STUDIA MATHEMATICA 215 (3) (2013)

Periodic solutions of an abstract third-order differential equation

by

VERÓNICA POBLETE and JUAN C. POZO (Santiago)

Abstract. Using operator valued Fourier multipliers, we characterize maximal regularity for the abstract third-order differential equation $\alpha u'''(t)+u''(t) = \beta Au(t)+\gamma Bu'(t)+f(t)$ with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, where A and B are closed linear operators defined on a Banach space X, $\alpha, \beta, \gamma \in \mathbb{R}_+$, and f belongs to either periodic Lebesgue spaces, or periodic Besov spaces, or periodic Triebel–Lizorkin spaces.

1. Introduction. In this paper we characterize the property of maximal regularity for a third-order differential equation. This type of equation describes several models arising from natural phenomena, such as wave propagation in viscous thermally relaxing fluids, flexible space structure, a thin uniform rectangular panel, like a solar cell array, and a spacecraft with flexible attachments. At present, the requirements for maximum performance of machines, at a minimum cost, have inevitably led to reducing the mass of their moving parts. This means that the structures lose rigidity and become much more flexible. Due to this, the study of flexible structures and their properties has recently been enjoying a great deal of interest. In the same manner, modelling acoustic wave propagation is also a field of research of great interest because it has a wide range of applications, such as the medical and industrial use of focused high intensity ultrasound in lithotrity, thermotherapy, ultrasound cleaning, and sonochemistry.

Kuznetsov's equation, the Westervelt equation, and the Kokhlov–Zabolotskaya–Kuznetsov equation are classical models of non-linear acoustics. These models involve second-order differential equations with respect to time. For well-posedness and stability analysis of several types of initial conditions for these models, see [33, 34, 44].

2010 *Mathematics Subject Classification*: Primary 34G10; Secondary 34C25.
Key words and phrases: third-order differential equation, Fourier multipliers, R-bounded operators.

DOI: 10.4064/sm215-3-1

Since the use of classical Fourier theory leads to an infinite signal speed paradox, several other alternatives for non-linear acoustics equations have been considered. For example, the equation governing one of the alternative models is given by

$$\tau\varphi_{ttt} + \varphi_{tt} - c^2\Delta\varphi - b\Delta\varphi_t = \frac{d}{dt}\left(\frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(\varphi_t)^2\right) \tag{1.1}$$

where $\tau > 0$ is a constant accounting for relaxation, c the speed of sound, δ the diffusivity of sound, B/A the non-linearity parameter, and $b = \delta + \tau c^2$, (see [32]). For a study of the decay rates of the natural energy function of the linear version of equation (1.1), see [35].

On the other hand, in general, the dynamics of linear vibrations of elastic structures is based on Hooke's law. The equation governing these vibrations is the wave equation. Further, the dynamics of flexible elastic structures is non-linear. All the same, the third-order differential equation

$$\lambda y'''(t) + y''(t) = c^2(\Delta y(t) + \mu\Delta y'(t)) \quad \text{for } t \in \mathbb{R}_+ \text{ and } \lambda < \mu \tag{1.2}$$

governing a realistic linear model is investigated in [9, 26, 27, 28, 29], where S. Bose and G. Gorain study boundary stabilization and obtain the explicit exponential energy decay rate for the solution subject to mixed boundary conditions.

The analysis of third-order differential equations dates back to the second half of the 1900's. At that time, Moore & Gibson [45] and Thompson [48] worked independently on models using these equations. In fact, the linear version of equation (1.1) is called the *Moore–Gibson–Thompson equation.* Under the influence of an external force, both this equation and the Bose–Gorain equation (1.2) take the abstract form

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t) \quad \text{for } t \in \mathbb{R}_+, \tag{1.3}$$

where A is a closed linear operator defined on a Banach space X, f is a given X-valued function, and $\alpha, \beta, \gamma \in \mathbb{R}_+$. Equation (1.3) has been studied in many aspects. For a characterization of solutions in Hölder spaces, see [19]. For the regularity of mild and strong solutions in Hilbert spaces defined on $\mathbb{R}_+$, see [21]. For a characterization of L^p-maximal regularity of solutions defined on $\mathbb{R}_+$, see [22]. Further, existence of mild bounded solutions of a semilinear version of this equation is studied in [3].

Here we study the third-order differential equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t) \quad \text{for } t \in [0, 2\pi] \tag{1.4}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, where f is a given X-valued function, A and B are closed linear operators defined on a Banach space X such that $D(A) \subseteq D(B)$, and $\alpha, \beta, \gamma \in \mathbb{R}_+$. We are interested in necessary and sufficient conditions which guarantee maximal regularity for this equation in the categories of periodic Lebesgue spaces, Besov spaces, and Triebel–Lizorkin spaces.

During the last decades, there has been an explosion of interest in the maximal regularity property due to its applications in theoretical mathematics, such as existence, uniqueness, and well-posedness of solutions of both linear and non-linear evolution equations.

Various techniques are used to study the problem of maximal regularity. We use Fourier multipliers or symbols. For operator-valued Fourier multipliers and maximal regularity for evolution equations, see, for example, [2, 4, 6, 7, 8, 11, 12, 14, 13, 17, 19, 18, 20, 24, 25, 31, 36, 43, 39, 40, 41, 47, 49]. Applications to physical problems, most notably viscoelasticity of materials with memory, are found in these works and the references therein.

Besov spaces are function spaces of special interest. They behave (in a sense we will clarify below) similarly to Sobolev spaces, and the property of maximal regularity can be stated elegantly for them. Moreover, they depend on three parameters (s, p, and q) and important spaces are identified with different choices of p, q, and s. For example, if $p = q = \infty$ and $0 < s < 1$, we recover the well known space of all Hölder continuous functions of index s. For further details, see [7]. However, the main reason for working in these spaces is that a certain form of Mikhlin's multiplier theorem holds for arbitrary Banach spaces, unlike the Lebesgue spaces $L^p(\mathbb{T}; X)$ in which this property holds if and only if $p = 2$. For further information, see [23]. Triebel–Lizorkin spaces have similar properties.

