theorem 1. Before beginning the proof we introduce the following two useful definitions. The closure of the set

$$\bigcup_{|\kappa| \leq p} \{x \in E : F^\kappa(x) \neq 0\}$$

in Ω will be called the *support* of the Whitney field F and denoted $\operatorname{supp} F$. A $\mathcal{C}^p$-Whitney field F on E is called *flat* on a subset $E' \subset E$ if $F^\kappa(x) = 0$, whenever $x \in E'$ and $|\kappa| \le p$.

We want to prove that there exists a function $f : \Omega \to R$ as in Theorem 1 and, in addition, of class $\mathcal{C}^q$ on $\Omega \setminus \operatorname{supp} F$. It will be convenient to formulate the following more general assertion. *If A is a definable closed subset of Ω such that* $\operatorname{supp} F \subset A \subset E$, *then there exists a function $f : \Omega \to R$ as in Theorem* 1 *and, in addition, of class $\mathcal{C}^q$ on $\Omega \setminus A$.* We will prove this by induction on $m = \dim A$. The case $m = 0$ is easy, so assume that $m > 0$. Take a $\mathcal{C}^q$-stratification $\mathcal{S}$ of A such that every stratum $S \in \mathcal{S}$ is a Λ_q-regular (thus, Λ_p-regular) cell in R^n in some linear coordinate system, and every F^κ ($|\kappa| \le p$) is of class $\mathcal{C}^q$ on S.

By induction hypothesis applied to F restricted to $E \setminus \bigcup\{S \in \mathcal{S} : \dim S = m\}$, we get an appropriate extension f_o. Replacing now F by $F - Tf_o|E$, we can assume without loss of generality that F is flat on every stratum from $\mathcal{S}$ of dimension $< m$. Similarly, using additional induction on the number of m-dimensional strata in $\mathcal{S}$ whose closures cover $\operatorname{supp} F$, we can assume that $\operatorname{supp} F$ is contained in the closure in Ω of just one stratum $S \in \mathcal{S}$ and F is flat on its boundary in Ω, i.e. on $\partial_\Omega S = (\overline{S} \setminus S) \cap \Omega$. In the case $m = n$, i.e. S is open in R^n, it suffices to define $f(x) = F^0(x)$ for $x \in S$, and $f(x) = 0$ for $x \in \Omega \setminus S$ (Hestenes' Lemma, [T, p. 80]), so let $1 \le m < n$. Then $S = \{(u, \varphi(u)) : u \in T\}$, where T is an open Λ_p-regular cell in R^m and $\varphi : T \to R^{n-m}$ is a Λ_p-regular mapping.

We will distinguish three cases.

Case I: $E = \overline{S} \cap \Omega = A \supset \operatorname{supp} F$ and $\varphi \equiv 0$, i.e. $S = T \times 0$ and $R^n \setminus \Omega \subset \overline{S} \setminus S = (\overline{T} \setminus T) \times 0$.

In this case set $\Gamma(T) := \{(u, w) \in T \times R^{n-m} : |w| < \operatorname{dist}(u, \partial T)\}$. *We shall construct a function f satisfying the conclusion of Theorem* 1 *and additionally such that $f = 0$ on $\Omega \setminus \Gamma(T)$*. Since, by Remark 5, F (as restricted to S) is the sum of the $\mathcal{C}^p$-Whitney fields

$$F_\beta(u, 0; X) = F_\beta(u, 0; U, W) = \sum_{|\alpha| \le p - |\beta|} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u) U^\alpha W^\beta,$$

where $\beta \in \mathbb{N}^{n-m}$, $|\beta| \le p$, $U = (U_1, \dots, U_m) = (X_1, \dots, X_m)$, $W = (W_1, \dots, W_{n-m}) = (X_{m+1}, \dots, X_n)$, we can assume that F is equal to one of them, i.e. $F(u, 0; X) = F_\beta(u, 0; X)$, for a fixed β.

