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Joint subnormality of n-tuples and C_0-semigroups of composition operators on L^2-spaces

by

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Abstract. Joint subnormality of a family of composition operators on L^2-space is characterized by means of positive definiteness of appropriate Radon–Nikodym derivatives. Next, simplified positive definiteness conditions guaranteeing joint subnormality of a C_0-semigroup of composition operators are supplied. Finally, the Radon–Nikodym derivatives associated to a jointly subnormal C_0-semigroup of composition operators are shown to be the Laplace transforms of probability measures (modulo a C_0-group of scalars) constituting a measurable family.

1. Introduction. The theory of subnormal operators is a vital part of Operator Theory (cf. [6]). The notion of a subnormal operator was introduced by Halmos in [12]. Roughly speaking, a subnormal operator is a restriction of a normal one to its invariant subspace. Halmos himself gave in [12] a two-condition criterion for subnormality of a single (bounded) operator. It was successively simplified by Bram (cf. [4]), Embry (cf. [10]) and Lambert (cf. [16]). In [15] Itô solved the problem of extending a family of commuting operators acting in a Hilbert space $\mathcal{H}$ to a family of commuting normal operators acting in a possibly larger Hilbert space $\mathcal{K}$. In particular, Itô proved that any C_0-semigroup of subnormal operators has an extension which is a C_0-semigroup of normal operators. This in turn enabled Nussbaum (cf. [23]) to show that the infinitesimal generator of a C_0-semigroup of subnormal operators is a subnormal operator (in general unbounded). A multioperator counterpart of the Embry–Lambert characterization of subnormality was proved by Lubin in [20].

The foundations of the theory of composition operators in abstract L^2-spaces are well developed. In particular, the questions of boundedness, nor-

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mality, quasinormality, subnormality, hyponormality etc. of such operators are entirely solved (cf. [9, 22, 27, 14, 8, 18, 19, 5]; see also [21, 25, 7] for special classes of composition operators). The present paper offers criteria, written in terms of Radon–Nikodym derivatives, for joint subnormality of n-tuples as well as C_0-semigroups of composition operators on L^2-spaces (see Theorem 3.4, Lemma 4.4 and Corollary 4.6). This generalizes in various ways Lambert's characterization of subnormality of a single composition operator (cf. [18]). For a particular class of composition operators induced by square matrices, joint subnormality is completely characterized by algebraic properties of symbols (cf. Theorem 3.6). It is shown that for every real $t \geq 0$, the Radon–Nikodym derivative h_t^{ϕ} attached to a jointly subnormal C_0-semigroup of composition operators $\{C_{\phi_u}\}_{u\geq 0}$ can be modified so as to coincide (modulo a C_0-group of scalars) with the Laplace transforms calculated at t of a measurable family of probability Borel measures, the family being independent of t (cf. Theorem 4.5). The paper concludes with an example of a C_0-semigroup of composition operators $\{C_{\phi_t}\}_{t\geq 0}$ which is not jointly subnormal, though the operator C_{ϕ_1} is subnormal. This shows that the criteria for joint subnormality contained in Lemma 4.4 are optimal in a sense.

A subsequent paper will be devoted to a general study of joint subnormality of C_0-groups of composition operators.

2. Preliminaries. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers, by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}_+$ the set of all nonnegative real numbers. If Q is a subset of $\mathbb{C}$ containing 0, then $Q^{(\mathbb{Z}_+^n)}$ stands for the set of all functions $\lambda\colon \mathbb{Z}_+^n \to Q$ for which the set $\lambda^{-1}(Q \setminus \{0\})$ is finite.

We say that an n-sequence $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ of real numbers is a *Stieltjes moment n-sequence* if there exists a positive Borel measure μ on $\mathbb{R}_+^n$ such that

$$t_\alpha = \int_{\mathbb{R}_+^n} s^\alpha \, d\mu(s), \qquad \alpha \in \mathbb{Z}_+^n; \tag{2.1}$$

such a μ is called a *representing measure* for $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$. If (2.1) holds and the closed support of μ is contained in a closed subset F of $\mathbb{R}_+^n$, then we say that $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ is a *Stieltjes moment n-sequence on F*. Let us recall a useful characterization of Stieltjes moment n-sequences on compact sets. Below $e_j = (\delta_{j,1}, \ldots, \delta_{j,n})$ for $j = 1, \ldots, n$, where $\delta_{k,l}$ stands for the Kronecker symbol (for simplicity, we suppress the dependence of e_j on n in the notation).

Theorem 2.1 ([26, Theorem 3]). *Assume that an n-sequence $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n} \subseteq \mathbb{R}$ satisfies the following three conditions*:

(i) $\sum_{\alpha,\beta\in\mathbb{Z}_+^n} t_{\alpha+\beta}\lambda(\alpha)\overline{\lambda(\beta)} \geq 0$ *for all* $\lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}$,

(ii) $\sum_{\alpha,\beta\in\mathbb{Z}_+^n} t_{\alpha+\beta+e_j}\lambda(\alpha)\overline{\lambda(\beta)} \geq 0$ *for all* $\lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}$ *and* $j = 1, \ldots, n$,

(iii) *there exists an n-tuple $(r_1, \ldots, r_n)$ of nonnegative real numbers such that*

$$t_{2\alpha+2e_j} \le r_j^2 t_{2\alpha}, \quad \alpha \in \mathbb{Z}_+^n,\ j = 1, \ldots, n.$$

Then $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ is a Stieltjes moment n-sequence on a compact subset of $\mathbb{R}_+^n$. Moreover, a representing measure μ for $\{t_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ is unique and its closed support is contained in the rectangle $[0, r_1] \times \cdots \times [0, r_n]$. If $[0, R_1] \times \cdots \times [0, R_n]$ is the least rectangle containing the closed support of μ, then

$$R_j = \lim_{n\to\infty} t_{2ne_j}^{1/2n}, \quad j = 1, \ldots, n.$$

It follows from Theorem 2.1 that a Stieltjes moment n-sequence which has a representing measure with compact support is *determinate*, i.e. the representing measure is unique (within the class of all Borel measures not necessarily compactly supported, cf. [11]).

A bounded (linear) operator S on a (complex) Hilbert space $\mathcal{H}$ is called *subnormal* if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a bounded normal operator N on $\mathcal{K}$ such that $S \subseteq N$, i.e. $Sh = Nh$ for all $h \in \mathcal{H}$. We say that a family $\{S_\omega : \omega \in \Omega\}$ of bounded operators on $\mathcal{H}$ is *jointly subnormal* if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a family $\{N_\omega : \omega \in \Omega\}$ of commuting bounded normal operators on $\mathcal{K}$ such that $S_\omega \subseteq N_\omega$ for all $\omega \in \Omega$. It is clear that a jointly subnormal family $\{S_\omega : \omega \in \Omega\}$ is commutative.

THEOREM 2.2 ([15]). *A family $\{S_\omega : \omega \in \Omega\}$ of bounded operators on a Hilbert space $\mathcal{H}$ is jointly subnormal if and only if for every finite subset Ω' of Ω the family $\{S_\omega : \omega \in \Omega'\}$ is jointly subnormal.*

Let us recall the Embry–Lambert–Lubin criterion for joint subnormality (cf. [20]): an n-tuple $\boldsymbol{S} = (S_1, \ldots, S_n)$ of commuting bounded operators on a Hilbert space $\mathcal{H}$ is jointly subnormal if and only if

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \|\boldsymbol{S}^{\alpha+\beta} f\|^2 \lambda(\alpha)\overline{\lambda(\beta)} \ge 0, \quad \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},\ f \in \mathcal{H}, \tag{2.2}$$

where $\boldsymbol{S}^\alpha = S_1^{\alpha_1} \cdots S_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$.