The paper is organized as follows. In Section 2, we establish notational conventions, and we introduce the concept of $\mathcal{M}$-boundedness. This concept is closely related to well-posedness. Sections 3–5 contain our principal results. We obtain results on maximal regularity for third-order differential equations in Lebesgue, Besov, and Triebel–Lizorkin spaces. In Section 6, we apply our results to interesting examples. In general, it is not easy to verify the R-boundedness condition, especially when two not necessarily commuting operators are involved. We use functional calculus and sectorial operators to establish boundedness and R-boundedness properties of certain families associated with equation (1.4); the scalar values α, β, and γ of this equation play an important role in proving boundedness and R-boundedness of these families.

2. Preliminaries. Let X and Y be complex Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all linear operators from X to Y. In the case $X = Y$, we write briefly $\mathcal{B}(X)$. Let A be an operator defined on X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Given $\alpha, \beta, \gamma > 0$, let A and B be closed linear operators with $D(A) \cap D(B) \neq \{0\}$. For $k \in \mathbb{Z}$, we will write

$$a_k = ik^3 \quad \text{and} \quad b_k = i\alpha k^3 + k^2 \tag{2.1}$$

and consider the operators

$$N_k = (b_k + i\gamma kB + \beta A)^{-1} \quad \text{and} \quad M_k = a_k N_k\,. \tag{2.2}$$

We denote

$$\rho(A,B) = \{k \in \mathbb{Z} : N_k \text{ exists and is bounded}\}, \qquad \sigma(A,B) = \mathbb{Z} \setminus \rho(A,B).$$

We denote by $E(\mathbb{T};X)$ the space of all 2π-periodic, X-valued functions, and by $E^n(\mathbb{T};X)$ the set of all functions in $E(\mathbb{T};X)$ which are n times differentiable. The following definitions will be used in subsequent sections for Lebesgue, Besov and Triebel–Lizorkin periodic spaces.

Definition 2.1. A function u is called a *strong E-solution* of equation (1.4) if $u \in E^3(\mathbb{T};X) \cap E^1(\mathbb{T};[D(B)]) \cap E(\mathbb{T};X)$ and equation (1.4) holds a.e. in $[0,2\pi]$.

Definition 2.2. We say that equation (1.4) has *E-maximal regularity* if for each $f \in E(\mathbb{T};X)$, equation (1.4) has a unique strong E-solution.

Definition 2.3. We say that the sequence $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ is an *$(E(X),E(Y))$-multiplier* if for each $f \in E(\mathbb{T};X)$, there exists a $u \in E(\mathbb{T};Y)$ such that

$$\widehat{u}(k) = L_k \widehat{f}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

In the case $X = Y$, we will say that $\{L_k\}_{k\in\mathbb{Z}}$ is an E-multiplier.

In order to give conditions which we will need later, we establish some notation. Let $\{L_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be a sequence of operators. Set

$$\Delta^0 L_k = L_k, \quad \Delta L_k = \Delta^1 L_k := L_{k+1} - L_k$$

and for $n = 2, 3, \ldots$, set

$$\Delta^n L_k = \Delta(\Delta^{n-1} L_k).$$

Definition 2.4. We say that a sequence $\{L_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is *$\mathcal{M}$-bounded of order n* ($n \in \mathbb{N} \cup \{0\}$) if

$$\sup_{0\le l\le n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l L_k\| < \infty. \tag{2.3}$$

Note that, for $j \in \mathbb{Z}$ fixed, we have

$$\sup_{0\le l\le n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l L_k\| < \infty \quad \text{if and only if} \quad \sup_{0\le l\le n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l L_{k+j}\| < \infty.$$

This follows directly from the binomial formula.

The $\mathcal{M}$-boundedness of order 0 for $\{L_k\}$ simply means that $\{L_k\}$ is bounded.

When $n = 1$, the $\mathcal{M}$-boundedness is equivalent to

$$\sup_{k\in\mathbb{Z}} \|L_k\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(L_{k+1} - L_k)\| < \infty. \tag{2.4}$$

When $n = 2$, in addition to (2.4), we must have

$$\sup_{k\in\mathbb{Z}} \|k^2(L_{k+2} - 2L_{k+1} + L_k)\| < \infty. \tag{2.5}$$

When $n = 3$, in addition to (2.5) and (2.4), we must have

$$\sup_{k\in\mathbb{Z}} \|k^3(L_{k+3} - 3L_{k+2} + 3L_{k+1} - L_k)\| < \infty. \tag{2.6}$$

In the scalar case, that is, $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$, we will write $\Delta^n a_k = \Delta(\Delta^{n-1}a_k)$.

DEFINITION 2.5. A sequence $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}\setminus\{0\}$ is called

- 1-*regular* if the sequence $\big\{k\,\frac{\Delta^1 a_k}{a_k}\big\}_{k\in\mathbb{Z}}$ is bounded;
- 2-*regular* if it is 1-regular and $\big\{k^2\frac{\Delta^2 a_k}{a_k}\big\}_{k\in\mathbb{Z}}$ is bounded;
- 3-*regular* if it is 2-regular and $\big\{k^3\frac{\Delta^3 a_k}{a_k}\big\}_{k\in\mathbb{Z}}$ is bounded.

For useful properties and further details about N-regularity, see [42, 46].

REMARK 2.6. Note that if $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, then for all $j \in \mathbb{Z}$ fixed, $\big\{k\frac{a_{k+j}-a_k}{a_{k+j}}\big\}_{k\in\mathbb{Z}}$ is bounded. If $n = 2, 3$, analogous properties hold.

3. Maximal regularity for a third-order differential equation in periodic Lebesgue spaces.

In order to introduce L^p-maximal regularity for equation (1.4), we define the following spaces.

DEFINITION 3.1. Let $p \in [1,\infty)$, and let $n \in \mathbb{N}$. Let X and Y be Banach spaces. We define the vector-valued function spaces

$$H^{n,p}_{\mathrm{per}}(X,Y) = \{u \in L^p(\mathbb{T};X) : \text{there exists } v \in L^p(\mathbb{T};Y) \text{ such that } \widehat{v}(k) = (ik)^n\widehat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

In the case $X = Y$, we just write $H^{n,p}_{\mathrm{per}}(X)$.

