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Some remarks on Gleason measures

by

P. De Nápoli (Buenos Aires) and M. C. Mariani (Las Cruces, NM)

Abstract. This work is devoted to generalizing the Lebesgue decomposition and the Radon–Nikodym theorem to Gleason measures. For that purpose we introduce a notion of integral for operators with respect to a Gleason measure. Finally, we give an example showing that the Gleason theorem does not hold in non-separable Hilbert spaces.

1. Introduction. Let H be a Hilbert space and $\mathcal{P}$ the family of orthogonal projections in $\mathcal{L}(H)$. A *Gleason measure* is a function $\mu : \mathcal{P} \to \mathbb{C}$ which is σ-additive on orthogonal families of projections in $\mathcal{P}$, i.e. if $(S_n)_{n\in\mathbb{N}}$ is a countable orthogonal family of subspaces of H with closed linear span S then

$$\mu(S) = \sum_{n\in\mathbb{N}} \mu(S_n).$$

Gleason measures have a natural quantum mechanical interpretation. In fact, in quantum mechanics the space of possible (pure) states of a physical system corresponds to a Hilbert space H. Then Gleason probability measures on H (i.e. $0 \le \mu(S) \le 1$ and $\mu(I) = 1$) correspond to mixed states, i.e. ones where the precise state of the system is not known, but a probability distribution μ of the "observable events" is given (a closed subspace of H corresponds to observable events).

A. M. Gleason [7] proved that if H is a separable Hilbert space of dimension greater than or equal to three, then every positive measure μ can be represented as

$$\mu(S) = \operatorname{Tr}(\varrho P_S)$$

with $\varrho \in \mathcal{L}(H)$ a positive self-adjoint trace class operator (see Remark 2.6 below). In the quantum mechanical interpretation, ϱ is called a density operator (see [11], [12]).

Gleason's theorem and its generalizations have been deeply studied by many authors (see for example [1]–[4], [8], [13]) and applied to the problem of hidden variables in quantum mechanics [9].

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This work is devoted to generalizing the Lebesgue decomposition and the Radon–Nikodym theorem to Gleason measures. It is organized as follows. In Section 2, we give some definitions and known results that will be useful for us. In Section 3, we introduce a notion of integral for operators with respect to a Gleason measure.

In Section 4, we obtain a Lebesgue decomposition for Gleason measures on commutative C^* algebras (Theorem 4.2). The proof is based on a representation theorem which is of independent interest (Theorem 4.1).

In Section 5, we give a Lebesgue decomposition for representable measures.

Using the integral previously introduced, we give in Section 6 a version of the Radon–Nikodym theorem for representable Gleason measures (Theorem 6.3).

In Section 7, we discuss a quantum-mechanical interpretation of the integral as the expected value of an observable in a mixed state, and the relationship between the Radon–Nikodym theorem and the conditional expectation of an observable with respect to another.

Finally, in Section 8, we present an example showing that Gleason's theorem does not hold for non-separable Hilbert spaces.

2. Definitions and previous results

Definition 2.1. Following F. Riesz and Sz.-Nagy ([14, Section 116]), we shall say that an unbounded operator T and a bounded operator B are *permutable* (or *commute*) if

$$BT \subset TB.$$

Let H be a Hilbert space, $\mathcal{A} \subset \mathcal{L}(H)$ a C^* algebra of bounded normal operators in H, and $\mathcal{P}$ the set of orthogonal projectors in H.

Definition 2.2 ([5, Chapter VII, Definition 2.E.1]). Let $(X, \mathcal{M})$ be a measurable space (i.e. $\mathcal{M}$ is a σ-algebra of subsets of X). A *spectral measure* is a mapping $E : \mathcal{M} \to \mathcal{P}$ such that

1. $E(U)$ is an orthogonal projector for every $U \in \mathcal{M}$.
2. $E(X) = I$, $E(\emptyset) = 0$.
3. If $U = \bigcup_{n\in\mathbb{N}} U_n$ and the sets U_n are disjoint, then $E(U) = \sum_{n\in\mathbb{N}} E(U_n)$ (where the series is convergent in the strong operator topology).
4. If $U_1 \supset U_2 \supset \cdots$ and $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$, then $E(U_n) \to 0$ in the strong operator topology, i.e. $E(U_n)x \to 0$ for all $x \in H$.
5. If $U = U_1 \cap U_2$, then $E(U) = E(U_1) \cdot E(U_2)$. In particular if U_1 and U_2 are disjoint, then $E(U_1)$ and $E(U_2)$ are orthogonal.

DEFINITION 2.3. Let $\mu : \mathcal{P} \to \mathbb{R}$ be a Gleason measure. Then μ is said to be *concentrated on a subspace* S_0 if $S \subset S_0^\perp$ implies that $\mu(S) = 0$. In terms of projections, we can express the same idea by saying that μ is *concentrated on a projector* P_0 if for any projector $S \in \mathcal{P}(H)$, $P_0P = 0$ implies that $\mu(P) = 0$. We then write $\mu \subset S_0$ or $\mu \subset P_0$. Furthermore, if the set $\{P \in \mathcal{P} : \mu(P) = 0\}$ has a greatest element, P_0, then $I - P_0$ is called the *strong support* of μ. Evidently, $\mu(P) = 0$ if and only if $P(I - P_0) = 0$ (see [10]).

DEFINITION 2.4. Let $\lambda, \alpha : \mathcal{P} \to \mathbb{R}$ be two Gleason measures. The measure λ is said to be *absolutely continuous with respect to* α, written $\lambda \ll \alpha$, if $\alpha(P) = 0$ implies $\lambda(P) = 0$. Two Gleason measures λ and α are said to be *mutually singular*, written $\lambda \perp \alpha$, if there exists an orthogonal decomposition $I = P_0 + Q_0$ with P_0, Q_0 orthogonal projections such that $P_0Q_0 = Q_0P_0 = 0$ and $\lambda \subset P_0$, $\mu \subset Q_0$.