3. Families of composition operators. Let (X, Σ, μ) be a σ-finite measure space. Consider a Σ-measurable transformation $\phi : X \to X$ such that the measure $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ. Then the operator $C_\phi : L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}, \quad C_\phi f = f \circ \phi \quad \text{for } f \in \mathcal{D}(C_\phi),$$

is well-defined and linear. We call it the *composition operator* induced by ϕ. We also say that ϕ is the *symbol* of C_ϕ. For every $n \in \mathbb{Z}_+$, we set

$$h_n^\phi = \frac{d\mu \circ (\phi^n)^{-1}}{d\mu}. \tag{3.1}$$

Notice that $h_0^\phi = 1$ a.e. $[\mu]$. Recall that C_ϕ is a bounded operator on $L^2(\mu)$ if and only if $h_1^\phi \in L^\infty(\mu)$. If $\psi\colon X \to X$ is a Σ-measurable transformation such that the mapping $L^2(\mu) \ni f \mapsto f \circ \psi \in L^2(\mu)$ is well-defined, then the measure $\mu \circ \psi^{-1}$ is absolutely continuous with respect to μ and

$$\|C_\psi\| = \|h_1^\psi\|_\infty^{1/2}, \tag{3.2}$$

where $\|h_1^\psi\|_\infty$ stands for the $L^\infty(\mu)$-norm of h_1^ψ. The interested reader is referred to [9] and [22] for further information on composition operators.

Consider now an n-tuple $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ of Σ-measurable transformations of X. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we define the measure $\mu_\alpha^{\boldsymbol{\phi}}$ on X by

$$\mu_\alpha^{\boldsymbol{\phi}}(\sigma) = \mu((\boldsymbol{\phi}^\alpha)^{-1}(\sigma)), \quad \sigma \in \Sigma,$$

where $\boldsymbol{\phi}^\alpha := \phi_1^{\alpha_1} \circ \cdots \circ \phi_n^{\alpha_n}$. It is a matter of routine to show that if the measures $\mu \circ \phi_j^{-1}$, $1 \le j \le n$, are absolutely continuous with respect to μ, then so is $\mu_\alpha^{\boldsymbol{\phi}}$ for every $\alpha \in \mathbb{Z}_+^n$. As a consequence, we may write the Radon–Nikodyn derivatives

$$h_\alpha^{\boldsymbol{\phi}} = \frac{d\mu_\alpha^{\boldsymbol{\phi}}}{d\mu}, \quad \alpha \in \mathbb{Z}_+^n,$$

and consider the composition operators C_{ϕ_j} in $L^2(\mu)$ for $j = 1, \ldots, n$. If no confusion can arise, we write μ_α and h_α instead of $\mu_\alpha^{\boldsymbol{\phi}}$ and $h_\alpha^{\boldsymbol{\phi}}$, respectively.

We now investigate under what conditions the equality $C_\phi = C_\psi$ holds.

LEMMA 3.1. *Assume that ϕ and ψ are Σ-measurable transformations of X inducing bounded composition operators C_ϕ and C_ψ on $L^2(\mu)$.*

(i) *If $\phi = \psi$ a.e. $[\mu]$ ([1]), then $C_\phi = C_\psi$.*

(ii) *If $C_\phi = C_\psi$, then $\mu \circ (\phi^n)^{-1} = \mu \circ (\psi^n)^{-1}$ and $h_n^\phi = h_n^\psi$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$.*

(iii) *$C_\phi \neq C_\psi$ if and only if there exist sets $Y, Z \in \Sigma$ such that $Y \cap Z = \emptyset$ and $\mu(\phi^{-1}(Y) \cap \psi^{-1}(Z)) > 0$.*

Proof. (i) is obvious.

(ii) If $\sigma \in \Sigma$ and $\mu(\sigma) < \infty$, then the characteristic function χ_σ of σ is in $L^2(\mu)$ and, by the measure transport theorem ([13, Theorem C, p. 163]),

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([1]) Note that in general the set $\{x \in X\colon \phi(x) \neq \psi(x)\}$ may not belong to Σ (see Example 3.2). Hence $\phi = \psi$ a.e. $[\mu]$ is understood to mean that there exists a set $Y \in \Sigma$ of full μ-measure such that $\phi(x) = \psi(x)$ for all $x \in Y$.

we have

$$\int_\sigma h_n^\phi \, d\mu = \|C_{\phi^n}\chi_\sigma\|^2 = \|C_\phi^n\chi_\sigma\|^2 = \|C_\psi^n\chi_\sigma\|^2 = \|C_{\psi^n}\chi_\sigma\|^2 = \int_\sigma h_n^\psi \, d\mu.$$

Since μ is σ-finite, we get $h_n^\phi = h_n^\psi$ a.e. $[\mu]$, which implies $\mu \circ (\phi^n)^{-1} = \mu \circ (\psi^n)^{-1}$.

(iii) To prove the "if" part of (iii), set $E = \phi^{-1}(Y) \cap \psi^{-1}(Z)$. Since the measure μ is σ-finite, there exists a Σ-measurable function $f\colon X \to \mathbb{R}_+$ such that $f(x) > 0$ for every $x \in Y$, $f(x) = 0$ for every $x \in X \setminus Y$ and $\int_X |f(x)|^2 \, d\mu(x) < \infty$. Combining this with the inclusions $\phi(E) \subseteq Y$ and $\psi(E) \subseteq Z \subseteq X \setminus Y$, we see that $f(\phi(x)) > 0$ and $f(\psi(x)) = 0$ for every $x \in E$. Since $\mu(E) > 0$, we get $C_\phi f \neq C_\psi f$.

Suppose now that $C_\phi f \neq C_\psi f$ for some $f \in L^2(\mu)$. Since simple functions belonging to $L^2(\mu)$ are dense in $L^2(\mu)$ and the operators C_ϕ and C_ψ are continuous, we deduce that there exists a simple function $h \in L^2(\mu)$ such that $C_\phi h \neq C_\psi h$. Then the set $F := \{x \in X\colon h(\phi(x)) \neq h(\psi(x))\}$ is in Σ and $\mu(F) > 0$. Since h is a simple function, it is of the form $h = \sum_{k=1}^n \alpha_k \chi_{Y_k}$, where $n \in \mathbb{N}$, $\{\alpha_k\}_{k=1}^n$ is a sequence of distinct complex numbers and $\{Y_k\}_{k=1}^n$ is a Σ-measurable partition of X. Clearly, $\{\phi^{-1}(Y_k) \cap \psi^{-1}(Y_l)\}_{k,l=1}^n$ is a Σ-measurable partition of X and (2)

$$F = \bigcup_{\substack{k,l=1 \\ k \neq l}}^n \phi^{-1}(Y_k) \cap \psi^{-1}(Y_l).$$

Since $\mu(F) > 0$, we conclude that there exist $k, l \in \{1, \ldots, n\}$ such that $k \neq l$ and $\mu(\phi^{-1}(Y_k) \cap \psi^{-1}(Y_l)) > 0$. This completes the proof. ∎

Note that if the sets Y and Z are as in (iii) of Lemma 3.1, then $\mu(Y) > 0$ and $\mu(Z) > 0$ (use the fact that $\mu \circ \phi^{-1} \ll \mu$ and $\mu \circ \psi^{-1} \ll \mu$).