We highlight two important properties of these spaces:

- Let $n, m \in \mathbb{N}$. If $n \le m$, then $H^{m,p}_{\mathrm{per}}(X,Y) \subseteq H^{n,p}_{\mathrm{per}}(X,Y)$.
- If $u \in H^{n,p}_{\mathrm{per}}(X)$, then $u^{(k)}(0) = u^{(k)}(2\pi)$ for all $0 \le k \le n-1$.

Let $\mathcal{S}(\mathbb{R};X)$ be the Schwartz space of all rapidly decreasing X-valued functions. A Banach space will be called a *UMD-space* if the Hilbert transform is bounded in $L^p(\mathbb{R};X)$ for some (and hence for all) $p \in (1,\infty)$. Examples of UMD-spaces include Hilbert spaces, Sobolev spaces $W^s_p(\Omega)$, with $1 < p < \infty$, the Lebesgue spaces $L^p(\Omega,\mu)$ and $L^p(\Omega,\mu;X)$, with $1 < p < \infty$ and X a UMD-space. For further information about these spaces, see [10, 15, 16].

Definition 3.2. Let X and Y be Banach spaces. A family $\mathcal{T} \subseteq \mathcal{B}(X,Y)$ of operators is called *R-bounded* if there exist $C > 0$ and $p \in [1,\infty)$ such that for each $n \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and all independent, symmetric, $\{-1,1\}$-valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$, the inequality

$$\Big\|\sum_{j=1}^{n} r_j T_j x_j\Big\|_{L^p(\Omega;Y)} \le C \Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{L^p(\Omega;X)}$$

holds. The smallest such $C \ge 0$ is called the *R-bound* of $\mathcal{T}$, denoted $R_p(\mathcal{T})$.

There are various classes of R-bounded families of operators (see [23] and the reference therein). For further properties of R-bounded families, see [20].

Proposition 3.3 ([6]). *Let $p \in (1,\infty)$, and let X and Y be UMD-spaces. Assume that $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If $\{L_k\}_{k\in\mathbb{Z}}$ is an $(L^p(X), L^p(Y))$-multiplier, then it is R-bounded.*

Theorem 3.4 ([6]). *Let $p \in (1,\infty)$, and let X and Y be UMD-spaces. Assume $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If $\{L_k\}_{k\in\mathbb{Z}}$ and $\{k\Delta^1 L_k\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{L_k\}_{k\in\mathbb{Z}}$ is an $(L^p(X), L^p(Y))$-multiplier.*

Lemma 3.5 ([6]). *Let $f, g \in L^p(\mathbb{T}; X)$ with $p \in [1,\infty)$. If A is a closed operator in a Banach space X, then the following assertions are equivalent:*

(i) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e.*
(ii) *$\widehat{f}(k) \in D(A)$ and $A\widehat{f}(k) = \widehat{g}(k)$, for all $k \in \mathbb{Z}$.*

Remark 3.6. For $1 \le p \le \infty$, by [6, Lemma 2.2], $\{k^n M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier if and only if $\{L_k\}_{k\in\mathbb{Z}}$ is an $(L^p(X), H^{n,p}_{\mathrm{per}}(X))$-multiplier for all $n \in \mathbb{N}$.

To prove Theorem 3.8 below, we will need the following. We use the notation given in (2.1) and (2.2).

Lemma 3.7. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on a Banach space X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded families of operators, then*

$$\{k a_k \Delta^1 N_k\}_{k\in\mathbb{Z}} \quad \text{and} \quad \{k^2 B \Delta^1 N_k\}_{k\in\mathbb{Z}}$$

are also R-bounded.

Proof. First note that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is R-bounded if and only if $\{b_k N_k\}$ is. Furthermore, for all $j \in \mathbb{Z}$ fixed, $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are R-bounded. For $k \in \mathbb{Z}$, we have

$$\Delta^1 N_k = N_{k+1}(b_k - b_{k+1} - i\gamma B)N_k = -(\Delta^1 b_k)N_{k+1}N_k - i\gamma N_{k+1}BN_k. \tag{3.1}$$

Hence

$$k a_k \Delta^1 N_k = -k\frac{\Delta^1 b_k}{b_{k+1}}\,\frac{b_{k+1}}{a_{k+1}} M_{k+1}M_k + \gamma a_k N_{k+1} kBN_k.$$

On the other hand, from (3.1) we obtain

$$\begin{aligned}k^2B\Delta^1N_k &= -k(\Delta^1 b_k)kBN_{k+1}N_k - i\gamma kBN_{k+1}kBN_k\\ &= -k\frac{\Delta^1 b_k}{b_k}\frac{b_k}{a_k}kBN_{k+1}M_k - i\gamma kBN_{k+1}kBN_k.\end{aligned}$$

Clearly, $\{b_k\}_{k\in\mathbb{Z}}$ is a 1-regular sequence. In addition, we have

$$\sup_{k\in\mathbb{Z}\setminus\{0\}}|b_k/a_k|<\infty,\quad \sup_{k\in\mathbb{Z}\setminus\{-1\}}|a_k/a_{k+1}|<\infty,\quad\text{and}\quad \sup_{k\in\mathbb{Z}\setminus\{-1\}}\left|\frac{k}{k+1}\right|<\infty.$$

The lemma results from the properties of R-bounded families. ■

Our two principal results in this section are Theorems 3.8 and 3.9 below.

THEOREM 3.8. *Let $p\in(1,\infty)$, and let X be a UMD-space. If $\alpha,\beta,\gamma>0$, and A and B are closed linear operators defined on X, then the following assertions are equivalent:*

(i) *The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are R-bounded.*
(ii) *The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are L^p-multipliers.*

Proof. (i)⇒(ii). By hypothesis, $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded. According to Theorem 3.4, it suffices to show that the families $\{k\Delta^1M_k\}_{k\in\mathbb{Z}}$ and $\{k\Delta^1(kBN_k)\}_{k\in\mathbb{Z}}$ are also R-bounded. For this, note that

$$k\Delta^1M_k = k\frac{\Delta^1 a_k}{a_{k+1}}M_{k+1} + ka_k\Delta^1N_k.$$

Similarly, we write $k\Delta^1(kBN_k) = k^2B\Delta^1N_k + kBN_{k+1}$. Statement (ii) results from Lemma 3.7 and the properties of R-bounded families.