DEFINITION 2.5. For $1 \leq p < \infty$, we denote by L_p the class of bounded operators $T \in \mathcal{L}(H)$ which satisfy the following condition: for each orthonormal system $\{\varphi_k : k \in K\}$ in H, $\sum_{k\in K} |\langle T\varphi_k, \varphi_k\rangle|^p < \infty$. As L_p is a two-sided ideal in $\mathcal{L}(H)$, it is contained in the ideal of compact operators [15].

REMARK 2.6. In order to define the trace of an operator T, we need the series

$$\mathrm{Tr}(T) = \sum_{k\in K} \langle T\varphi_k, \varphi_k\rangle$$

to be absolutely convergent. So it is natural to define the trace for operators in L_1. We call the operators in L_1 *operators of trace class*. If A is a trace class operator and B is bounded, then AB is also of trace class. Moreover,

$$|\mathrm{Tr}(AB)| \leq \|B\| \, \mathrm{Tr}(|A|).$$

3. Integral with respect to a Gleason measure. It is natural to ask if it is possible to associate some sort of integral to a Gleason measure μ. Consider a self-adjoint operator that is a finite linear combination of projections,

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

where $P_i \in \mathcal{P}$ and $P_iP_j = 0$ if $i \neq j$. In analogy with standard measure theory, we shall call such operators *simple*. Then it is natural to define the integral of a simple operator with respect to μ by

$$\int A \, d\mu = \sum_{i=1}^{n} \lambda_i \mu(P_i).$$

We shall extend this notion of integral to the class of self-adjoint bounded operators. To do that we shall make use of the spectral theorem, in the following formulation:

THEOREM 3.1 ([5, Chapter VII, 2.F.2]). *To each (possibly unbounded) self-adjoint operator A in a Hilbert space H corresponds a spectral measure $E = E_A$ (defined on the Borel sets of $\mathbb{R}$) such that:*

1. $A = \int_{-\infty}^{\infty} \lambda \, dE$ *in the sense that*
$$Ax = \lim_{n\to\infty} \int_{-n}^{n} \lambda E(d\lambda)x$$
and
$$D(A) = \Big\{ x \in H : \int_{-\infty}^{\infty} \lambda^2 \langle E(d\lambda)x, x\rangle < \infty \Big\}.$$
2. *For each Borel set $U \subset \mathbb{R}$, $E(U)$ commutes with any bounded operator that commutes with A, and*
$$E(U)A = \int_U \lambda \, dE.$$
3. *For any real Borel measurable function $f(\lambda)$ we have*
$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE$$
with
$$D(f(A)) = \Big\{ x \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2 \langle E(d\lambda)x, x\rangle < \infty \Big\}.$$
4. *The spectral measure E is supported in the spectrum $\sigma(A)$ of A, i.e. for every Borel set $U \subset \mathbb{R}$, $E(U) = E(U \cap \sigma(A))$.*

In the case of a simple operator, the spectral measure E_A is given by
$$E_A(U) = \sum_{i:\, \lambda_i \in U} P_i.$$
Consider the measure $\mu \circ E_A$ on the Borel subsets of $\mathbb{R}$. Then if A is a simple operator,

$$\int A \, d\mu = \int_{-\infty}^{\infty} \lambda \, d(\mu \circ E_A). \tag{1}$$

So for any self-adjoint operator A we may define $\int A\, d\mu$ using equation (1). In a similar way, using more general versions of the spectral theorem, it is possible to define the integral $\int A \, d\mu$ when A is a normal operator.

Remark 3.2. Following F. Riesz and Sz.-Nagy ([14, Section 130]), if A and B are two self-adjoint operators, we shall say that A and B are *permutable* if $E_A(U)$ and $E_B(V)$ are permutable for any measurable sets $U, V \subset \mathbb{R}$, with E_A, E_B the spectral measures associated to A, B. In that case there exists a spectral measure $E_{A,B}$ defined on the Borel subsets of $\mathbb{R}^2$ such that

$$E_{A,B}(U \times V) = E_A(U)E_B(V)$$

for all "measurable rectangles". It follows that

$$A = \iint_{\mathbb{R}^2} \lambda_1 \, dE_{A,B}(\lambda_1, \lambda_2), \qquad B = \iint_{\mathbb{R}^2} \lambda_2 \, dE_{A,B}(\lambda_1, \lambda_2).$$

Proposition 3.1. *If A and B are permutable self-adjoint operators, then*

$$\int (A + B) \, d\mu = \int A \, d\mu + \int B \, d\mu. \tag{2}$$

Proof. Let $E_{A,B}$ be the spectral measure on $\mathbb{R}^2$ associated to the pair (A, B). Then

$$\begin{aligned}\int (A + B) \, d\mu &= \iint (\lambda_1 + \lambda_2) \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2) \\ &= \iint \lambda_1 \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2) + \iint \lambda_2 \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2) \\ &= \int \lambda_1 \, d(\mu \circ E_A)(\lambda_1) + \int \lambda_2 \, d(\mu \circ E_B)(\lambda_2) = \int A \, d\mu + \int B \, d\mu.\end{aligned}$$

Remark 3.3. This property does not hold if A and B are not permutable, as can be seen from the following example: We consider the Hilbert space $H = \mathbb{R}^2$, and denote by S_θ the 1-dimensional subspace generated by the vector $(\cos\theta, \sin\theta)$. Given a function $f : [0, \pi/2) \to [0, 1]$ we can define a Gleason measure μ in H by

$$\mu(S_\theta) = \begin{cases} f(\theta) & \text{if } 0 \le \theta < \pi/2, \\ 1 - f(\theta - \pi/2) & \text{if } \pi/2 \le \theta < \pi, \end{cases}$$

and $\mu(0) = 0$, $\mu(H) = 1$. If we take A to be the projection onto S_0 and B to be the projection onto $S_{\pi/4}$, it can be easily seen that (2) does not hold for general f.