EXAMPLE 3.2. It is not true in general that the equality $C_\phi = C_\psi$ implies $\phi = \psi$ a.e. $[\mu]$. This can be illustrated by various examples built on σ-algebras generated by finite (or infinite) partitions of a nonempty set X. Here is a sample of what is possible in this matter. Consider the set $X = \{1, 2, 3, 4, 5\}$, the σ-algebra (= algebra) Σ generated by the partition $\{1, 2\}, \{3\}, \{4, 5\}$ of X, and a finite positive measure μ on Σ such that $\mu(\{1, 2\}) > 0$, $\mu(\{3\}) > 0$ and $\mu(\{4, 5\}) > 0$. Let ϕ and ψ be the transformations of X given by $\phi(1) = 4$, $\phi(2) = 5$, $\phi(3) = 5$, $\psi(1) = 5$, $\psi(2) = 5$, $\psi(3) = 4$ and $\phi(k) = \psi(k) = k$ for $k = 4, 5$. Then ϕ and ψ are Σ-measurable transformations of X such that C_ϕ and C_ψ are well-defined on $L^2(\mu)$ and $C_\phi = C_\psi$, though the equality $\phi = \psi$ a.e. $[\mu]$ does not hold; in this particular case the set $\{x \in X\colon \phi(x) \neq \psi(x)\}$ does not belong to Σ.

(2) Note that $C_\phi h \neq C_\psi h$ implies $n \geq 2$.

COROLLARY 3.3. *Let X be a topological Hausdorff space, Σ be a σ-algebra of all Borel subsets of X and μ be a σ-finite positive Borel measure on X which is inner regular* ([3]) *with respect to compact sets. Assume that ϕ and ψ are continuous transformations of X inducing bounded composition operators C_ϕ and C_ψ on $L^2(\mu)$. Then $C_\phi = C_\psi$ if and only if $\phi = \psi$ a.e. $[\mu]$. Moreover, if $\mu(U) > 0$ for every nonempty open subset U of X, then $C_\phi = C_\psi$ if and only if $\phi = \psi$.*

Proof. We only have to show that $C_\phi = C_\psi$ implies $\phi = \psi$ a.e. $[\mu]$ (the "moreover" part is a direct consequence of this implication). Suppose, contrary to our claim, that $\mu(X_0) > 0$, where $X_0 = \{x \in X \colon \phi(x) \neq \psi(x)\}$ (as X is Hausdorff, the set $X \setminus X_0$ is closed). Take $x \in X_0$. Since X is Hausdorff, there exist open neighbourhoods Y_x and Z_x of $\phi(x)$ and $\psi(x)$ respectively such that $Y_x \cap Z_x = \emptyset$. Then $E_x := \phi^{-1}(Y_x) \cap \psi^{-1}(Z_x)$ is an open neighbourhood of x and $E_x \subseteq X_0$. This implies that $X_0 = \bigcup_{x \in X_0} E_x$. In view of Lemma 3.1(iii), it is enough to show that there exists $x_0 \in X_0$ such that $\mu(E_{x_0}) > 0$. Suppose, contrary to our claim, that $\mu(E_x) = 0$ for every $x \in X_0$. If K is a compact subset of X_0, then there exists a finite subset $\{x_1, \ldots, x_n\}$ of X_0 such that $K \subseteq \bigcup_{k=1}^n E_{x_k}$. This implies that $\mu(K) = 0$. It follows from the inner regularity of μ that $\mu(X_0) = 0$, a contradiction. This completes the proof. ■

Jointly subnormal n-tuples of composition operators can be characterized as follows (see [18] for a single operator case).

THEOREM 3.4. *An n-tuple $(C_{\phi_1}, \ldots, C_{\phi_n})$ of commuting bounded composition operators on $L^2(\mu)$ is jointly subnormal if and only if one of the following three equivalent conditions holds*:

(i) *for μ-almost every $x \in X$,*

$$\sum_{\alpha,\beta \in \mathbb{Z}_+^n} h_{\alpha+\beta}(x)\lambda(\alpha)\overline{\lambda(\beta)} \geq 0 \quad \textit{for all } \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},$$

(ii) *for μ-almost every $x \in X$, $\{h_\alpha(x)\}_{\alpha \in \mathbb{Z}_+^n}$ is a Stieltjes moment n-sequence*,

(iii) *for μ-almost every $x \in X$, $\{h_\alpha(x)\}_{\alpha \in \mathbb{Z}_+^n}$ is a Stieltjes moment n-sequence on the compact set $[0, \|C_{\phi_1}\|^2] \times \cdots \times [0, \|C_{\phi_n}\|^2]$.*

Proof. Set $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ and $C_{\boldsymbol{\phi}} = (C_{\phi_1}, \ldots, C_{\phi_n})$. Applying the commutativity of $C_{\boldsymbol{\phi}}$ and the measure transport theorem, we get

$$\|C_{\boldsymbol{\phi}}^\alpha f\|^2 = \|C_{\boldsymbol{\phi}^\alpha} f\|^2 = \int |f|^2 h_\alpha \, d\mu, \quad f \in L^2(\mu),\ \alpha \in \mathbb{Z}_+^n. \tag{3.3}$$

([3]) We do not assume that μ is finite on compact subsets of X.

Suppose that $C_{\boldsymbol{\phi}}$ is jointly subnormal. By (2.2) and (3.3), we have

$$0 \le \sum_{\alpha,\beta\in\mathbb{Z}_+^n} \|C_{\boldsymbol{\phi}}^{\alpha+\beta} f\|^2 \lambda(\alpha)\overline{\lambda(\beta)} = \int |f|^2 g_\lambda \, d\mu, \quad \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},\ f \in L^2(\mu), \tag{3.4}$$

where $g_\lambda = \sum_{\alpha,\beta\in\mathbb{Z}_+^n} h_{\alpha+\beta}\lambda(\alpha)\overline{\lambda(\beta)}$. Since f is an arbitrary member of $L^2(\mu)$ and μ is σ-finite, we deduce that $g_\lambda \ge 0$ a.e. $[\mu]$ for all $\lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}$. Hence

$$\mu(X \setminus g_\lambda^{-1}(\mathbb{R}_+)) = 0, \quad \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}. \tag{3.5}$$

Let Q be any countable dense subset of $\mathbb{C}$ containing 0. Set

$$\tau = \bigcap_{\lambda\in Q^{(\mathbb{Z}_+^n)}} g_\lambda^{-1}(\mathbb{R}_+).$$

It follows from (3.5) that

$$\mu(X \setminus \tau) = 0. \tag{3.6}$$

Since Q is dense in $\mathbb{C}$ and $g_\lambda(x) \ge 0$ for all $x \in \tau$ and $\lambda \in Q^{(\mathbb{Z}_+^n)}$, we see that

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} h_{\alpha+\beta}(x)\lambda(\alpha)\overline{\lambda(\beta)} \ge 0, \quad x \in \tau,\ \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}. \tag{3.7}$$

Repeating the above reasoning with $f \circ \phi_j$ in place of f, we get

$$\begin{gathered} \mu(X \setminus \tau_j) = 0, \quad j = 1, \dots, n, \\ \sum_{\alpha,\beta\in\mathbb{Z}_+^n} h_{\alpha+\beta+e_j}(x)\lambda(\alpha)\overline{\lambda(\beta)} \ge 0, \quad x \in \tau_j,\ \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},\ j = 1, \dots, n, \end{gathered} \tag{3.8}$$

where $\tau_j = \bigcap_{\lambda\in Q^{(\mathbb{Z}_+^n)}} g_{j,\lambda}^{-1}(\mathbb{R}_+)$ with $g_{j,\lambda} = \sum_{\alpha,\beta\in\mathbb{Z}_+^n} h_{\alpha+\beta+e_j}\lambda(\alpha)\overline{\lambda(\beta)}$. Moreover, by (3.3), the following inequality holds for all $f \in L^2(\mu)$, $\alpha \in \mathbb{Z}_+^n$ and $j = 1, \dots, n$:

$$\int |f|^2 h_{2\alpha+2e_j}\, d\mu = \|C_{\boldsymbol{\phi}}^{2\alpha+2e_j} f\|^2 \le \|C_{\phi_j}\|^4 \|C_{\boldsymbol{\phi}}^{2\alpha} f\|^2 = \|C_{\phi_j}\|^4 \int |f|^2 h_{2\alpha}\, d\mu.$$

By σ-finiteness of μ this implies that for μ-almost every $x \in X$,

$$h_{2\alpha+2e_j}(x) \le \|C_{\phi_j}\|^4 h_{2\alpha}(x), \quad \alpha \in \mathbb{Z}_+^n,\ j = 1, \dots, n. \tag{3.9}$$

Combining (3.6)–(3.9), we conclude that for μ-almost every $x \in X$, the n-sequence $\{h_\alpha(x)\}_{\alpha\in\mathbb{Z}_+^n}$ satisfies the assumptions of Theorem 2.1. Hence condition (iii) holds.