(ii)⇒(i). Apply Proposition 3.3. ■

THEOREM 3.9. *Let $p\in(1,\infty)$, and let X be a UMD-space. The following assertions are equivalent:*

(i) *Equation* (1.4) *has L^p-maximal regularity.*
(ii) *$\sigma(A,B)=\emptyset$, and the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded.*

Proof. (i)⇒(ii). Fix $k\in\mathbb{Z}$, and let $x\in X$. Define $h(t)=e^{ikt}x$. A simple computation shows that $\widehat{h}(k)=x$.

By hypothesis, there exists $u\in H^{3,p}_{\mathrm{per}}(X)\cap H^{1,p}_{\mathrm{per}}(X;[D(B)])\cap L^p(\mathbb{T};[D(A)])$ such that, for almost all $t\in[0,2\pi]$,

$$\alpha u'''(t)+u''(t)=\beta Au(t)+\gamma Bu'(t)+h(t).$$

Applying the Fourier transform to both sides, we obtain

$$(-i\alpha k^3-k^2-i\gamma kB-\beta A)\widehat{u}(k)=x.$$

Since x is arbitrary, we see that $-i\alpha k^3-k^2-i\gamma kB-\beta A$ is surjective.

On the other hand, let $z \in D(A) \cap D(B)$, and assume $(-b_k - i\gamma kB - \beta A)z = 0$. Substituting $u(t) = e^{ikt}z$ in (1.4), we see that u is a periodic solution of this equation when $f \equiv 0$. The uniqueness of the solution implies that $z = 0$.

Now suppose $b_k + i\gamma kB + \beta A$ has no bounded inverse. Then for each $k \in \mathbb{Z}$, there exists a sequence $\{y_{k,n}\}_{n\in\mathbb{Z}} \subseteq X$ such that

$$\|y_{n,k}\| \leq 1 \quad \text{and} \quad \|N_k y_{k,n}\| \geq n^2, \qquad \text{for all } n \in \mathbb{Z}.$$

Define $x_k = y_{k,k}$. We obtain $\|N_k x_k\| \geq k^2$ for all $k \in \mathbb{Z}$. Let

$$g(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{x_k}{k^2} e^{ikt}.$$

Note that $g \in L^p(\mathbb{T}; X)$. By hypothesis, there exists a unique strong L^p-solution $u \in L^p(\mathbb{T}; X)$. Applying the Fourier transform to (1.4), we have $\widehat{u}(k) = -N_k \widehat{g}(k)$ for all $k \in \mathbb{Z}$. We know

$$u(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} -\frac{x_k}{k^2} e^{ikt} N_k.$$

For all $k \in \mathbb{Z}$, we have $\|(x_k/k^2)N_k\| \geq 1$ and conclude that $u \notin L^p(\mathbb{T}; X)$. This is a contradiction, since u is a strong L^p-solution of (1.4). Hence $N_k \in \mathcal{B}(X)$ for all $k \in \mathbb{Z}$. Therefore, $\sigma(A, B) = \emptyset$.

Next let $f \in L^p(\mathbb{T}; X)$. By hypothesis, there exists a unique function $u \in H^{3,p}_{\mathrm{per}}(X) \cap H^{1,p}_{\mathrm{per}}(X; [D(B)]) \cap L^p(\mathbb{T}; [D(A)])$ such that

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma B u'(t) + f(t)$$

for almost all $t \in [0, 2\pi]$. Applying the Fourier transform to both sides yields

$$(-b_k - i\gamma kB - \beta A)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Since $\sigma(A, B) = \emptyset$, we have

$$\widehat{u}(k) = (-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

Multiplying the preceding equality by $i\gamma k$, we obtain

$$i\gamma k\widehat{u}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k).$$

Since $u \in H^{1,p}_{\mathrm{per}}(X; [D(B)])$, there is a function $v \in L^p(\mathbb{T}; [D(B)])$ satisfying $\widehat{v}(k) = i\gamma k\widehat{u}(k)$ for all $k \in \mathbb{Z}$. Therefore,

$$\widehat{v}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

Define $w = Bv$. Since $v \in L^p(\mathbb{T}; [D(B)])$, we conclude $w \in L^p(\mathbb{T}; X)$.

Since B is a closed linear operator, it follows from Lemma 3.5 that

$$\widehat{w}(k) = -i\gamma kBN_k\widehat{f}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

This implies that $\{kBN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.

On the other hand, since $u \in L^p(\mathbb{T};[D(A)])$, defining $r = -\beta Au$ we have $r \in L^p(\mathbb{T};X)$. Since A is linear and closed, Lemma 3.5 yields

$$\widehat{r}(k) = -\beta AN_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $\{-\beta AN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Now for all $k \in \mathbb{Z}$, we have $b_kN_k = I - i\gamma kBN_k - \beta AN_k$. Since the sum of L^p-multipliers is also an L^p-multiplier, we conclude $\{b_kN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. The sequence $\{a_k/b_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded. Hence, $(a_k/b_k)b_kN_k = M_k$ is an L^p-multiplier. It now follows from Proposition 3.3 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded.

(ii)$\Rightarrow$(i). By hypothesis, the conditions of Theorem 3.8 are satisfied. Therefore, $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are L^p-multipliers. From Remark 3.6 we conclude that $\{(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an $(L^p(X), H^{3,p}_{\mathrm{per}}(X))$-multiplier. Given $f \in L^p(\mathbb{T};X)$, there exists $u \in H^{3,p}_{\mathrm{per}}(X)$ such that

$$\widehat{u}(k) = (-b_k - \beta A - i\gamma kB)^{-1}\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{3.2}$$

Moreover, Lemma 3.5 shows that $u(t) \in D(A) \cap D(B)$ for almost all $t \in [0, 2\pi]$.

By hypothesis, $\{ikB(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Then there exists $v \in L^p(\mathbb{T};X)$ satisfying

$$\widehat{v}(k) = ikB(-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

According to (3.2), we have $\widehat{v}(k) = ikB\widehat{u}(k)$ for all $k \in \mathbb{Z}$.