Remark 3.4. Let $(X, \mathcal{M})$ be a measurable space. If $E : \mathcal{M} \to P(H)$ is a spectral measure and $A = \int_X \lambda \, dE$ then

$$\langle Ax, y \rangle = \int \lambda \, dE[x, y] \tag{3}$$

where $E[x, y](U) = \langle E(U)x, y \rangle$.

Proposition 3.2. *Let $S(H)$ be the set of subspaces of H, let $\mu : S(H) \to \mathbb{R}$ be a Gleason measure, and assume that μ is represented by a trace class operator ϱ, i.e.*

$$\mu(S) = \mathrm{Tr}(\varrho P_S) \qquad \forall S \in S(H).$$

Then for any (not necessarily bounded) μ-integrable self-adjoint operator A in H we have

$$\int A\,d\mu = \mathrm{Tr}(\varrho A).$$

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis of H such that $e_i \in D(A)$ for all $i \in I$ (this can be done since $D(A)$ is dense in H). Then

$$\mathrm{Tr}(\varrho A) = \sum_{i\in I} \langle \varrho A e_i, e_i\rangle = \sum_{i\in I} \langle A e_i, \varrho e_i\rangle.$$

From (3), we have

$$\langle A e_i, \varrho e_i\rangle = \int \lambda\, dE_A[e_i, \varrho e_i]$$

where E_A is the spectral measure associated with A. Hence,

$$\mathrm{Tr}(\varrho A) = \int \lambda\, d\Big(\sum_{i\in I} E_A[e_i, \varrho e_i]\Big).$$

On the other hand,

$$\begin{aligned}\Big(\sum_{i\in I} E_A[e_i, \varrho e_i]\Big)(U) &= \sum_{i\in I} E_A[e_i, \varrho e_i](U) = \sum_{i\in I} \langle E_A(U) e_i, \varrho e_i\rangle \\ &= \sum_{i\in I} \langle \varrho E_A(U) e_i, e_i\rangle = \mathrm{Tr}(\varrho E_A(U)) = \mu(E_A(U)).\end{aligned}$$

It follows that

$$\mathrm{Tr}(\varrho A) = \int \lambda\, d(\mu \circ E_A) = \int A\, d\mu.$$

In order to justify these formal computations, we may assume first that A is a positive operator, and then for the general case, we use the decomposition $A = A^+ - A^-$.

Remark 3.5. It follows that when the Gleason measure μ is represented by a trace class operator, the linearity property (2) holds for any operators A, B (even if they are not permutable).

Lemma 3.6. *Let μ be a finite non-negative Gleason measure and A a bounded self-adjoint operator. Then if*

$$m = m(A) = \inf_{\|x\|=1} \langle Ax, x\rangle, \qquad M = M(A) = \sup_{\|x\|=1} \langle Ax, x\rangle.$$

are the lower and upper bounds of A, we have

$$m(A)\mu(I) \le \int A\, d\mu \le M(A)\mu(I)$$

In particular,

$$\Big|\int A\, d\mu\Big| \le \|A\|\, \mu(I).$$

Proof. Let E_A be the spectral measure associated with A. Then E_A is concentrated in the interval $[m, M]$. It follows that

$$\int A\,d\mu = \int_m^M \lambda\,d(\mu \circ E_A)$$

and hence

$$\mu(E_A([m,M]))m \le \int_m^M \lambda\,d(\mu \circ E_A) \le \mu(E_A([m,M]))M$$

since $\mu \circ E_A$ is a non-negative Lebesgue–Stieltjes measure. We observe that $\mu(E_A([m,M])) = \mu(I)$ and the result follows.

Definition 3.7. Let μ be a (non-negative) Gleason measure and A a self-adjoint operator. We say that $A=0$ *a.e. with respect to* μ if $\mu(\mathrm{Ker}(A)^{\perp})=0$.

Lemma 3.8. *If $A = 0$ a.e. with respect to μ, then $\int A\,d\mu = 0$.*

Proof. Let E_A be the spectral measure associated with A. Then

$$\mathrm{Ker}(A) = \{x \in H : Ax = 0\} = E_A(\{0\}).$$

Hence $\mathrm{Ker}(A)^{\perp} = E_A(\{x \in \mathbb{R} : x \neq 0\})$. It follows that

$$\int A\,d\mu = \int_{-\infty}^{\infty} \lambda\,d(\mu \circ E_A) = \int_{\{x\neq 0\}} \lambda\,d(\mu \circ E_A) = 0$$

since $\mu \circ E_A(\{x \neq 0\}) = 0$.

4. Lebesgue decomposition for measures on commutative von Neumann algebras. The following result says that Gleason measures on commutative von Neumann algebras can be represented by ordinary measure spaces. We recall that a *von Neumann algebra* $\mathcal{A} \subset \mathcal{L}(H)$ is a C^* algebra that is closed in the strong operator topology.

Theorem 4.1. *Let H be a Hilbert space, $\mathcal{A} \subset \mathcal{L}(H)$ a commutative von Neumann algebra with identity of normal operators in H, and $\mathcal{P}(\mathcal{A})$ the set of orthogonal projectors in $\mathcal{A}$. If $\mu : \mathcal{P}(\mathcal{A}) \to \mathbb{R}$ is a Gleason measure, then there exists a measure space $(X, \mathcal{M}, \overline{\mu})$ and a mapping $\Phi : \mathcal{P} \to \mathcal{M}$ such that*

1. *$\Phi(P_1P_2) = \Phi(P_1) \cap \Phi(P_2)$.*
2. *If P_1 and P_2 are orthogonal (i.e. $P_1P_2 = 0$), then $\Phi(P_1 + P_2) = \Phi(P_1) \cup \Phi(P_2)$.*
3. *$\Phi(I - P) = X - \Phi(P)$.*
4. *$\mu(P) = \overline{\mu}(\Phi(P))\ \forall P \in \mathcal{P}(\mathcal{A})$.*

Proof. We follow the lines of the proof of the Bochner–Weil–Raikov theorem in [5, Chapter VII, 2.D.11].