Implications (iii)⇒(ii) and (ii)⇒(i) are clear.

If (i) holds, then we can go back from (3.6) and (3.7) to (3.4). Applying the Embry–Lambert–Lubin criterion completes the proof. ∎

Consider now a positive Borel measure μ on $\mathbb{R}^\varkappa$ of the form $d\mu = \varrho\, d\nu_\varkappa$, where $\varrho : \mathbb{R}^\varkappa \to [0, \infty)$ is a Borel function and $\nu_\varkappa$ is the $\varkappa$-dimensional Lebesgue measure. It is left to the reader to check that μ is σ-finite and inner regular with respect to compact sets. Assume that $\nu_{\varkappa}(\varrho^{-1}(\{0\})) = 0$. Suppose that $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ is an n-tuple of invertible linear transformations of $\mathbb{R}^{\varkappa}$ such that the composition operators $C_{\phi_1}, \ldots, C_{\phi_n}$ are bounded on $L^2(\varrho d\nu_{\varkappa})$. Write $\boldsymbol{\phi}^{\alpha} = \phi_1^{\alpha_1} \cdots \phi_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$.

Corollary 3.5. *Let ϱ and $\boldsymbol{\phi}$ be as above. The n-tuple $(C_{\phi_1}, \ldots, C_{\phi_n})$ is jointly subnormal if and only if one of the following three equivalent conditions holds*:

1° *the transformations $\phi_1, \ldots, \phi_n$ commute and for $\nu_{\varkappa}$-almost every x in $\mathbb{R}^{\varkappa}$,*
$$\sum_{\alpha,\beta \in \mathbb{Z}_+^n} \varrho(\boldsymbol{\phi}^{-(\alpha+\beta)}(x))\lambda(\alpha)\overline{\lambda(\beta)} \geq 0 \quad \text{for all } \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},$$

2° *the transformations $\phi_1, \ldots, \phi_n$ commute and for $\nu_{\varkappa}$-almost every x in $\mathbb{R}^{\varkappa}$, $\{\varrho(\boldsymbol{\phi}^{-\alpha}(x))\}_{\alpha \in \mathbb{Z}_+^n}$ is a Stieltjes moment n-sequence,*

3° *the transformations $\phi_1, \ldots, \phi_n$ commute and for $\nu_{\varkappa}$-almost every x in $\mathbb{R}^{\varkappa}$, $\{\varrho(\boldsymbol{\phi}^{-\alpha}(x))\}_{\alpha \in \mathbb{Z}_+^n}$ is a Stieltjes moment n-sequence on the compact set $[0, \|C_{\phi_1}\|^2] \times \cdots \times [0, \|C_{\phi_n}\|^2]$.*

Moreover, if $(C_{\phi_1}, \ldots, C_{\phi_n})$ is jointly subnormal and $\sigma \neq \emptyset$ is an open subset of $\mathbb{R}^{\varkappa}$ such that ϱ is positive and continuous on (4) *σ and $\phi_j(\sigma) = \sigma$ for all $j = 1, \ldots, n$, then 1°–3° hold with "for $\nu_{\varkappa}$-almost every $x \in \mathbb{R}^{\varkappa}$" replaced by "for every $x \in \sigma$".*

Proof. By the assumption on ϱ, the measures μ and $\nu_{\varkappa}$ are mutually absolutely continuous. Clearly, $\nu_{\varkappa}$ does not vanish on nonempty open subsets of $\mathbb{R}^{\varkappa}$ and so neither does μ. Since μ is inner regular with respect to compact sets, we deduce from Corollary 3.3 that the operators $C_{\phi_1}, \ldots, C_{\phi_n}$ commute if and only if the transformations $\phi_1, \ldots, \phi_n$ commute.

It is a matter of routine to verify that

$$h_{\alpha} = \frac{\varrho \circ \boldsymbol{\phi}^{-\alpha}}{\varrho |\det \boldsymbol{\phi}|^{\alpha}} \quad \text{a.e. } [\mu],\ \alpha \in \mathbb{Z}_+^n, \tag{3.10}$$

where $|\det \boldsymbol{\phi}| = (|\det \phi_1|, \ldots, |\det \phi_n|)$. This enables us to show that conditions 1°–3° correspond to conditions (i)–(iii) of Theorem 3.4 respectively.

For the proof of the "moreover" part, notice that in view of (3.10) all the Radon–Nikodym derivatives h_{α}, $\alpha \in \mathbb{Z}_+^n$, are continuous on σ. This, the mutual absolute continuity of μ and $\nu_{\varkappa}$, and the fact that $\nu_{\varkappa}$ does not vanish on nonempty open subsets of $\mathbb{R}^{\varkappa}$ imply that the inequalities in (3.7)–(3.9) are valid for all $x \in \sigma$. Hence the same argument as in the proof of Theorem 3.4 yields the conclusion. ∎

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(4) This part of the conclusion of Corollary 3.5 is patterned upon Proposition 2.4 of [25]. We take this opportunity to mention that the density function r appearing in Proposition 2.4 of [25] has to be assumed to be positive on the set σ.

We conclude this section with a generalization of [25, Theorem 2.5] to the case of families of composition operators. Let $\|\cdot\|$ be a norm on $\mathbb{R}^\varkappa$ induced by an inner product. Denote by $\mathcal{R}_{\|\cdot\|}$ the class of all functions $\varrho\colon \mathbb{R}^\varkappa \to [0,\infty)$ of the form

$$\varrho(x) = \sum_{m=0}^{\infty} a_m \|x\|^{2m}, \qquad x \in \mathbb{R}^\varkappa,$$

where a_m are nonnegative real numbers and $a_k > 0$ for some $k \geq 1$. A density function $\varrho \in \mathcal{R}_{\|\cdot\|}$ is said to be of *polynomial type* if there exists $k \geq 2$ such that $a_m = 0$ for all $m \geq k$. We refer the reader to [25, Proposition 2.2] for a criterion which guarantees the boundedness of the composition operator C_ϕ on $L^2(\varrho d\nu_\varkappa)$ (resp. on $L^2((1/\varrho)d\nu_\varkappa)$), where ϕ is an invertible linear transformation of $\mathbb{R}^\varkappa$.