On the other hand, since $H^{3,p}_{\mathrm{per}}(X) \subseteq H^{1,p}_{\mathrm{per}}(X)$, there exists $w \in L^p(\mathbb{T};X)$ such that $\widehat{w}(k) = ik\widehat{u}(k)$ for all $k \in \mathbb{Z}$. Since B is a closed linear operator, we have

$$\widehat{v}(k) = B(ik\widehat{u}(k)) = B\widehat{w}(k) = \widehat{Bw}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By the uniqueness of the Fourier coefficients, $v = Bw$. This implies that $w \in L^p(\mathbb{T};[D(B)])$. Therefore, $u \in H^{1,p}_{\mathrm{per}}(X;[D(B)])$. We claim that $u \in L^p(\mathbb{T};[D(A)])$. In fact, using the identity

$$\beta A(b_k + i\gamma kB + \beta A)^{-1} = I - b_k(b_k + i\gamma kB + \beta A)^{-1} - i\gamma kB(b_k + i\gamma kB + \beta A)^{-1}$$

we see that $\{\beta A(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Thus, there exists a function $h \in L^p(\mathbb{T};X)$ satisfying

$$\widehat{h}(k) = A(b_k + i\gamma B + \beta A)^{-1}\widehat{f}(k) \quad \text{for all } k.$$

It follows from (3.2) that $\widehat{h}(k) = A\widehat{u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, we have $h = Au$. This implies that $u \in L^p(\mathbb{T};[D(A)])$ as asserted, so $u \in H^{3,p}_{\mathrm{per}}(X) \cap H^{1,p}_{\mathrm{per}}(X;[D(B)]) \cap L^p(\mathbb{T};[D(A)])$.

As $u \in H^{3,p}_{\mathrm{per}}(X)$, we have $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, and $u''(0) = u''(2\pi)$. Since A and B are closed linear operators, it now follows from (3.2) that

$$\alpha\widehat{u'''}(k) + \widehat{u''}(k) = \beta\widehat{Au}(k) + \gamma\widehat{Bu}(k) + \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

From the uniqueness of the Fourier coefficients we conclude that (1.4) holds a.e. in $[0, 2\pi]$. Therefore, u is a strong L^p-solution of (1.4).

It remains to show that this solution is unique. Indeed, let $f \in L^p(\mathbb{T}; X)$. Suppose (1.4) has two strong L^p-solutions, u_1 and u_2. A direct computation shows that

$$(-b_k - i\gamma kB - \beta A)(\widehat{u_1}(k) - \widehat{u_2}(k)) = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Since $-b_k - i\gamma kB - \beta A$ is invertible, we have $\widehat{u_1}(k) = \widehat{u_2}(k)$ for all k. By the uniqueness of the Fourier coefficients, $u_1 \equiv u_2$. Therefore, (1.4) has L^p-maximal regularity. ∎

We define the operators

$$S_k = \left(-\frac{b_k}{\beta} - A\right)^{-1} \quad \text{and} \quad T_k = \left(I - \frac{\gamma}{\beta} ikBS_k\right)^{-1}, \qquad \text{for all } k \in \mathbb{Z}.$$

We use this notation in our next result.

Corollary 3.10. *Let $1 < p < \infty$, and let X be a UMD-space. Assume that the families of operators*

$$\mathcal{F}_1 = \{a_k S_k : k \in \mathbb{Z}\} \quad \textit{and} \quad \mathcal{F}_2 = \left\{ik\frac{\gamma}{\beta} BS_k : k \in \mathbb{Z}\right\}$$

are R-bounded. If $\mathcal{R}_p(\mathcal{F}_2) < 1$, then equation (1.4) has L^p-maximal regularity.

Proof. According to [30, Lemma 3.17], the family $\{T_k\}_{k\in\mathbb{Z}}$ is R-bounded. Since $M_k = a_k S_k T_k$ and $kBN_k = kBS_kT_k$, for all $k \in \mathbb{Z}$, we conclude that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded by the properties of R-boundedness. The corollary now follows from Theorem 3.9. ∎

For all $k \in \mathbb{Z}$, we define

$$c_k = \frac{-i\alpha k^3 - k^2}{\beta} \quad \text{and} \quad d_k = -\frac{i\alpha k^3 + k^2}{i\gamma k + \beta}. \tag{3.3}$$

We use this notation in our next results.

Corollary 3.11. *Let $p \in (1, \infty)$, and let X be a UMD-space. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv 0$ has L^p-maximal regularity.*
(ii) *$\{c_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{a_k(c_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. Note that (i) is equivalent to condition (i) of Theorem 3.9 with $B \equiv 0$, and (ii) is equivalent to condition (ii) of Theorem 3.9 with $B \equiv 0$. ∎

COROLLARY 3.12. *Let $p \in (1, \infty)$, and let X be a UMD-space. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv A$ has L^p-maximal regularity.*
(ii) *$\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$, and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. (i)$\Rightarrow$(ii). By Theorem 3.9, we have $\sigma(A, A) = \emptyset$ and $(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} \in \mathcal{B}(X)$ for all $k \in \mathbb{Z}$. In addition, $\{ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded, hence bounded, so there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{Z}} \|ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1}\| \le C.$$

This implies

$$\|(d_k - A)^{-1}\| \le \frac{|i\gamma k + \beta|}{|ik^3|} C \quad \text{for all } k \in \mathbb{Z} \setminus \{0\}.$$

Since $0 \in \rho(A, A)$ if and only if $0 \in \rho(A)$, we have $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$. Properties of R-bounded families and the equality

$$d_k(d_k - A)^{-1} = \frac{i\alpha k^3 + k^2}{ik^3} ik^3(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}$$

show that $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

(ii)$\Rightarrow$(i). Note that (ii) guarantees that condition (ii) of Theorem 3.9 is satisfied. In fact, $d_k \in \rho(A)$ implies that $(d_k - A)^{-1}$ is well defined in $\mathcal{B}(X)$. Since $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded, there exists a constant $C \ge 0$ such that

$$\sup_{k\in\mathbb{Z}} \|d_k(d_k - A)^{-1}\| = \sup_{k\in\mathbb{Z}} |i\alpha k^3 + k^2| \, \|(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}\| \le C.$$

Then, for all $k \in \mathbb{Z} \setminus \{0\}$, we obtain

$$\|(-i\alpha k^3 - k^2 - (i\gamma k + \beta)A)^{-1}\| \le \frac{C}{|i\alpha k^3 + k^2|}.$$

Since $0 \in \rho(A)$ if and only if $0 \in \rho(A, A)$, we have $\sigma(A, A) = \emptyset$.