The algebra $\mathcal{A}$ is isomorphic to the algebra $C(M)$, where M is the spectral space of $\mathcal{A}$, i.e. the set of real positive linear and multiplicative functionals over $\mathcal{A}$ (see [5, Chapter VII, 2.D.6]). Then M is a compact Hausdorff space with the weak-$*$ topology (it is a subset of the dual space of $\mathcal{A}$). The isomorphism is given by the correspondence $\mathcal{A} \ni A \mapsto \widetilde{A} \in C(M)$ where

$$\widetilde{A}(t) = t(A) \quad \forall t \in M.$$

We take $X = M$, define $\mathcal{M}$ to be the σ-algebra of Borel subsets of M, and $\Phi : \mathcal{P} \to \mathcal{M}$ by

$$\Phi(P) = \{t \in M : \widetilde{P}(t) = 1\}.$$

Since $P^2 = P$, it follows that $\widetilde{P}(t) = 0$ or $\widetilde{P}(t) = 1$ for all $t \in M$. Moreover, since $\widetilde{P}$ is continuous, the set $\Phi(P)$ is closed and hence Borel.

We note that the integral

$$\varphi(A) = \int A \, d\overline{\mu} \tag{4}$$

defines a continuous linear functional on the algebra $\mathcal{A}$ (as seen using Proposition 3.1). Since $\mathcal{A}$ is by assumption a von Neumann algebra, it contains the projectors $E_A(U)$ for every $U \in \mathcal{M}$, where E_A is the spectral measure associated with A. It follows that the integral (4) is well defined. Using the isomorphism with $C(M)$, we see that for any $\widetilde{A} \in C(M)$ we can define

$$\widetilde{\varphi}(\widetilde{A}) = \varphi(A)$$

and $\widetilde{\varphi}$ is a continuous bounded functional on $C(M)$. By the Riesz representation theorem, there exists a Borel measure $\overline{\mu}$ on M such that

$$\widetilde{\varphi}(\widetilde{A}) = \varphi(A) = \int \widetilde{A}(t) \, d\overline{\mu}$$

for any $A \in \mathcal{A}$. Choosing $A = P$, an orthogonal projector, we see that

$$\mu(P) = \overline{\mu}(\Phi(P)).$$

We remark that if μ is positive, so is $\overline{\mu}$.

The following results give a Lebesgue decomposition for the case of a commutative von Neumann algebra:

THEOREM 4.2. *Let H be a Hilbert space, $\mathcal{A} \subset \mathcal{L}(H)$ a commutative von Neumann algebra with identity of normal operators in H, and $\mathcal{P}$ the set of orthogonal projectors in $\mathcal{A}$. If $\mu, \lambda : \mathcal{P} \to \mathbb{R}$ are two Gleason measures, and μ is positive, then there exist two Gleason measures λ_{a} and λ_{s} such that*

$$\lambda = \lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}$$

with $\lambda_{\mathrm{a}} \ll \mu$ and $\lambda_{\mathrm{s}} \perp \mu$.

Proof. We apply the previous result to the measures μ and λ. We obtain a measure space $(X,\mathcal{M})$ and a mapping $\Phi:\mathcal{P}\to\mathcal{M}$, which depends only on the algebra $\mathcal{A}$, and two measures $\overline{\mu},\overline{\lambda}:\mathcal{M}\to\mathbb{R}$. Using the Lebesgue decomposition for ordinary measures we get

$$\overline{\lambda}=\overline{\lambda}_{\mathrm{s}}+\overline{\lambda}_{\mathrm{a}}$$

with $\overline{\lambda}_{\mathrm{a}}\ll\overline{\mu}$ and $\overline{\lambda}_{\mathrm{s}}\perp\overline{\mu}$. Then we define

$$\lambda_{\mathrm{s}}(P)=\overline{\lambda}_{\mathrm{s}}(\Phi(P)),\qquad \lambda_{\mathrm{a}}(P)=\overline{\lambda}_{\mathrm{a}}(\Phi(P)),$$

and we get the desired decomposition.

5. Lebesgue decomposition with respect to a representable measure. In this section, we present a different approach to obtaining a Lebesgue decomposition from Gleason measures, which applies when μ is a representable measure.

THEOREM 5.1. *Let μ, λ be two Gleason measures defined on a Hilbert space S and assume that μ is represented by a positive trace class operator ϱ_1. Then there exist two Gleason measures λ_{a} and λ_{s} such that*

$$\lambda(P)=\lambda_{\mathrm{a}}(P)+\lambda_{\mathrm{s}}(P)\tag{5}$$

for any projector that commutes with the projector P_R onto the range $R(\varrho_1)$ of ϱ_1, $\lambda_{\mathrm{a}}\perp\lambda_{\mathrm{s}}$, $\lambda_{\mathrm{a}}\ll\mu$ and λ_{s} is singular with respect to μ. Moreover, if λ is also a representable measure, this decomposition holds for any $P\in\mathcal{P}(H)$.