THEOREM 3.6. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^\varkappa$ induced by an inner product, ϱ be a member of $\mathcal{R}_{\|\cdot\|}$ and $\mathfrak{A}$ be a nonempty family of invertible linear transformations of $\mathbb{R}^\varkappa$ inducing bounded composition operators $\{C_\phi\colon \phi \in \mathfrak{A}\}$ on $L^2(\varrho d\nu_\varkappa)$ (resp. on $L^2((1/\varrho)d\nu_\varkappa)$). Then the family $\{C_\phi\colon \phi \in \mathfrak{A}\}$ (resp. $\{C_\phi^*\colon \phi \in \mathfrak{A}\}$) is jointly subnormal if and only if $\mathfrak{A}$ consists of commuting normal operators in $(\mathbb{R}^\varkappa, \|\cdot\|)$.*

Proof. If $\{C_\phi\colon \phi \in \mathfrak{A}\}$ is jointly subnormal, then by Corollary 3.5, $\mathfrak{A}$ is commutative, and by Theorem 2.5 of [25] each $\phi \in \mathfrak{A}$ is normal in $(\mathbb{R}^\varkappa, \|\cdot\|)$.

In view of Theorem 2.2, the proof of the converse reduces to the case of $\mathfrak{A}$ finite, say $\mathfrak{A} = \{\phi_1, \ldots, \phi_n\}$. Set $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$. Since $\phi_1, \ldots, \phi_n$ are normal and commuting, so are their inverses. This in turn implies that $\phi_1^{-1}, \ldots, \phi_n^{-1}, (\phi_1^{-1})^*, \ldots, (\phi_n^{-1})^*$ commute. Hence for all $x \in \mathbb{R}^\varkappa$ and all $\lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}$,

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \|\boldsymbol{\phi}^{-(\alpha+\beta)}(x)\|^2 \lambda(\alpha)\overline{\lambda(\beta)} = \Big\| \sum_{\alpha\in\mathbb{Z}_+^n} \lambda(\alpha)(\boldsymbol{\phi}^{-\alpha})^*\boldsymbol{\phi}^{-\alpha}(x)\Big\|^2 \geq 0.$$

Using the Schur theorem [2, Theorem 3.1.12], we obtain

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \|\boldsymbol{\phi}^{-(\alpha+\beta)}(x)\|^{2m} \lambda(\alpha)\overline{\lambda(\beta)} \geq 0, \qquad x \in \mathbb{R}^\varkappa,\ \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)},\ m \in \mathbb{Z}_+,$$

which yields

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \varrho(\boldsymbol{\phi}^{-(\alpha+\beta)}(x)) \lambda(\alpha)\overline{\lambda(\beta)} \geq 0, \qquad x \in \mathbb{R}^\varkappa,\ \lambda \in \mathbb{C}^{(\mathbb{Z}_+^n)}.$$

Thus Corollary 3.5 implies that the n-tuple $(C_{\phi_1}, \ldots, C_{\phi_n})$ is jointly subnormal. The case of $\{C_\phi^*\colon \phi \in \mathfrak{A}\}$ is similar. ∎

4. C_0-semigroups of composition operators. The following characterization of joint subnormality of C_0-semigroups is due to Itô (see [15, Theorem 1 and the proof of Lemma 5]).

THEOREM 4.1. *Let a be a positive real number. A C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is jointly subnormal if and only if the operator $S(a/n)$ is subnormal for every integer $n \geq 1$.*

It is worth noting that Theorem 4.1 is no longer true if "every integer $n \geq 1$" is replaced by "some integer $n \geq 1$". A counterexample in two-dimensional Hilbert space has been given by R. Mathias (cf. [1]); see also Example 5.4 below for the case of C_0-semigroups of composition operators.

Suppose that

(4.1) (X, Σ, μ) is a σ-finite measure space with $\mu \neq 0$ (equivalently: $L^2(\mu) \neq \{0\}$) and $\boldsymbol{\phi} = \{\phi_t\}_{t\geq 0}$ is a family of Σ-measurable transformations of X indexed by nonnegative real numbers such that every ϕ_t induces a bounded composition operator C_{ϕ_t} on $L^2(\mu)$ and $\{C_{\phi_t}\}_{t\geq 0}$ is a C_0-semigroup.

Define

$$h_t^{\boldsymbol{\phi}} = \frac{d\mu \circ \phi_t^{-1}}{d\mu}, \quad t \in \mathbb{R}_+. \tag{4.2}$$

Since $C_{\phi_0} = C_I$ (I is the identity transformation of X) and $C_{\phi_t^n} = C_{\phi_t}^n = C_{\phi_{nt}}$, we infer from (3.1) and Lemma 3.1(ii) that $h_0^{\boldsymbol{\phi}} = 1$ a.e. $[\mu]$ and

$$h_n^{\phi_t} = h_{nt}^{\boldsymbol{\phi}} \quad \text{a.e. } [\mu] \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}_+. \tag{4.3}$$

REMARK 4.2. Obviously, for each $t \geq 0$ the function $h_t^{\boldsymbol{\phi}}$ can be redefined on a set of measure zero (depending on t) without affecting the validity of (4.2). This may improve the properties of the function $t \mapsto h_t^{\boldsymbol{\phi}}(x)$ (cf. Theorem 4.5).

LEMMA 4.3. *If (4.1) holds, then the C_0-semigroup $\{C_{\phi_t}\}_{t\geq 0}$ is jointly subnormal if and only if one of the following three equivalent conditions holds:*

(i) *for μ-almost every $x \in X$,*

$$\sum_{m,n\in\mathbb{Z}_+} h_{(m+n)/k}^{\boldsymbol{\phi}}(x)\lambda(m)\overline{\lambda(n)} \geq 0 \quad \textit{for all } \lambda \in \mathbb{C}^{(\mathbb{Z}_+)} \textit{ and } k \in \mathbb{N},$$

(ii) *for μ-almost every $x \in X$ and every $k \in \mathbb{N}$, $\{h_{n/k}^{\boldsymbol{\phi}}(x)\}_{n\in\mathbb{Z}_+}$ is a Stieltjes moment sequence*,

(iii) *for μ-almost every $x \in X$ and every $k \in \mathbb{N}$, $\{h_{n/k}^{\boldsymbol{\phi}}(x)\}_{n\in\mathbb{Z}_+}$ is a Stieltjes moment sequence on $[0, \|C_{\phi_{1/k}}\|^2]$.*

Proof. Apply Theorem 4.1, equality (4.3) and Lambert's criterion for subnormality of composition operators (cf. [18]; see also Theorem 3.4) to $C_{\phi_{1/k}}$. ∎

By Lambert's criterion, the operator C_{ϕ_t} is subnormal if and only if for μ-almost every $x \in X$, there exists a (unique) positive Borel measure ϑ_x^t on $\mathbb{R}_+$ with compact support such that

$$h_n^{\phi_t}(x) = \int_0^\infty s^n \, d\vartheta_x^t(s), \qquad n \in \mathbb{Z}_+. \tag{4.4}$$

Notice that for μ-almost every $x \in X$, the closed support of ϑ_x^t is contained in $[0, \|C_{\phi_t}\|^2]$. Substituting $n = 0$ into (4.4), we deduce that for μ-almost every $x \in X$, ϑ_x^t is a probability measure. Moreover, since for μ-almost every $x \in X$ and all $n \in \mathbb{Z}_+$, $h_n^{\phi_0}(x) = 1$, we see that for such x's the closed support of ϑ_x^0 equals $\{1\}$.