We combine properties of R-bounded families with the identities

$$ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{ik^3}{i\alpha k^3 + k^2} d_k(d_k - A)^{-1}$$

and

$$kA(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{-k}{i\gamma k + \beta}(d_k(d_k - A)^{-1} - I)$$

to find that $\{ik^3(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{kA(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded. ∎

4. Maximal regularity for a third-order differential equation in periodic Besov spaces. Before introducing the $B^s_{p,q}$-maximal regularity for equation (1.4), we recall the definition of periodic Besov space. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$, $\mathcal{S}'(\mathbb{R})$ the space of all tempered distributions on $\mathbb{R}$, and $\mathcal{D}'(\mathbb{T})$ the space of 2π-periodic distributions. Let $\mathcal{D}'(\mathbb{T};X) = \mathcal{B}(\mathcal{D}(\mathbb{T});X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. The elements of $\mathcal{D}'(\mathbb{T};X)$ are called *X-valued distributions* on $\mathbb{T}$. Let $\Phi(\mathbb{R})$ be the set of all systems $\phi = \{\phi_j\}_{j\geq 0} \subseteq \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi_0) \subseteq [-2,2]$, and

$$\operatorname{supp}(\phi_j) \subseteq [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \qquad \sum_{j\geq 0} \phi_j(t) = 1 \quad \text{for } t \in \mathbb{R},$$

and, for $\alpha \in \mathbb{N} \cup \{0\}$, there is a $C_\alpha > 0$ such that

$$\sup_{j\geq 0,\, x\in\mathbb{R}} 2^{\alpha j} \|\phi_j^{(\alpha)}(x)\| \leq C_\alpha.$$

That such systems exist is a well known fact which is related to the Littlewood–Paley decomposition. For further information, see [1, 2, 5, 7].

Definition 4.1. Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $\phi = (\phi_j)_{j\geq 0} \in \Phi(\mathbb{R})$. The *$X$-valued periodic Besov space* is defined by

$$B^{s,\phi}_{p,q}(\mathbb{T};X) = \{f \in \mathcal{D}'(\mathbb{T};X) : \|f\|_{B^{s,\phi}_{p,q}} < \infty\}$$

where

$$\|f\|_{B^{s,\phi}_{p,q}} = \Big(\sum_{j\geq 0} 2^{jsq} \Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k)\Big\|_p^q\Big)^{1/q}$$

with the usual modification when $p = \infty$ or $q = \infty$. The space $B^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$, and the norms $\|\cdot\|_{B^{s,\phi}_{p,q}}$ for different ϕ are equivalent. We will denote $\|\cdot\|_{B^{s,\phi}_{p,q}}$ simply by $\|\cdot\|_{B^s_{p,q}}$.

For further references on these spaces and their properties, see [7].

Theorem 4.2 ([7]). *Let $1 \leq p, q \leq \infty$, and $s \in \mathbb{R}$. Let X and Y be Banach spaces. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ is $\mathcal{M}$-bounded of order 2, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

Recall that Theorem 4.2 does not impose any conditions on the Banach spaces X and Y.

Lemma 4.3. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families of operators, then*

$$\{k^2 a_k \Delta^2 N_k\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \{k^3 B \Delta^2 N_k\}_{k\in\mathbb{Z}}$$

are also bounded.

Proof. We follow the proof of Lemma 3.7. Note that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j\in\mathbb{Z}$ fixed, $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded. For all $k \in \mathbb{Z}$, we have

$$k^2 a_k \Delta^2 N_k = i\gamma k a_k (N_k - N_{k+2}) kBN_{k+1} - M_k k^2 \frac{\Delta^2 b_k}{b_{k+1}} \frac{b_{k+1}}{a_{k+1}} M_{k+1} + k a_k (N_{k+2} - N_k) k \frac{\Delta^1 b_{k+1}}{b_{k+1}} \frac{b_{k+1}}{a_{k+1}} M_{k+1}$$

and

$$k^3 B \Delta^2 N_k = k^2 B (N_k - N_{k+2}) kBN_{k+1} - kBN_k k^2 \frac{\Delta^2 b_k}{b_{k+1}} \frac{b_{k+1}}{a_{k+1}} M_{k+1} - k^2 B (N_{k+2} - N_k) k \frac{\Delta^1 b_{k+1}}{b_{k+1}} \frac{b_{k+1}}{a_{k+1}} M_{k+1}.$$

Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 2-regular sequence, Lemma 3.7 shows that $\{k^2 a_k \Delta^2 N_k\}_{k\in\mathbb{Z}}$ and $\{k^3 B \Delta^2 N_k\}_{k\in\mathbb{Z}}$ are bounded. ∎

Our two principal results in this section are Theorems 4.4 and 4.5 below.

THEOREM 4.4. *Let $1 \le p, q \le \infty$, and $s > 0$. Let $\alpha, \beta, \gamma \in \mathbb{R}_+$, and let A and B be closed linear operators defined on a Banach space X. The following assertions are equivalent:*

(i) *$\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.*
(ii) *$\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers.*

Proof. (i)⇒(ii). According to Theorem 4.2, we need to show that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathcal{M}$-bounded of order 2. Exactly the same calculation made in Theorem 3.8 displays that $k\Delta^1 M_k$ and $k\Delta^1(kBN_k)$ are uniformly bounded. Now note that

$$k^2 \Delta^2 M_k = k^2 a_k \Delta^2 N_k + k^2 \frac{\Delta^2 a_k}{a_{k+1}} M_{k+1} - k \frac{\Delta^1 a_k}{a_k} k a_k (N_k - N_{k+2}).$$

Also

$$k^2 \Delta^2 (kBN_k) = k^3 B \Delta^2 N_k + k^2 B (N_{k+2} - N_k).$$

From Lemmas 3.7 and 4.3 we conclude that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathcal{M}$-bounded of order 2.