Let Δ be the trace of Ω on R^m, i.e. $\Delta \times 0 = \Omega \cap (R^m \times 0)$. Notice that $\partial_\Omega S = \partial_\Delta T \times 0$. There exists a finite decomposition

$$\overline{T} \cap \Delta = \bigcup_\nu Q_\nu \cup Z$$

such that

(4.1) every Q_ν is an open Λ_q-regular cell in R^m in some linear coordinate system;

(4.2) every Q_ν is contained in T, while Z is closed in Δ and $\dim Z < m$,

(4.3) on every Q_ν all the functions $F^{(\alpha,\beta)}$ are of class $\mathcal{C}^q$,

and, by Proposition 4,

(4.4) for each $\gamma \in \mathbb{N}^m$ such that $|\gamma| \le p$, we have the estimates

$$|D^\gamma F^{(\alpha,\beta)}(u)| \le C \sup\{|F^{(\alpha,\beta)}(v)| : v \in Q_\nu,\ |u-v| < \operatorname{dist}(u, \partial Q_\nu)\}/\operatorname{dist}(u,\partial Q_\nu)^{|\gamma|},$$

for each $u \in Q_\nu$.

Since $\overline{T} \times 0$ and $R^n \setminus \Gamma(T)$ are simply separated, $Z \times 0$ and $\Omega \setminus \Gamma(T)$ are locally simply separated in Ω (see [Paw3] for the definition and properties of simply separated subsets). It follows that the formula

$$G(x,X) = \begin{cases} F(x,X) & \text{when } x \in Z \times 0, \\ 0 & \text{when } x \in \Omega \setminus \Gamma(T), \end{cases}$$

defines a definable $\mathcal{C}^p$-Whitney field on $(Z\times 0) \cup (\Omega \setminus \Gamma(T))$ (see [M, Chapter I, Remark 5.6]). By the induction hypothesis there exists a definable $\mathcal{C}^p$-extension $g : \Omega \to R$ of G which is of class $\mathcal{C}^q$ outside $Z \times 0$. It suffices to get an extension for $F - T(g)|E$ instead of that for F. Since $\Gamma(Q_\nu) \subset \Gamma(T)$ and $\Gamma(Q_\nu) \cap \Gamma(Q_\mu) \subset Z \times 0$ when $\nu \ne \mu$, it is enough to get an extension of every field

$$F|(\overline{Q}_\nu \cap \Delta) \times 0 - T(g)|(\overline{Q}_\nu \cap \Delta) \times 0, \tag{4.5}$$

for every ν separately.

Fix ν and put $Q = Q_\nu$. Put

$$h(u,w) = \frac{1}{\beta!} F^{(0,\beta)}(u) w^\beta - g(u,w)$$

for each $(u,w) \in T \times R^{n-m}$. This is a definable $\mathcal{C}^p$-function, which is $\mathcal{C}^q$ on $\Omega \cap (Q \times R^{n-m})$. Now we have the following

LEMMA 5. *Let $\kappa = (\varepsilon, \theta) \in \mathbb{N}^m \times \mathbb{N}^{n-m}$ with $|\kappa| \le p$ and let $a \in \partial_\Delta Q = (\overline{Q} \setminus Q) \cap \Delta$. Then*

$$D^\kappa h(u,w) = o(\operatorname{dist}(u,\partial Q)^{p-|\kappa|})$$

when $\Gamma(Q) \ni (u,w) \to (a,0)$.

Proof. When $a \in T$ this is immediate by the Taylor formula , since h is $\mathcal{C}^p$ and p-flat on $Z \cap T \supset (\partial Q) \cap T$. Suppose now that $a \in (\partial T) \cap \Delta$. We distinguish two possibilities.

(I) $\Gamma(Q) \ni (u,w) \to (a,0)$ and $\operatorname{dist}(u,\partial Q) < \operatorname{dist}(u,\partial T)$.