Proof. Let us define the required measures by

$$\lambda_{\mathrm{a}}(P)=\lambda(P_RP),\qquad \lambda_{\mathrm{s}}(P)=\lambda((I-P_R)P).$$

Since P commutes with P_R, P_RP is a projector. Moreover P_RP and $(I-P_R)P$ are orthogonal, so that (5) holds. If a subspace S can be written as an orthogonal direct sum $S=\sum_{n\in\mathbb{N}}S_n$ then if P_S is the projector onto S, and P_{S_n} the projector onto S_n, we have the orthogonal decomposition

$$P_RP_S=\sum_{n\in\mathbb{N}}P_RP_{S_n}.$$

Hence,

$$\lambda_{\mathrm{a}}(S)=\lambda(P_RP_S)=\sum_{n\in\mathbb{N}}\lambda(P_RP_{S_n})=\sum_{n\in\mathbb{N}}\lambda_{\mathrm{a}}(S_n).$$

It is clear from the definition of λ_{a} and λ_{s} that $\lambda_{\mathrm{a}}\subset P_R$, whereas $\lambda_{\mathrm{s}}\subset P_R^{\perp}$. It follows from Definition 2.4 that $\lambda_{\mathrm{a}}\perp\lambda_{\mathrm{s}}$.

We claim that if $\mu(P)=0$, then $P_RP=0$, hence $\lambda_{\mathrm{a}}\ll\mu$.

Indeed, if $P_RP\neq 0$, then there exists $x\in R(P)$ such that $P_Rx\neq 0$, hence $\varrho_1P_Rx=\varrho_1x\neq 0$, since $H=R(\varrho_1)\oplus\operatorname{Ker}(\varrho_1)$. From $\langle\varrho_1x,x\rangle=\|\varrho_1^{1/2}x\|^2$, with $\varrho_1^{1/2}$ the positive square root of ϱ_1, we deduce that $\langle \varrho_1 x, x\rangle \neq 0$. Therefore, $\mu(P) = \mathrm{Tr}(\varrho_1 P) \neq 0$.

It remains to show that λ_{s} is singular with respect to μ. But it is clear from the definitions that $\lambda_{\mathrm{s}} \subset I - P_R$, whereas $\mu \subset P_R$, and since P_R and $I - P_R$ give an orthogonal decomposition of the identity, this shows that $\mu \perp \lambda_{\mathrm{s}}$.

If λ is also a representable measure, represented by a trace class operator ϱ_2, we can define

$$\lambda_{\mathrm{a}}(P) = \mathrm{Tr}(\varrho_2 P_R P), \qquad \lambda_{\mathrm{s}}(P) = \mathrm{Tr}(\varrho_2 (I - P_R) P),$$

and since the trace is a linear operator, in that case (5) holds for any projector P.

6. A version of the Radon–Nikodym theorem. Let A be a normal operator, μ a positive Gleason measure, and define

$$\lambda_A(S) = \int A_{|S}\, d\mu = \int_S A\, d\mu$$

where $A_{|S}$ is the operator AP_S. Then λ_{a} is a Gleason measure on the set of A-invariant subspaces (with the identification of S with P_S we may view it as the set of projectors such that $P_S A \subset AP_S$). In fact, if $S = \bigoplus_{n\in\mathbb{N}} S_n$, then $P_S = \sum_{n\in\mathbb{N}} P_{S_n}$, and using Proposition 3.1 we see that

$$\int_S A\, d\mu = \sum_{n\in\mathbb{N}} \int_{S_n} A\, d\mu$$

since AP_{S_i} and AP_{S_j} commute, because S_i, S_j are A-invariant subspaces.

We remark that in the special case where μ is a Gleason measure represented by a trace class operator, we may consider λ_{a} to be defined for all closed subspaces of H, since as observed before, in that case the linearity property (2) holds without restrictions.

Lemma 6.1. *λ_A is absolutely continuous with respect to μ.*

Proof. Let S be a closed subspace such that $\mu(S) = 0$. We will show that $AP_S = 0$ a.e. with respect to μ. In fact, if $x \in S^\perp$ then $AP_S x = 0$, that is,

$$S^\perp \subset \mathrm{Ker}(AP_S),$$

and so

$$\mathrm{Ker}(AP_S)^\perp \subset S.$$

Since $\mu(S) = 0$, it follows that $\mu(\mathrm{Ker}(AP_S)^\perp) = 0$. This means that $AP_S = 0$ a.e. with respect to μ, and hence $\int_S A\, d\mu = 0$.

Now suppose that we are given two Gleason measures λ, μ such that λ is absolutely continuous with respect to μ. It is natural to ask if $\lambda = \lambda_A$ for some self-adjoint operator A.

Consider the special case where λ and μ are positive representable measures (for example if H is a separable space of dimension ≥ 3). Then we have

LEMMA 6.2. *Assume that λ and μ are Gleason measures represented by the trace class operators ϱ_1 and ϱ_2. Then $\lambda \ll \mu$ if and only if* $\mathrm{Ker}(\varrho_2) \subset \mathrm{Ker}(\varrho_1)$.

Proof. Assume first that $\lambda \ll \mu$, and let $x \in \mathrm{Ker}(\varrho_2)$. Then if $S = \langle x \rangle$ is the one-dimensional subspace generated by x, we have

$$\mu(S) = \mathrm{Tr}(\varrho_2 P_S) = \frac{\langle \varrho_2 x, x\rangle}{\|x\|^2} = 0.$$

Hence $\lambda(S) = 0$, and from

$$\lambda(S) = \mathrm{Tr}(\varrho_1 P_S) = \frac{\langle \varrho_1 x, x\rangle}{\|x\|^2}$$

we see that $\langle \varrho_1 x, x\rangle = 0$. Since ϱ_1 is a positive operator, it has a unique positive square root $\varrho_1^{1/2}$. It follows that $\|\varrho_1^{1/2} x\| = 0$, or $\varrho_1^{1/2} x = 0$. Hence $\varrho_1 x = 0$. Thus, we have shown that $\mathrm{Ker}(\varrho_2) \subset \mathrm{Ker}(\varrho_1)$.