(ii)⇒(i). It follows from the Closed Graph Theorem that there exists a $C \ge 0$ (independent of f) such that, for $f \in B^s_{p,q}(\mathbb{T}; X)$, we have

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \widehat{f}(k) \Big\|_{B^s_{p,q}} \le C \|f\|_{B^s_{p,q}}.$$

Let $x \in X$, and define $f(t) = e^{ikt} x$ for $k \in \mathbb{Z}$ fixed. Then the preceding inequality implies

$$\|e_k\|_{B^s_{p,q}} \|M_k x\|_{B^s_{p,q}} = \|e_k M_k x\|_{B^s_{p,q}} \le C \|e_k\|_{B^s_{p,q}} \|x\|_{B^s_{p,q}}.$$

Hence $\|M_k\| \leq C$ for all $k \in \mathbb{Z}$, and $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Similarly, $\sup_{k\in\mathbb{Z}} \|kBN_k\| < \infty$. ∎

THEOREM 4.5. *Let $1 \leq p, q \leq \infty$, and $s > 0$. Let X be a Banach space. The following assertions are equivalent:*

(i) *Equation* (1.4) *has $B^s_{p,q}$-maximal regularity.*
(ii) *$\sigma(A, B) = \emptyset$, and the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded.*

Proof. (i)⇒(ii). The same proof as that of Theorem 3.9 shows that, for all $k \in \mathbb{Z}$, $b_k + i\gamma kB + \beta A$ has an inverse. Suppose $b_k + i\gamma kB + \beta A$ has no bounded inverse. Then for each $k \in \mathbb{Z}$, there exists a sequence $\{y_{k,n}\}_{n\in\mathbb{Z}} \subseteq X$ such that

$$\|y_{n,k}\| \leq 1 \quad \text{and} \quad \|(b_k + i\gamma kB + \beta A)^{-1} y_{k,n}\| \geq |n|^{2+s}, \qquad \text{for all } n \in \mathbb{Z}.$$

Defining $x_k = y_{k,k}$, we have

$$\|(b_k + i\gamma kB + \beta A)^{-1} x_k\| \geq |k|^{2+s} \qquad \text{for all } k \in \mathbb{Z}.$$

Let

$$g(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{x_k}{|k|^{2+s}} e^{ikt}.$$

Note that $g \in B^s_{p,q}(\mathbb{T}; X)$. In fact,

$$\begin{aligned}\sum_{j\geq 0} 2^{jsq} \Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{g}(k)\Big\|_p^q &= \sum_{j\geq 0} 2^{jsq} \bigg\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \frac{x_k}{|k|^{2+s}}\bigg\|_p^q \\ &= \sum_{j\geq 0} 2^{jsq} \bigg\|\sum_{k\in\mathbb{Z}} e_k \otimes \frac{1}{|k|^2}\phi_j(k) \frac{x_k}{|k|^s}\bigg\|_p^q.\end{aligned}$$

Since $\operatorname{supp}(\phi_j) \subseteq [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$ and by the estimation made in the construction of Besov spaces, we have the inequality

$$\sum_{j\geq 0} 2^{jsq} \Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{g}(k)\Big\|_p^q \leq \sum_{j\geq 0} 2^{jsq} \frac{C}{2^{jq}} \frac{1}{2^{q(j-1)s}} < \infty.$$

By hypothesis, there exists a unique strong $B^s_{p,q}$-solution u of (1.4). Since (1.4) holds for almost $t \in [0, 2\pi]$, taking the Fourier transform we obtain

$$\widehat{u}(k) = -(b_k + i\gamma kB + \beta A)^{-1}\widehat{g}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

We know that $u(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} -\frac{x_k}{|k|^{2+s}}(b_k + i\gamma kB + \beta A)^{-1} e^{ikt}$, and since

$$\bigg\|\frac{x_k}{|k|^{2+s}}(b_k + i\gamma kB + \beta A)^{-1}\bigg\| \geq 1$$

we have $u \notin B^s_{p,q}(\mathbb{T}; X)$, a contradiction. Hence $(b_k + i\gamma kB + \beta B)^{-1} \in \mathcal{B}(X)$ for all $k \in \mathbb{Z}$. Therefore, $\sigma(A, B) = \emptyset$.

By an analogous idea to the proof of Theorem 3.9, we deduce that the families $\{a_k(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{kB(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. The result follows from Theorem 4.4.

(ii)⇒(i). By (ii) and Theorem 4.4, $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. Given $f \in B^s_{p,q}(\mathbb{T};X)$, there exists $u \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{u}(k) = (-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{4.1}$$

Since $1 \le p, q \le \infty$ and $s > 0$, we have $B^s_{p,q}(\mathbb{T};X) \subseteq L^p(\mathbb{T};X)$. Lemma 3.5 shows that $u(t) \in D(A) \cap D(B)$ for almost $t \in [0, 2\pi]$.

Define $I_k = (1/a_k)I$ if $k \ne 0$ and $I_0 = I$. According to Theorem 4.2, the family $\{I_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Hence $\{I_kM_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. In other words, $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Thus, there exists a function u_3 such that, for all integers k, we have

$$\widehat{u_3}(k) = -ik^3(-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k).$$

By (4.1), for all $k \in \mathbb{Z}$, we have $\widehat{u_3}(k) = -ik^3\widehat{u}(k)$. Thus, $u \in B^{s+3}_{p,q}(\mathbb{T};X)$.

On the other hand, since $\{kBN_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, there exists a function $v \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{v}(k) = ikB(-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

It follows from (4.1) that $\widehat{v}(k) = ikB\widehat{u}(k)$ for all $k \in \mathbb{Z}$.

Moreover, since $B^{s+3}_{p,q}(\mathbb{T};X) \subseteq B^{s+1}_{p,q}(\mathbb{T};X)$, we have $u' \in B^s_{p,q}(\mathbb{T};X)$ and $\widehat{u'}(k) = ik\widehat{u}(k)$ for all $k \in \mathbb{Z}$. Since B is a closed linear operator, we have

$$\widehat{v}(k) = B(ik\widehat{u}(k)) = B\widehat{u'}(k) = \widehat{Bu'}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By the uniqueness of the Fourier coefficients, $v = Bu'$. This implies that $u' \in B^s_{p,q}(\mathbb{T};[D(B)])$. Accordingly $u \in B^{s+1}_{p,q}(\mathbb{T};[D(B)])$.