Let $\tilde{u} \in \partial Q$ be such that $|u - \tilde{u}| = \mathrm{dist}(u, \partial Q)$. By the Taylor formula for $D^\kappa h$ on the line segment $L \subset Q \times R^{n-m}$ joining points (u, w) and $(\tilde{u}, 0)$ (we take L without the endpoints), we have

$$\begin{aligned}|D^\kappa h(u,w)| &\le \sum_{|\lambda|=p-|\kappa|} \sup_{x\in L} |D^{\kappa+\lambda} h(x)| \sqrt{|u-\tilde{u}|^2 + |w|^2}^{|\lambda|} \\ &\le \sqrt{2}^{p-|\kappa|}(p+1)^n \sup_{x\in L,\, |\omega|=p} |D^\omega h(x)| \,\mathrm{dist}(u,\partial Q)^{p-|\kappa|}.\end{aligned}$$

Obviously,

$$\sup_{x\in L,\, |\omega|=p} |D^\omega h(x)| \le (*) + (**),$$

where

$$(*) = \sup_{x\in L,\, |\omega|=p} \left| D^\omega\left(\frac{1}{\beta!} F^{(0,\beta)}(u) w^\beta\right)(x)\right|, \qquad (**) = \sup_{x\in L,\, |\omega|=p} |D^\omega g(x)|.$$

It is clear that $(**) \to 0$, when $(u, w) \to (a, 0)$, because g is p-flat at $(a, 0)$. Now we will show the same about $(*)$.

Put $\omega = (\sigma, \tau)$. We can assume that $\tau \le \beta$, because otherwise $(*) = 0$. Then

$$D^\omega\left(\frac{1}{\beta!} F^{(0,\beta)}(u) w^\beta\right) = \frac{1}{(\beta-\tau)!} D^\gamma F^{(\alpha,\beta)}(u) w^{\beta-\tau},$$

where $\sigma = \alpha + \gamma$ and $|\alpha| + |\beta| = p$.

Let $x = (u_*, w_*) \in L$. Then, by (4.4),

$$\begin{aligned}&|D^\gamma F^{(\alpha,\beta)}(u_*) w_*^{\beta-\tau}| \\ &\quad \le C \frac{\sup\{|F^{(\alpha,\beta)}(v)| : v \in Q,\ |u_* - v| < \mathrm{dist}(u_*, \partial Q)\} |w_*|^{|\beta|-|\tau|}}{\mathrm{dist}(u_*, \partial Q)^{|\gamma|}} \\ &\quad \le C \sup\{|F^{(\alpha,\beta)}(v)| : v \in Q,\ |u_* - v| < \mathrm{dist}(u_*, \partial Q)\},\end{aligned}$$

because $|w_*| < \mathrm{dist}(u_*, \partial Q)$ and $|\gamma| = |\sigma| - |\alpha| = p - |\tau| - |\alpha| = |\beta| - |\tau|$. But the last tends to 0 when $\Gamma(Q) \ni (u, w) \to (a, 0)$, since F is p-flat at $(a, 0)$.

(II) $\Gamma(Q) \ni (u, w) \to (a, 0)$ and $\mathrm{dist}(u, \partial Q) = \mathrm{dist}(u, \partial T)$.

By the Taylor formula for g on a line segment $L \subset Q \times R^{n-m}$ joining points (u, w) and $(\tilde{u}, 0)$, where $\tilde{u} \in \partial T$ and $|u - \tilde{u}| = \mathrm{dist}(u, \partial T)$, we have

$$\begin{aligned}|D^\kappa g(u,w)| &\le \sum_{|\lambda|=p-|\kappa|} \sup_{x\in L} |D^{\kappa+\lambda} g(x)| \sqrt{|u-\tilde{u}|^2 + |w|^2}^{|\lambda|} \\ &\le \sqrt{2}^{p-|\kappa|}(p+1)^n \sup_{x\in L,\, |\omega|=p} |D^\omega g(x)| \,\mathrm{dist}(u,\partial Q)^{p-|\kappa|},\end{aligned}$$

and observe that

$$\sup_{x\in L,\,|\omega|=p} |D^\omega g(x)| \to 0, \qquad \text{when } \varGamma(Q) \ni (u,w) \to (a,0),$$

since g is p-flat at $(a,0)$.