For $t \in \mathbb{R}_+$, we define the function $\xi_t \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\xi_t(s) = s^t, \qquad s \in \mathbb{R}_+ \quad (\text{with } 0^0 = 1).$$

LEMMA 4.4. *If* (4.1) *holds, then the following conditions are equivalent*:

(i) *$\{C_{\phi_t}\}_{t \geq 0}$ is jointly subnormal*,

(ii) *C_{ϕ_1} is subnormal and for μ-almost every $x \in X$*,

$$h_{n/k}^{\boldsymbol{\phi}}(x) = \int_0^\infty s^{n/k} \, d\vartheta_x^1(s) \quad \textit{for all } n \in \mathbb{Z}_+ \textit{ and } k \in \mathbb{N}, \tag{4.5}$$

(iii) *for μ-almost every $x \in X$ there exists a positive Borel measure $\widetilde{\vartheta}_x$ on $\mathbb{R}_+$ such that*

$$h_{n/k}^{\boldsymbol{\phi}}(x) = \int_0^\infty s^{n/k} \, d\widetilde{\vartheta}_x(s) \quad \textit{for all } n \in \mathbb{Z}_+ \textit{ and } k \in \mathbb{N}. \tag{4.6}$$

Moreover, if $\{C_{\phi_t}\}_{t \geq 0}$ is jointly subnormal, then

(iv) *for μ-almost every $x \in X$, $\vartheta_x^1 = \widetilde{\vartheta}_x$*,

(v) *for every $t > 0$ and μ-almost every $x \in X$, $\vartheta_x^t(\{0\}) = 0$*,

(vi) *for every $t > 0$ and μ-almost every $x \in X$, $\vartheta_x^t = \vartheta_x^1 \circ \xi_{1/t}$*,

(vii) *for every $t \geq 0$ and μ-almost every $x \in X$, $h_t^{\boldsymbol{\phi}}(x) = \int_0^\infty s^t \, d\vartheta_x^1(s)$.*

Proof. (i)⇒(ii). It follows from (4.3), (4.4) and the measure transport theorem that for μ-almost every $x \in X$ and all $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$,

$$\int_0^\infty s^n \, d\vartheta_x^1(s) = h_n^{\phi_1}(x) = h_{kn}^{\phi_{1/k}}(x) = \int_0^\infty s^{kn} \, d\vartheta_x^{1/k}(s) = \int_0^\infty s^n \, d\vartheta_x^{1/k} \circ \xi_{1/k}(s),$$

hence $\vartheta_x^1 = \vartheta_x^{1/k} \circ \xi_{1/k}$, and consequently $\vartheta_x^{1/k} = \vartheta_x^1 \circ \xi_k$. By (4.3) this implies that for μ-almost every $x \in X$ and all $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$,

$$h^{\boldsymbol{\phi}}_{n/k}(x) = h_n^{\phi_{1/k}}(x) = \int_0^\infty s^n \, d\vartheta_x^{1/k}(s) = \int_0^\infty s^n \, d\vartheta_x^1 \circ \xi_k(s) = \int_0^\infty s^{n/k} \, d\vartheta_x^1(s).$$

This means that for μ-almost every $x \in X$, the equality in (vii) is valid for all rational numbers $t \geq 0$.

Now we show that (vii) holds in full generality. Let t be a positive real number. Then there exists a sequence $\{t_j\}_{j=1}^\infty$ of positive rational numbers such that $t_j \to t$ as $j \to \infty$. Since for μ-almost every $x \in X$, the probability measure ϑ_x^1 is compactly supported, we infer from Lebesgue's dominated convergence theorem that

$$\begin{aligned}(4.7) \qquad \int_0^\infty s^t d\vartheta_x^1(s) &= \lim_{j\to\infty} \int_0^\infty s^{t_j} d\vartheta_x^1(s) \\ &= \lim_{j\to\infty} h^{\boldsymbol{\phi}}_{t_j}(x) \quad \text{for } \mu\text{-almost all } x \in X.\end{aligned}$$

Employing (4.3), (3.2) and the continuity of $\{C_{\phi_s}\}_{s\geq 0}$, we see that there exists a constant $M > 0$ such that for μ-almost every $x \in X$,

$$|h^{\boldsymbol{\phi}}_{t_j}(x)| = |h_1^{\phi_{t_j}}(x)| \leq \|C_{\phi_{t_j}}\|^2 \leq M, \quad j \geq 1.$$

Lebesgue's dominated convergence theorem applied to (4.7) now yields

$$\begin{aligned}(4.8) \qquad \int_\tau h_1^{\phi_t}(x)\, d\mu(x) &= \|C_{\phi_t}(\chi_\tau)\|^2 = \lim_{j\to\infty} \|C_{\phi_{t_j}}(\chi_\tau)\|^2 \\ &= \lim_{j\to\infty} \int_\tau h_1^{\phi_{t_j}}(x)\, d\mu(x) \\ &\overset{(4.3)}{=} \lim_{j\to\infty} \int_\tau h^{\boldsymbol{\phi}}_{t_j}(x)\, d\mu(x) \overset{(4.7)}{=} \int_\tau \int_0^\infty s^t \, d\vartheta_x^1(s)\, d\mu(x)\end{aligned}$$

for every measurable subset τ of X of finite measure (χ_τ is the characteristic function of τ). Since μ is σ-finite, (4.8) implies that for μ-almost every $x \in X$,

$$h^{\boldsymbol{\phi}}_t(x) \overset{(4.3)}{=} h_1^{\phi_t}(x) = \int_0^\infty s^t \, d\vartheta_x^1(s),$$

which proves (vii). Hence for every real $t > 0$ and μ-almost every $x \in X$,

$$\begin{aligned}(4.9) \qquad \int_0^\infty s^n \, d\vartheta_x^t(s) &\overset{(4.4)}{=} h_n^{\phi_t}(x) \overset{(4.3)}{=} h^{\boldsymbol{\phi}}_{nt}(x) \overset{\text{(vii)}}{=} \int_0^\infty s^{nt}\, d\vartheta_x^1(s) \\ &= \int_0^\infty s^n \, d\vartheta_x^1 \circ \xi_{1/t}(s), \quad n \in \mathbb{Z}_+.\end{aligned}$$

Since for μ-almost every $x \in X$, the Stieltjes moment sequence defined by the left hand side of (4.9) is determinate, we get (vi). Substituting $k = 1$ into (4.5) and (4.6), and using determinacy again, we obtain (iv).

In view of (vi), to prove (v) it suffices to show that

$$\vartheta_x^1(\{0\}) = 0 \quad \text{for } \mu\text{-almost every } x \in X. \tag{4.10}$$

As in the proof of (4.7) and (4.8), we see that for μ-almost every $x \in X$,

$$\vartheta_x^1((0,\infty)) = \lim_{j\to\infty} \int_0^\infty s^{1/j}\, d\vartheta_x^1(s) = \lim_{j\to\infty} h_{1/j}^{\phi}(x),$$

and hence for every measurable subset τ of X of finite measure,

$$\mu(\tau) = \lim_{j\to\infty} \|C_{\phi_{1/j}}(\chi_\tau)\|^2 = \lim_{j\to\infty} \int_\tau h_{1/j}^{\phi}(x)\, d\mu(x) = \int_\tau \vartheta_x^1((0,\infty))\, d\mu(x).$$

As a consequence, $\vartheta_x^1((0,\infty)) = 1$ for μ-almost every $x \in X$. Since for μ-almost every $x \in X$, ϑ_x^1 is a probability measure, we get (4.10).

(ii)$\Rightarrow$(iii). Evident.

(iii)$\Rightarrow$(i). Verify condition (i) of Lemma 4.3. ∎

The Laplace transform $\mathfrak{L}(\zeta)\colon \mathbb{R}_+ \to \mathbb{R}_+$ of a finite positive Borel measure ζ on $\mathbb{R}_+$ is defined by

$$\mathfrak{L}(\zeta)(t) = \int_0^\infty e^{-ts}\, d\zeta(s), \qquad t \geq 0.$$

The function $\mathfrak{L}(\zeta)$ is always continuous (see [28] for the foundations of the theory of the Laplace transform). Below $\mathfrak{B}(J)$ stands for the σ-algebra of all Borel subsets of a Borel set $J \subseteq \mathbb{R}$. The ring of all complex polynomials in formal indeterminate Z is denoted by $\mathbb{C}[Z]$.