Conversely, assume that $\mathrm{Ker}(\varrho_2) \subset \mathrm{Ker}(\varrho_1)$ and let $S \subset H$ be a closed subspace such that $\mu(S) = 0$. We want to show that $\lambda(S) = 0$. Consider an orthonormal basis $\{e_\alpha\}_{\alpha \in I}$ of S, and complete it to obtain an orthonormal basis $\{e_\alpha\}_{\alpha \in J}$ of H. Then

$$\mu(S) = \mathrm{Tr}(\varrho_2 P_S) = \sum_{\alpha \in J} \langle \varrho_2 P_S e_\alpha, e_\alpha\rangle = \sum_{\alpha \in I} \langle \varrho_2 e_\alpha, e_\alpha\rangle = 0.$$

Since ϱ_2 is a positive operator, it follows that

$$\langle \varrho_2 e_\alpha, e_\alpha\rangle = 0 \qquad \forall \alpha \in I.$$

As before, we deduce that $\varrho_2 e_\alpha = 0$ for all $\alpha \in I$ and then, by hypothesis, $\varrho_1 e_\alpha = 0$ for all $\alpha \in I$. It follows that

$$\lambda(S) = \mathrm{Tr}(\varrho_1 P_S) = \sum_{\alpha \in J} \langle \varrho_1 P_S e_\alpha, e_\alpha\rangle = \sum_{\alpha \in I} \langle \varrho_1 e_\alpha, e_\alpha\rangle = 0.$$

Hence we conclude that $\lambda \ll \mu$.

THEOREM 6.3. *Let λ, μ be two positive representable Gleason measures, and ϱ_1, ϱ_2 be their respective density operators, so that*

$$\lambda(S) = \mathrm{Tr}(\varrho_1 P_S), \qquad \mu(S) = \mathrm{Tr}(\varrho_2 P_S)$$

with ϱ_1, ϱ_2 positive operators. Assume that $\lambda \ll \mu$. Then there exists a (not necessarily bounded) self-adjoint operator A such that

$$\lambda(T) = \int A P_T \, d\mu$$

for any closed subspace T of H.

Remark 6.4. Since μ is by hypothesis a Gleason measure represented by a trace class operator, λ_{a} is defined for any closed subspace.

Remark 6.5. Under the assumptions of the lemma, λ and μ are positive Gleason measures. A similar result holds if λ is assumed to be a complex Gleason measure, represented by a normal operator ϱ_2. In that case, A should be a normal operator (see [6, XII.9.10] for the notion of a normal unbounded operator).

Proof of Theorem 6.3. Let S be the range of ϱ_2. We define a self-adjoint operator A by

$$Ax = \begin{cases} 0 & \text{if } x \in \mathrm{Ker}(\varrho_2) = S^\perp, \\ \varrho_1(\varrho_{2|S})^{-1}x & \text{if } x \in S, \end{cases}$$

or

$$A = \varrho_1(\varrho_{2|S})^{-1}P_S.$$

Therefore, the domain of A is

$$D(A) = S + S^\perp,$$

which is a dense subspace, and A is a (possibly) unbounded self-adjoint operator. It follows from the definition of A that

$$A\varrho_2 x = \varrho_1 x \quad \forall x \in H.$$

Let T be a closed subspace of H. Then

$$\lambda(T) = \mathrm{Tr}(\varrho_1 P_T) = \mathrm{Tr}(\varrho_1 P_{S^\perp} P_T) + \mathrm{Tr}(\varrho_1 P_S P_T).$$

Note that $P_{S^\perp} P_T x \in S^\perp = \mathrm{Ker}(\varrho_2) \subset \mathrm{Ker}(\varrho_1)$. Hence $\varrho_1 P_{S^\perp} P_T = 0$. Therefore

$$\lambda(T) = \mathrm{Tr}(\varrho_1 P_S P_T) = \mathrm{Tr}(A\varrho_2 P_S P_T) = \mathrm{Tr}(P_T A\varrho_2 P_S) = \mathrm{Tr}(P_T A P_S \varrho_2).$$

In order to check the last identity, we take an orthonormal basis $(e_i)_{i\in J}$ of S, and complete it to an orthonormal basis $(e_i)_{i\in I}$ of H. Then

$$\mathrm{Tr}(P_T A\varrho_2 P_S) = \sum_{i\in I}\langle P_T A\varrho_2 P_S e_i, e_i\rangle = \sum_{i\in J}\langle P_T A\varrho_2 e_i, e_i\rangle$$

since $P_S e_i = 0$ if $i \notin J$, and

$$\mathrm{Tr}(P_T A P_S \varrho_2) = \sum_{i\in I}\langle P_T A P_S \varrho_2 e_i, e_i\rangle = \sum_{i\in J}\langle P_T A\varrho_2 e_i, e_i\rangle$$

since $P_S\varrho_2 e_i = \varrho_2 e_i$ and $\varrho_2 e_i = 0$ if $i \notin J$.

Using Proposition 3.2, we conclude that

$$\lambda(T) = \mathrm{Tr}(\varrho_1 P_T) = \mathrm{Tr}(P_T A\varrho_2) = \mathrm{Tr}(\varrho_2 A P_T) = \int A P_T\, d\mu = \int_T A\, d\mu.$$

In order to prove that $\mathrm{Tr}(P_T A\varrho_2) = \mathrm{Tr}(\varrho_2 A P_T)$ we take a basis $\{e_i\}_{i\in J}$ of S such that $\varrho_2(e_i) = \lambda_i e_i$, and we complete it to a basis $\{e_i\}_{i\in I}$ of H with

$e_i \in D(A)$. Then we have

$$\begin{aligned}\mathrm{I} := \sum_{i\in I}\langle \varrho_2 A P_T(e_i), e_i\rangle &= \sum_{i\in I}\langle A P_T(e_i), \varrho_2(e_i)\rangle \\ = \sum_{i\in J}\langle A P_T(e_i), \lambda_i e_i\rangle &= \sum_{i\in J}\lambda_i\langle P_T(e_i), A(e_i)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{II} := \sum_{i\in I}\langle P_T A \varrho_2(e_i), e_i\rangle &= \sum_{i\in J}\langle P_T A(\lambda_i e_i), e_i\rangle \\ = \sum_{i\in J}\lambda_i\langle A(e_i), P_T(e_i)\rangle.\end{aligned}$$

and we get the desired equality: $\mathrm{I} = \mathrm{II}$.