Now we have to estimate

$$D^\kappa\left[\frac{1}{\beta!}F^{(0,\beta)}(u)w^\beta\right].$$

We can assume that $\theta \le \beta$, because otherwise this is 0. Then

$$D^\kappa\left[\frac{1}{\beta!}F^{(0,\beta)}(u)w^\beta\right] = \frac{1}{(\beta-\theta)!}D^\epsilon F^{(0,\beta)}(u)w^{\beta-\theta}.$$

Now two cases are possible: either $|\epsilon| > p-|\beta|$ or $|\epsilon| \le p - |\beta|$.

In the first case $\epsilon = \epsilon' + \epsilon''$, where $|\epsilon'| = p - |\beta|$. By (4.4),

$$\begin{aligned}&|D^\epsilon F^{(0,\beta)}(u)w^{\beta-\theta}|\\ &\quad\le \sup\{|F^{(\epsilon',\beta)}(v)| : v\in Q,\ |u-v| < \operatorname{dist}(u,\partial Q)\}|w|^{|\beta|-|\theta|}/\operatorname{dist}(u,\partial Q)^{|\epsilon''|}\\ &\quad\le \sup\{|F^{(\epsilon',\beta)}(v)| : v\in Q,\ |u-v| < \operatorname{dist}(u,\partial Q)\}\operatorname{dist}(u,\partial Q)^{|\beta|-|\theta|-|\epsilon''|},\end{aligned}$$

and the desired conclusion follows, since $|\beta| - |\theta| - |\epsilon''| = p - |\kappa|$ and $F^{(\epsilon',\beta)}(a) = 0$.

In the second case it follows from (1.5), (1.1) and the flatness of F on $\partial_\Omega S$ that

$$D^\kappa\left[\frac{1}{\beta!}F^{(0,\beta)}(u)w^\beta\right] = \frac{1}{(\beta-\theta)!}F^{(\epsilon,\beta)}(u)w^{\beta-\theta} = o(|u-\tilde u|^{p-|\beta|-|\epsilon|})|w|^{|\beta|-|\theta|},$$

where $\tilde u \in \partial T$ is such that $|u-\tilde u| = \operatorname{dist}(u,\partial T) = \operatorname{dist}(u,\partial Q)$.

Consequently,

$$D^\kappa\left[\frac{1}{\beta!}F^{(0,\beta)}(u)w^\beta\right] = o(\operatorname{dist}(u,\partial Q)^{p-|\kappa|}),$$

since $p-|\beta|-|\epsilon|+|\beta|-|\theta| = p-|\kappa|$. The proof of the lemma is complete.

Now we will define the desired extension of the Whitney field (4.5). Take a semialgebraic $\mathcal{C}^q$-function $\psi : R \to [0,1]$ such that $\psi(t) = 1$ near 0 and $\psi(t) = 0$ if $|t| \ge 1$. Let $\rho_1,\ldots,\rho_{2m}$ denote the functions associated with the cell Q. Define

$$f(u,w) = \prod_{i=1}^{n-m}\prod_{j=1}^{2m}\psi(w_iK\sqrt{n-m}/\rho_j(u))h(u,w),$$

where K is as in Remark 9. This is a definable $\mathcal{C}^q$-function on $Q\times R^{n-m}$ coinciding with h in a neighborhood of $Q\times 0$. Combining Lemma 1, Lemma 4 (where we set $r(u) = \operatorname{dist}(u,\partial Q)$), Lemma 5 and Lemma 3, we see that

$$\text{(4.6)}\qquad D^\kappa f(u,w) = o(\operatorname{dist}(u,\partial Q)^{p-|\kappa|}), \qquad \text{when } \varGamma(Q)\ni(u,w)\to(a,0),$$

for each $a \in (\overline{Q} \setminus Q) \cap \Delta$ and $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$. On the other hand, $f(u,w) = 0$ if $(u,w) \in (Q \times R^{n-m}) \setminus \Gamma(Q)$, due to Remark 9; hence, f extends to a $\mathcal{C}^p$-function on Ω vanishing outside $\Gamma(Q)$ and $\mathcal{C}^q$ outside $\overline{S}$. This completes the proof of Theorem 1 in Case I.