We now show that if $\{C_{\phi_t}\}_{t\geq 0}$ is a jointly subnormal C_0-semigroup of composition operators on $L^2(\mu)$, then the functions h_t^{ϕ} can be modified so as to satisfy the equality $h_t^{\phi}(x) = e^{\delta t}\mathfrak{L}(P(x,\cdot))(t)$ for all $x \in X$ and $t \in \mathbb{R}_+$, where $x \mapsto P(x,\cdot)$ is a Σ-measurable family of probability Borel measures on $\mathbb{R}_+$ and δ is a real number.

THEOREM 4.5. *If* (4.1) *holds and the C_0-semigroup $\{C_{\phi_t}\}_{t\geq 0}$ is jointly subnormal, then there exists a function $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ such that*:

1° *for every $x \in X$, $P(x,\cdot)$ is a probability measure*,
2° *for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, $P(\cdot,\sigma)$ is Σ-measurable*,
3° *for every $t \in \mathbb{R}_+$, the function $X \ni x \mapsto \mathfrak{L}(P(x,\cdot))(t) \in \mathbb{R}_+$ is Σ-measurable*,
4° *for μ-almost every $x \in X$ and all $t \in \mathbb{R}_+$, $h_t^{\phi}(x) = e^{\delta t}\mathfrak{L}(P(x,\cdot))(t)$, where* (5) $\delta := 2\log\|C_{\phi_1}\|$.

(5) Since $L^2(\mu) \neq \{0\}$, Proposition 1 of [23] implies that $\delta \in \mathbb{R}$ and $e^{\delta t} = \|C_{\phi_t}\|^2$ for $t \geq 0$.

Moreover, for μ-almost every $x \in X$,

$$P(x,\sigma) = \vartheta_x^1(\omega^{-1}(\sigma)), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{4.11}$$

where ω is a function from $(0, e^\delta]$ to $[0,\infty)$ defined by $\omega(s) = \delta - \log s$ for $s \in (0, e^\delta]$.

Proof. Set $J = [0, e^\delta]$. It follows from Lemma 4.4(v),(vii) that there exists a set $X_0 \in \Sigma$ of full μ-measure such that for every $x \in X_0$, ϑ_x^1 is a probability measure, $\vartheta_x^1(\{0\}) = 0$, the closed support of ϑ_x^1 is contained in J and

$$h_j^\phi(x) = \int_J s^j \, d\vartheta_x^1(s), \quad j \in \mathbb{Z}_+, \, x \in X_0.$$

This implies that for every polynomial $p = \sum_{j=0}^k c_j Z^j \in \mathbb{C}[Z]$,

$$\int_J p(s)\, d\vartheta_x^1(s) = \sum_{j=0}^k c_j h_j^\phi(x), \quad x \in X_0. \tag{4.12}$$

Take a continuous function $f\colon J \to \mathbb{C}$. By the Weierstrass theorem, there exists a sequence $\{p_n\}_{n=1}^\infty \subseteq \mathbb{C}[Z]$ which converges to f uniformly on J. This leads to

$$\int_J f\, d\vartheta_x^1 = \lim_{n\to\infty} \int_J p_n \, d\vartheta_x^1, \quad x \in X_0,$$

which, together with (4.12), guarantees that the function $X_0 \ni x \mapsto \int_J f\, d\vartheta_x^1 \in \mathbb{C}$ is Σ-measurable. Denote by $\mathfrak{A}$ the class of all Borel sets $\sigma \subseteq J$ such that the function $X_0 \ni x \mapsto \vartheta_x^1(\sigma) \in \mathbb{R}_+$ is Σ-measurable. It is clear that $\mathfrak{A}$ is a monotone class which contains $\emptyset$ and J. We claim that $[0,a) \in \mathfrak{A}$ for every $a \in J$ such that $a > 0$. Indeed, we can find a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions on J pointwise converging to $\chi_{[0,a)}$ as $n \to \infty$, and such that $0 \le f_n \le 1$ for all $n \ge 1$. Then, by Lebesgue's dominated convergence theorem, we have

$$\vartheta_x^1([0,a)) = \lim_{n\to\infty} \int_J f_n \, d\vartheta_x^1, \quad x \in X_0,$$

which proves our claim. Since the class $\mathfrak{A}$ is closed under the operation of taking set-theoretic proper difference and finite disjoint unions, we see that the algebra $\mathfrak{A}_0$ generated by the class $\{[0,a)\colon a \in J, \, a > 0\}$ is contained in $\mathfrak{A}$. Applying the monotone class theorem (cf. [3, Theorem 3.4]), we conclude that $\mathfrak{A} = \mathfrak{B}(J)$. Since the measure μ is nonzero, there is no loss of generality in assuming that $X_0 = X$. Hence ϑ_x^1 is a probability measure and $\vartheta_x^1(\mathbb{R}_+ \setminus (0, e^\delta]) = 0$ for every $x \in X$; moreover, for every $\sigma \in \mathfrak{B}(J)$, the function $X \ni x \mapsto \vartheta_x^1(\sigma) \in \mathbb{R}$ is Σ-measurable. It is now easily seen that the function $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ defined by (4.11) satisfies 1° and 2°. By a standard measure theory argument, it follows that for every Borel function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$, the function $X \ni x \mapsto \int_0^\infty f(s)\, P(x, ds) \in [0, \infty]$ is Σ-measurable. This implies 3°. Since $\vartheta_x^1(\mathbb{R}_+ \setminus (0, e^\delta]) = 0$ for all $x \in X$, we get

$$\text{(4.13)} \qquad \int_0^\infty s^t\, d\vartheta_x^1(s) = \int_{(0,e^\delta]} e^{t \log s}\, d\vartheta_x^1(s) = e^{\delta t} \int_{[0,\infty)} e^{-tu}\, d\vartheta_x^1 \circ \omega^{-1}(u) \overset{(4.11)}{=} e^{\delta t} \mathcal{L}(P(x,\cdot))(t), \quad x \in X,\ t \in \mathbb{R}_+.$$

Set $\widetilde{h}_t^{\phi}(x) = e^{\delta t}\mathcal{L}(P(x,\cdot))(t)$ for $x \in X$ and $t \in \mathbb{R}_+$. By 3°, the function $\widetilde{h}_t^{\phi}$ is Σ-measurable for every $t \in \mathbb{R}_+$. It follows from (4.13) and Lemma 4.4(vii) that $\widetilde{h}_t^{\phi} = h_t^{\phi}$ a.e. $[\mu]$ for every $t \in \mathbb{R}_+$. Replacing h_t^{ϕ} by $\widetilde{h}_t^{\phi}$, we get 4° (cf. Remark 4.2). This completes the proof. ∎

Corollary 4.6. *If* (4.1) *holds and* $\delta := 2 \log \|C_{\phi_1}\|$, *then the following conditions are equivalent*:

(i) $\{C_{\phi_t}\}_{t\geq 0}$ *is jointly subnormal*,
(ii) *for μ-almost every $x \in X$ there exists a finite positive Borel measure ζ_x on $\mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$, $h_t^{\phi}(x) = e^{\delta t}\mathcal{L}(\zeta_x)(t)$.*

Moreover, *if* (ii) *holds*, *then*

$$\text{(4.14)} \qquad \zeta_x = \vartheta_x^1 \circ \omega^{-1} \quad \textit{for } \mu\textit{-almost every } x \in X,$$

where ω is as in Theorem 4.5.