Remark 6.6. Gleason's theorem can be seen as a version of the Radon–Nikodym theorem. In fact, consider the Gleason measure Δ given by

$$\Delta(S) = \dim(S).$$

It is clear that Δ is a non-negative Gleason measure (though it may take the value $+\infty$). Then if $A = \sum \lambda_i P_{S_i}$ is a simple self-adjoint operator,

$$\int A\, d\Delta = \sum \lambda_i \dim(S_i) = \mathrm{Tr}(A).$$

This identity also holds for any operator A of trace class (i.e. Δ-integrable). If μ is another Gleason measure, it is clear that μ is absolutely continuous with respect to Δ since $\mu(\{0\}) = 0$. Gleason's theorem says that there exists a self-adjoint operator ϱ such that

$$\mu(S) = \int \varrho P_S\, d\Delta$$

for any closed subspace S. The condition that H should be separable means that Δ should be a σ-finite Gleason measure (a hypothesis of the usual Radon–Nikodym theorem).

7. A quantum-mechanical interpretation. In this section we present some heuristic remarks on the quantum-mechanical interpretation of the integral of an operator with respect to a Gleason measure, and the application of the Radon–Nikodym theorem (Theorem 6.3 from the previous section) to define the conditional expectation of an observable with respect to another one.

Gleason measures have a natural quantum-mechanical interpretation. In fact, in quantum mechanics the space of possible (pure) states of a physical system corresponds to a Hilbert space H. Then Gleason probability measures on H (i.e. $0 \le \mu(S) \le 1$ and $\mu(I) = 1$) correspond to mixed states: the precise state of the system is not known, but some probability distribution

μ of the "observable events" is given (a closed subspace of H corresponds to observable events).

Then a self-adjoint operator corresponds to an observable, and the integral $\int A\,d\mu$ that we have introduced is the expected value $E_\mu(A)$ of the observable A (a random variable) in the mixed state μ.

Let A, B two observables (self-adjoint operators). Then we can give a meaning to $E_\mu(A \mid B)$ (the expected value of A in the B state μ).

We remark that the set $\mathcal{F}_B$ of operators that are functions of B is the set of observables that can be deduced from B. This set is a commutative subalgebra of $\mathcal{L}(H)$. We call the elements of $\mathcal{F}_B$ *B-observables.*

The conditional expectation $E_\mu(A \mid B)$ should be a B-observable (which means that its value should be known if the value of B is known). Moreover, if we compute

$$E(A_{|S}) = \int A P_S\,d\mu$$

for a B-invariant subspace (a B-observable event), then we should have

$$E(A_{|S}) = \int E_\mu(A \mid B) P_S\,d\mu.$$

If μ is a representable Gleason measure, the conditional expectation $E_\mu(A \mid B)$ can be defined using our version of the Radon–Nikodym theorem (Theorem 6.3). Indeed, if μ is a Gleason measure represented by a density operator ϱ, then $\lambda : \mathcal{S}_B \to \mathbb{R}$ (where $\mathcal{S}_B$ is the set of closed B-invariant subspaces) given by

$$\lambda(S) = \int A P_S\,d\mu$$

is a Gleason measure on $\mathcal{F}_B$, represented by the density operator ϱA, and the existence of $E_\mu(A \mid B)$ follows from the Radon–Nikodym theorem ([16]).

Pure states. Let $x \in H$ a pure state. We can define the Gleason measure δ_x (that corresponds to the mixed state such that the system is in state x with probability 1) by

$$\delta_x(S) = \langle P_S x, x \rangle$$

for any closed subspace $S \subset H$. Let us check that δ_x is a Gleason measure. If S is the closed linear span of an orthogonal family $(S_i)_{i\in\mathbb{N}}$ of subspaces then $P_S = \sum_{i=1}^{\infty} P_{S_i}$. Therefore

$$\delta_x(S) = \sum_{i=1}^{\infty} \langle P_{S_i} x, x \rangle = \sum_{i=1}^{\infty} \delta_x(S_i).$$

Let us compute $\int A\,d\delta_x$. First we consider the case of a simple operator $A = \sum_{i=1}^{n} \lambda_i P_{S_i}$. Then

$$\int A\,d\delta_x = \sum_{i=1}^{n} \lambda_i \delta_x(S_i) = \sum_{i=1}^{n} \lambda_i \langle P_{S_i}x, x\rangle = \Big\langle \sum_{i=1}^{n} \lambda_i P_{S_i}x, x\Big\rangle = \langle Ax, x\rangle.$$

This can be extended to any self-adjoint operator A.

So for Gleason measures corresponding to pure states, our definition of the expected value agrees with the usual one.

8. The non-separable case. In this section, we present an example showing that if H is not separable, Gleason's theorem does not hold.

Let H be a non-separable Hilbert space with an orthonormal basis $\{e_\alpha\}_{\alpha\in\mathbb{R}}$. We fix a function $f \in L^1(\mathbb{R})$ with $0 \le f(x) \le 1$ a.e. and $\int_{\mathbb{R}} f(x)\,dx = 1$ (a probability density on $\mathbb{R}$).

For a closed subspace $S \subset H$, define a Gleason measure

$$\mu(S) = \int \langle P_S e_\alpha, e_\alpha\rangle f(\alpha)\,d\alpha.$$

There is, however, a difficulty here: the function $\langle P_S e_\alpha, e_\alpha\rangle$ may not be measurable. In fact, let $V \subset \mathbb{R}$ be the Vitali set and S the closed linear span of $\{e_\alpha : \alpha \in V\}$. Then $\langle P_S e_\alpha, e_\alpha\rangle$ is the characteristic function of the Vitali set, hence it is not Lebesgue measurable.