CASE II: As in Case I, $\overline{S} \cap \Omega = A \supset \operatorname{supp} F$ and $\varphi \equiv 0$, i.e. $S = T \times 0$ and $R^n \setminus \Omega \subset \overline{S} \setminus S = (\overline{T} \setminus T) \times 0$, but A is a proper subset of E.

Take the definable function $r : T \to (0, +\infty)$ defined by

$$r(u) := \begin{cases} \inf\{|w| : (u,w) \in E \setminus S\} & \text{when } \{w : (u,w) \in E \setminus S\} \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Since F is flat on $E \setminus S$,

$$F^\kappa(u,0) = o(r(u)^{p-|\kappa|}) \quad \text{when } T \ni u \to a,\ \kappa \in \mathbb{N}^m,\ |\kappa| \le p, \tag{4.7}$$

for any $a \in (\partial T) \cap \Delta$. By Theorem 3, there exists a finite decomposition

$$\overline{T} \cap \Delta = \bigcup_\nu Q_\nu \cup Z$$

such that

(4.8) every Q_ν is an open Λ_q-regular cell in R^m in some linear coordinate system;
(4.9) every Q_ν is contained in T, while Z is closed in Δ and $\dim Z < m$,
(4.10) on every Q_ν the function r is of class $\mathcal{C}^q$,

and either

(4.11.1) for all $j \in \{1, \ldots, m\}$, $|\partial r/\partial u_j| \le 1$ on Q_ν, and (by Corollary to Proposition 4 after perhaps a subdivision of Q_ν)

$$|D^\kappa r(u)| \operatorname{dist}(u, \partial Q_\nu)^{|\kappa|-1}$$

is bounded on Q_ν for each $\kappa \in \mathbb{N}^m$ with $|\kappa| \le p$,

or

(4.11.2) for some $j \in \{1, \ldots, m\}$, $|\partial r/\partial u_j| > 1$ on Q_ν.

By the induction hypothesis we can assume in addition that F is flat on $Z \times 0$, and hence on every $(\partial Q_\nu \cap \Delta) \times 0$. Notice that it is enough to have, for every ν, an appropriate extension $f_\nu : \Omega \to R$ of the $\mathcal{C}^p$-Whitney field $F|E \cap (\overline{Q}_\nu \times R^{n-m})$ such that f_ν vanishes outside $\Gamma(Q_\nu)$, since then we will glue all f_ν together to get a final extension. Fix ν. We will write Q instead of Q_ν. Let $g : \Omega \to R$ denote the extension of the $\mathcal{C}^p$-Whitney field $F|(\overline{Q} \cap \Delta) \times 0$ constructed in Case I. By the Taylor formula, (4.7) implies

(4.12) $D^\kappa g(u,w) = o(r(u)^{p-|\kappa|})$, when $Q \times R^{n-m} \ni (u,w) \to (a,0)$ with $|w| < Cr(u)$,

for each $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$, $a \in (\partial Q) \cap \Delta$ and any positive constant C.

We distinguish two subcases depending on whether (4.11.1) or (4.11.2) holds.

Subcase II.1: (4.11.1) holds.

Define

$$f(u,w) := \prod_{i=1}^{n-m} \psi(w_i\sqrt{n-m}/r(u))g(u,w)$$

for each $(u,w) \in Q \times R^{n-m}$.

Set $\Gamma_*(Q) = \{(u,w) \in Q \times R^{n-m} : |w| < \min(r(u), \operatorname{dist}(u, \partial Q))\}$. We will check that

$$D^\kappa f(u,w) = o(\min(r(u), \operatorname{dist}(u,\partial Q))^{p-|\kappa|}) \tag{4.13}$$

when $\Gamma_*(Q) \ni (u,w) \to (a,0)$, for each $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$ and $a \in (\partial Q) \cap \Delta$.