Proof. (i)⇒(ii). Apply Theorem 4.5.

(ii)⇒(i). Verify condition (i) of Lemma 4.3.

Assume that (ii) holds. Then by Lemma 4.4(v) and equalities (4.3) and (4.4), we see that for μ-almost every $x \in X$,

$$\int_{(0,e^\delta]} u^n\, d\vartheta_x^1(u) = h_n^{\phi}(x) \overset{\text{(ii)}}{=} \int_0^\infty \omega^{-1}(s)^n\, d\zeta_x(s) = \int_{(0,e^\delta]} u^n\, d\zeta_x \circ \omega(u), \quad n \in \mathbb{Z}_+.$$

Since the above Stieltjes moment sequence is determinate, we get $\vartheta_x^1 = \zeta_x \circ \omega$ for μ-almost every $x \in X$, which completes the proof. ∎

Note that if (ii) of Corollary 4.6 holds and $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is as in Theorem 4.5, then by (4.11) and (4.14), we have

$$\zeta_x = P(x,\cdot) \quad \text{and} \quad \mathcal{L}(\zeta_x) = \mathcal{L}(P(x,\cdot)) \quad \text{for } \mu\text{-almost every } x \in X.$$

5. An example. We begin by discussing a particular class of C_0-semigroups of composition operators induced by linear transformations of $\mathbb{R}^\varkappa$.

Proposition 5.1. *Let μ be a positive Borel measure on $\mathbb{R}^\varkappa$ which is finite on each compact subset of $\mathbb{R}^\varkappa \setminus \{0\}$ and $\mu(\{0\}) = 0$. Suppose that A*

is a linear transformation of $\mathbb{R}^\varkappa$ *such that for every* $t \in \mathbb{R}_+$, *the composition operator* $C_{e^{tA}}$ *is bounded on* $L^2(\mu)$, *and*

$$\sup_{0\le t\le t_0} \|C_{e^{tA}}\| < \infty \tag{5.1}$$

for some $t_0 > 0$. *Then* $\{C_{e^{tA}}\}_{t\ge 0}$ *is a* C_0*-semigroup.*

Proof. Take a sequence $\{t_n\}_{n=1}^\infty$ of positive real numbers converging to 0. Fix real numbers $0 < m < M < \infty$. Let $f\colon \mathbb{R}^\varkappa \to \mathbb{C}$ be a continuous function vanishing off the set $\Delta_{m,M} := \{x \in \mathbb{R}^\varkappa \colon m \le \|x\| \le M\}$ ($\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^\varkappa$). Take $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon$ for all $x, y \in \mathbb{R}^\varkappa$ such that $\|x - y\| \le \delta$. As the group $\{e^{tA}\}_{t\in\mathbb{R}}$ is uniformly continuous, there exists an integer $n_0 \ge 1$ such that $\|e^{\pm t_n A}\| \le 2$ and $\|e^{t_n A} - I\| \le \delta/2M$ for all $n \ge n_0$. This implies that for all $n \ge n_0$,

$$\begin{aligned} &\|e^{t_n A}x\| \ge \tfrac{1}{2}\|x\| && \text{for all } x \in \mathbb{R}^\varkappa,\\ &\|e^{t_n A}x\| < m && \text{for all } x \in \mathbb{R}^\varkappa \text{ such that } \|x\| < m/2,\\ &\|e^{t_n A}x - x\| \le \delta && \text{for all } x \in \mathbb{R}^\varkappa \text{ such that } \|x\| \le 2M. \end{aligned}$$

Thus, we have

$$|f(e^{t_n A}x) - f(x)| \le \begin{cases} \varepsilon & \text{if } x \in \Delta_{m/2,2M},\\ 0 & \text{otherwise,} \end{cases} \qquad n \ge n_0,$$

and consequently

$$\|C_{e^{t_n A}} f - f\|^2 = \int_{\Delta_{m/2,2M}} |f(e^{t_n A}x) - f(x)|^2 \, d\mu(x) \le \varepsilon^2 \mu(\Delta_{m/2,2M})$$

for all $n \ge n_0$. Summarizing, we have proved that $\lim_{t\to 0+} C_{e^{tA}} f = f$ for every continuous function $f\colon \mathbb{R}^\varkappa \to \mathbb{C}$ with compact support contained in $\mathbb{R}^\varkappa \setminus \{0\}$. Since μ is finite on each compact subset of $\mathbb{R}^\varkappa \setminus \{0\}$ and $\mu(\{0\}) = 0$, the set of all such functions is dense in $L^2(\mu)$ (use [24, Theorems 2.18 and 3.14]). This together with (5.1) implies that $\lim_{t\to 0+} C_{e^{tA}} f = f$ for every $f \in L^2(\mu)$, which means that $\{C_{e^{tA}}\}_{t\ge 0}$ is a C_0-semigroup. ∎

COROLLARY 5.2. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^\varkappa$ *induced by an inner product,* ϱ *be a member of* $\mathcal{R}_{\|\cdot\|}$ *and* μ *be any of the measures* $\varrho d\nu_\varkappa$ *or* $(1/\varrho) d\nu_\varkappa$. *Suppose that* A *is a linear transformation of* $\mathbb{R}^\varkappa$ *such that for every* $t \in \mathbb{R}_+$, *the composition operator* $C_{e^{tA}}$ *is bounded on* $L^2(\mu)$. *Then* $\{C_{e^{tA}}\}_{t\ge 0}$ *is a* C_0*-semigroup.*

Proof. It follows from [25, Lemma 2.1 and Proposition 2.2] and the continuity of the function $\mathbb{R} \ni t \mapsto \det e^{-tA} \in \mathbb{C} \setminus \{0\}$ that (5.1) holds for every $t_0 > 0$. Applying Proposition 5.1 completes the proof. ∎

REMARK 5.3. It is a matter of routine to verify that Corollary 3.5, Theorem 3.6, Proposition 5.1 and Corollary 5.2 remain valid for $\mathbb{C}$-linear transformations of $\mathbb{C}^{\varkappa}$ (see also Section 3 of [25]).

We now show that the implication (ii)$\Rightarrow$(i) of Lemma 4.4 is no longer true if the hypothesis (4.5) is dropped.

EXAMPLE 5.4. Denote by $|\cdot|_2$ the Euclidean norm on $\mathbb{C}^2$, i.e. $|x|_2^2 = |x_1|^2 + |x_2|^2$ for $x = (x_1, x_2) \in \mathbb{C}^2$. Let $\varrho \in \mathcal{R}_{|\cdot|_2}$ be a density function on $\mathbb{C}^2$ of polynomial type and let $d\mu = \varrho\, d\nu_4$. Following R. Mathias (cf. [1]), we define the nonsingular 2×2 complex matrix $A = \pi \left[\begin{smallmatrix} i & 1 \\ 0 & -i \end{smallmatrix}\right]$. Consider the semigroup $\{\phi_t\}_{t \geq 0}$ of transformations of $\mathbb{C}^2$ given by $\phi_t = e^{tA}$. According to a complex version of [25, Proposition 2.2], the composition operator C_{ϕ_t} is bounded on $L^2(\mu)$ for every $t \in \mathbb{R}_+$. Hence, by a complex version of Corollary 5.2, $\{C_{\phi_t}\}_{t \geq 0}$ is a C_0-semigroup. Since ϕ_1 is normal in $(\mathbb{C}^2, |\cdot|_2)$ and ϕ_t is not normal in $(\mathbb{C}^2, |\cdot|_2)$ for some $t > 0$, we infer from a complex version of [25, Theorem 2.5] that C_{ϕ_1} is subnormal and $\{C_{\phi_t}\}_{t \geq 0}$ is not jointly subnormal.

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