So we restrict our attention to the class $\mathcal{S}_{\mathrm{m}}$ of closed subspaces of H such that the function $\langle P_S e_\alpha, e_\alpha\rangle$ is Lebesgue measurable (as a function of α); we call them *measurable subspaces*. This class clearly depends on the choice of the orthonormal basis $\{e_\alpha\}$.

We see that if $S \in \mathcal{S}_{\mathrm{m}}$, then $\mu(S)$ is well defined and is a Gleason measure on $\mathcal{S}_{\mathrm{m}}$ since if S is the orthogonal direct sum of a countable family $(S_n)_{n\in\mathbb{N}}$ of measurable subspaces, then S is measurable and $P_S = \sum_{n\in\mathbb{N}} P_{S_n}$ (in the strong operator topology). Hence

$$\begin{aligned}\mu(P_S) &= \int \langle P_S e_\alpha, e_\alpha\rangle f(\alpha)\,d\alpha = \int \sum_{n\in\mathbb{N}} \langle P_{S_n} e_\alpha, e_\alpha\rangle f(\alpha)\,d\alpha \\ &= \sum_{n\in\mathbb{N}} \int \langle P_{S_n} e_\alpha, e_\alpha\rangle f(\alpha)\,d\alpha = \sum_{n\in\mathbb{N}} \mu(S_n).\end{aligned}$$

We claim that it is not possible to find a trace class operator ϱ such that

$$\mu(S) = \mathrm{Tr}(\varrho P_S) \tag{6}$$

for every measurable subspace $S \subset H$. So Gleason's theorem does not hold in the non-separable case. To see this, note that if ϱ is a trace class operator then

$$\mathrm{Tr}(\varrho) = \sum_{\alpha} \langle \varrho e_\alpha, e_\alpha\rangle < \infty.$$

Hence $\langle \varrho e_\alpha, e_\alpha\rangle \ne 0$ for an at most countable set of values of α. Let S be the closed linear span of $\{e_\alpha : \langle \varrho e_\alpha, e_\alpha\rangle = 0\}$. Then $S^\perp$ is the closed linear span of $\{e_\alpha : \langle \varrho e_\alpha, e_\alpha \rangle \neq 0\}$. We see that $P_{S^\perp} e_\alpha = 0$ if $\langle \varrho e_\alpha, e_\alpha \rangle = 0$, and this happens for almost all $\alpha \in \mathbb{R}$, hence

$$\mu(S^\perp) = \int \langle P_{S^\perp} e_\alpha, e_\alpha \rangle f(\alpha)\, d\alpha = 0.$$

We shall show that $\varrho P_S = 0$. In fact, if $x \in S$ then $x = \sum_\alpha x_\alpha e_\alpha$ with $x_\alpha = 0$ if $\langle \varrho e_\alpha, e_\alpha \rangle \neq 0$, hence

$$\langle \varrho x, x \rangle = \sum_\alpha x_\alpha^2 \langle \varrho e_\alpha, e_\alpha \rangle = 0.$$

Since $\varrho \geq 0$, there exists a positive self-adjoint square root $\varrho^{1/2}$ and we have

$$\langle \varrho x, x \rangle = \|\varrho^{1/2} x\|^2 = 0.$$

It follows that $\varrho^{1/2} x = 0$ and so $\varrho x = 0$. Hence, we have shown that if $x \in S$ then $\varrho x = 0$. Therefore $\varrho P_S = 0$ and thus $\mu(S) = \mathrm{Tr}(\varrho P_S) = 0$. We conclude that

$$\mu(I) = \mu(S) + \mu(S^\perp) = 0;$$

but $\mu(I) = \int_{\mathbb{R}} f(\alpha)\, d\alpha = 1$. This contradiction shows that the Gleason measure μ cannot be represented in the form (6).

A more general construction. In this subsection, we present a more general construction of Gleason measures in the non-separable case and we conjecture a more general version of Gleason's theorem that may apply to these examples.

Let H be a Hilbert space. We assume that we are given a finite measurable space $(X, \mathcal{M}, \mu)$ and a function $e : X \to H$ such that $e(\alpha) = e_\alpha$ is an orthonormal basis of H. We call an operator $A : H \to H$ *measurable* if the function $\langle A e_\alpha, e_\alpha \rangle$ is measurable (i.e. it belongs to the σ-algebra $\mathcal{M}$). We call a subspace S *measurable* if the projector P_S is measurable.

Then for a measurable operator $\varrho : H \to H$ we may define the *generalized trace* (or *μ-trace*) by

$$\mathrm{Tr}_\mu(\varrho) = \int \langle \varrho e_\alpha, e_\alpha \rangle\, d\mu.$$

We remark that if $X = \mathbb{N}$, $\mathcal{M}$ is the σ-algebra of all subsets of $\mathbb{N}$ and μ is the counting measure, we get the usual definition of trace. Note also that this generalized trace depends on the choice of the orthonormal basis $\{e_\alpha\}$.

We can define a Gleason measure $\widetilde{\mu}$ on the measurable subspaces by

$$\widetilde{\mu}(S) = \mathrm{Tr}_\mu(\varrho P_S) = \int \langle P_S e_\alpha, e_\alpha \rangle\, d\mu.$$

We can ask if all Gleason measures arise this way for a suitable choice of the measure μ. An affirmative answer would give a generalization of Gleason's theorem to the non-separable case.

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Departamento de Matemática
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
1428 Buenos Aires, Argentina
E-mail: pdnapo@dm.uba.ar

Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-0001, U.S.A.
E-mail: mmariani@nmsu.edu

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