When $r(u) < \operatorname{dist}(u,\partial Q)$, $|D^\alpha r(u)| r(u)^{|\alpha|-1}$ ($|\alpha| \le p$) are bounded and, by the formula (2.4) in the proof of Lemma 1, $|D^\alpha(1/r)(u)| r(u)^{|\alpha|+1}$ are bounded too. Hence, (4.13) follows from Lemma 3 combined with Lemma 4 and (4.12).

When $r(u) \ge \operatorname{dist}(u,\partial Q)$, by (2.4) and (4.11.1),

$$|D^\alpha(1/r)| \operatorname{dist}(u,\partial Q)^{|\alpha|+1} \quad (|\alpha| \le p)$$

are bounded. Hence, (4.13) again follows from Lemma 3 combined with Lemma 4 and (4.6).

Since $f = 0$ on $(Q \times R^{n-m}) \setminus \Gamma_*(Q)$ and $E \setminus S \subset (Q \times R^{n-m}) \setminus \Gamma_*(Q)$, f extends to a $\mathcal{C}^p$-function $f : \Omega \to R$ flat on $E \setminus S$.

Subcase II.2: (4.11.2) holds.

Choose $j \in \{1, \ldots, m\}$ such that $|\partial r/\partial u_j| > 1$. We shall check that $r(u) \ge \operatorname{dist}(u,\partial Q)$, for each $u \in Q$. To see this take any point $a = (a_1, \ldots, a_m)$ in T. Then

$$\{t \in R : (a_1, \ldots, a_{j-1}, t, a_{j+1} \ldots, a_m) \in T\} = (b_1, c_1) \cap \cdots \cap (b_k, c_k),$$

where $b_1 < c_1 \le b_2 < \cdots \le b_k < c_k$. For some $l \in \{1, \ldots, k\}$, $a_j \in (b_l, c_l)$. By the Mean Value Theorem, for each $u_j \in (b_l, c_l)$,

$$\begin{aligned} r(a_1, \ldots, a_{j-1}, u_j, a_{j+1}, \ldots, a_m) &\ge \min(u_j - b_l, c_l - u_j) \\ &\ge \operatorname{dist}((a_1, \ldots, a_{j-1}, u_j, a_{j+1}, \ldots, a_m), \partial Q); \end{aligned}$$

hence, $r(a) \ge \operatorname{dist}(a, \partial Q)$.

It follows that $E \setminus S \subset R^n \setminus \Gamma(Q)$, hence $f(u,w) := g(u,w)$ is the desired extension. This completes the proof in Subcase II.2 and the proof in Case II.

Case III (general): As in Case II, $E \supset \overline{S} \cap \Omega = A \supset \operatorname{supp} F$, but $S = \{(u, \varphi(u)) : u \in T\}$, where $\varphi \not\equiv 0$.

It is enough to prove the theorem with Ω replaced by the maximal possible open subset of R^n in which $\overline{S}\cap\Omega$ is closed, i.e. $R^n\setminus(\overline{S}\setminus\Omega)$. Therefore, one can assume that $R^n\setminus\Omega\subset\overline{S}\setminus S$. Let $\Delta:=R^m\setminus\pi(R^n\setminus\Omega)$, where $\pi: R^n=R^m\times R^{n-m}\ni(u,w)\mapsto u\in R^m$ denotes the natural projection.

Set $r(x):=\min(\mathrm{dist}(x,E\setminus S),\mathrm{dist}(x,\partial S))$, for each $x\in S$.

Consider the following Λ_p-regular automorphism:

$$\Phi: T\times R^{n-m}\ni(u,w)\mapsto(u,w+\varphi(u))\in T\times R^{n-m}$$

of $T\times R^{n-m}$, which is of class $\mathcal{C}^q$. Since F is flat on $E\setminus S\supset\partial S\cap\Omega$, for each $b\in(\partial S)\cap\Omega$ and each $\kappa\in\mathbb{N}^n$ with $|\kappa|\le p$,

$$F^\kappa(x)=o(r(x)^{p-|\kappa|}),\qquad\text{when } S\ni x\to b.$$

It follows from Proposition 3, Remark 9 and the Hestenes Lemma that $G:=(F|S)\circ(T\Phi)$ is a $\mathcal{C}^p$-Whitney field on $T\times 0$, which extends to a $\mathcal{C}^p$-Whitney field on $(\overline{T}\cap\Delta)\times 0$ flat on $(\partial T)\cap\Delta\times 0$ with G^κ of class $\mathcal{C}^q$ on $T\times 0$ and such that

$$G^\kappa(u,0)=o(r(u,\varphi(u))^{p-|\kappa|}),\qquad\text{when } T\ni u\to a,\tag{4.14}$$

for any $a\in(\partial T)\cap\Delta$ and $\kappa\in\mathbb{N}^n$ with $|\kappa|\le p$. Since Φ extends to a bi-Lipschitz homeomorphism $\overline{\Phi}:\overline{T}\times R^{n-m}\to\overline{T}\times R^{n-m}$, (4.14) implies

$$G^\kappa(u,0)=o(\mathrm{dist}((u,0),\tilde{E}\setminus(T\times 0))^{p-|\kappa|}),\qquad\text{when } T\ni u\to a,\tag{4.15}$$

where $\tilde{E}:=\overline{\Phi}^{-1}(E\cap(\overline{T}\times R^{n-m}))$.

It follows from (4.15) that G extends to a $\mathcal{C}^p$-Whitney field on $\tilde{E}$ flat on $\tilde{E}\setminus(T\times 0)$. Hence, we are in the situation of Case II if $\tilde{E}\neq\emptyset$, or Case I if $\tilde{E}=\emptyset$. Therefore, there exists a definable $\mathcal{C}^p$-function $g: R^n\setminus((R^m\setminus\Delta)\times 0)\to R$ such that $Tg=G$ on $\tilde{E}$ and $g\equiv 0$ outside $\Gamma(T)$. By Proposition 3 (see Remark 7), the function $f(x):=g\circ\Phi^{-1}$ extends by 0 to Ω to the desired function. This completes the proof of Theorem 1.

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References

[C] M. Coste, *An Introduction to O-minimal Geometry*, Dottorato di Ricerca in Matematica, Dipartimento di Matematica, Università di Pisa, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[G] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Anal. Math. 6 (1958), 1–124.

[K] K. Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent 1*, in: Real Algebraic Geometry (Rennes, 1991), Lecture Notes in Math. 1524, Springer, 1992, 316–322.

[KP] K. Kurdyka and A. Parusiński, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture*, in: Singularity Theory and its Applications, Adv. Stud. Pure Math. 43, Math. Soc. Japan, Tokyo, 2006, 137–177.

[KPaw] K. Kurdyka and W. Pawłucki, *Subanalytic version of Whitney's extension theorem*, Studia Math. 124 (1997), 269–280.

[M] B. Malgrange, *Ideals of Differentiable Functions*, Oxford Univ. Press, 1966.

[Par] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Ann. Sci. École Norm. Sup. 27 (1994), 661–696.

[Paw1] W. Pawłucki, *Lipschitz cell decomposition in o-minimal structures. I*, Illinois J. Math. 52 (2008), 1045–1063.

[Paw2] W. Pawłucki, *A linear extension operator for Whitney fields on closed o-minimal sets*, Ann. Inst. Fourier (Grenoble) 58 (2008), 383–404.

[Paw3] W. Pawłucki, *A decomposition of a set definable in an o-minimal structure into perfectly situated sets*, Ann. Polon. Math. 79 (2002), 171–184.

[Th] A. Thamrongthanyalak, *Whitney's extension theorem in o-minimal structures*, MODNET Preprint 626.

[T] J.-Cl. Tougeron, *Idéaux de fonctions différentiables*, Springer, Berlin, 1972.

[vdD] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge Univ. Press, 1998.

[W] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

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