Following the lines of the proof of Theorem 3.9 we note that the family $\{\beta A(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Hence, there exists $w \in B^s_{p,q}(\mathbb{T};X)$ satisfying

$$\widehat{w}(k) = A(-b_k - i\gamma B - \beta A)^{-1}\widehat{f}(k) \quad \text{for all } k,$$

hence $\widehat{w}(k) = A\widehat{u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of Fourier coefficients, we conclude that $w = Au$, so $u \in B^s_{p,q}(\mathbb{T};[D(A)])$. Therefore $u \in B^{s+3}_{p,q}(\mathbb{T};X) \cap B^{s+1}_{p,q}(\mathbb{T};[D(B)]) \cap B^s_{p,q}(\mathbb{T};[D(A)])$. As $u \in B^{s+3}_{p,q}(\mathbb{T};X)$, we have $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$. Since A and B are closed linear operators, it now follows from (4.1) that

$$\alpha\widehat{u'''}(k) + \widehat{u''}(k) = \beta\widehat{Au}(k) + \gamma\widehat{Bu}(k) + \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

From the uniqueness of Fourier coefficients, we conclude that (1.4) holds a.e. in $[0, 2\pi]$. Therefore u is a strong $B^s_{p,q}$-solution of (1.4). Using the same argument as for Theorem 3.9 we find that this solution is unique. ∎

In our next corollaries, we use the notations $\{S_k\}_{k\in\mathbb{Z}}$, $\{c_k\}_{k\in\mathbb{Z}}$ and $\{d_k\}_{k\in\mathbb{Z}}$, introduced in Section 3. The proofs are similar to the corresponding ones of Section 3, so we omit them.

Corollary 4.6. *Let $1 \le p,q \le \infty$, $s > 0$ and X a Banach space. Assume that the families $\{a_k S_k\}_{k\in\mathbb{Z}}$ and $\{(i\gamma k/\beta)BS_k\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}} \|a_k S_k\| < 1$, then equation* (1.4) *has $B^s_{p,q}$-maximal regularity.*

Corollary 4.7. *Let X be a Banach space and $1 \le p,q \le \infty$ and $s > 0$. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv 0$ has $B^s_{p,q}$-maximal regularity.*
(ii) *$\{c_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{a_k(c_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.*

Corollary 4.8. *Let X be a Banach space and $1 \le p,q \le \infty$ and $s > 0$. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv A$ has $B^s_{p,q}$-maximal regularity.*
(ii) *$\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.*

5. Maximal regularity for a third-order differential equation in periodic Triebel–Lizorkin spaces. In this section, we study maximal regularity for equation (1.4) in periodic Triebel–Lizorkin spaces. We briefly recall their definition in the vector-valued case (see [14]). We use the notations $\mathcal{S}(\mathbb{R};X)$, $\mathcal{S}'(\mathbb{R};X)$, $\mathcal{D}'(\mathbb{T};X)$ and $\Phi(\mathbb{R})$ of the preceding section.

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \Phi(\mathbb{R})$ be fixed, for $1 \le p,q \le \infty$, and $s \in \mathbb{R}$. The *X-valued periodic Triebel–Lizorkin spaces* is defined by

$$F^{s,\phi}_{p,q}(\mathbb{T};X) = \{f \in \mathcal{D}'(\mathbb{T};X) : \|f\|_{F^{s,\phi}_{p,q}} < \infty\}$$

where

$$\|f\|_{F^{s,\phi}_{p,q}} = \Big\| \Big(\sum_{j\ge 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \widehat{f}(k) \Big\|_X^q \Big)^{1/q} \Big\|_p$$

with the usual modification when $p = \infty$ or $q = \infty$. The space $F^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$, and the norms $\|\cdot\|_{F^{s,\phi}_{p,q}}$ for different ϕ are equivalent. Consequently, we simply denote $\|\cdot\|_{F^{s,\phi}_{p,q}}$ by $\|\cdot\|_{F^s_{p,q}}$.

Theorem 5.1 ([14]). *Let $1 \le p,q \le \infty$, $s \in \mathbb{R}$, and let X, Y be Banach spaces. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ is $\mathcal{M}$-bounded of order* 3, *then $\{L_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.*

Recall that Theorem 5.1, as in the case of Theorem 4.2, does not impose any conditions on the underlying Banach spaces X and Y.

The proof of Theorem 5.3 below will depend on our next result.

LEMMA 5.2. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families, then so are*

$$\{k^3 a_k \Delta^3 N_k\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \{k^4 B\Delta^3 N_k\}_{k\in\mathbb{Z}}.$$

Proof. We follow the proofs of Lemmas 3.7 and 4.3. We note that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j \in \mathbb{Z}$ fixed, $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded. Using the calculations of Lemma 4.3, we see that for all $k \in \mathbb{Z}$,

$$\Delta^2 N_k = (N_{k+2} - N_k)(-\Delta^1 b_{k+2} - i\gamma B)N_{k+1} - N_k(\Delta^2 b_k)N_{k+1}.$$

Therefore,

$$\begin{aligned}
(5.1)\quad & k^3 a_k \Delta^3 N_k \\
&= k^2 a_k(\Delta^2 N_{k+1})k(-\Delta b_{k+2} - i\gamma B)N_{k+2} \\
&\quad + k^3 a_k(\Delta^2 N_k)k(-\Delta b_{k+2} - i\gamma B)N_{k+2} \\
&\quad - k a_k(N_{k+2} - N_k)k^2 \frac{\Delta^2 b_{k+1}}{b_{k+2}} b_{k+2} N_{k+2} \\
&\quad + k a_k(N_{k+2} - N_k)k^2(-\Delta b_{k+1} - i\gamma B)\Delta^1 N_{k+1} \\
&\quad - k^3 \frac{\Delta^3 b_k}{b_{k+2}} a_k N_{k+1} b_{k+2} N_{k+2} - k^2 \frac{\Delta^2 b_k}{b_{k+2}} k a_k(\Delta^1 N_k) b_{k+2} N_{k+2} \\
&\quad - k^2 \frac{\Delta^2 b_k}{b_k} b_k N_k k a_k(N_{k+2} - N_k).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(5.2)\quad & k^4 B\Delta^3 N_k \\
&= k^3 B(\Delta^2 N_{k+1})k(-\Delta^1 b_{k+2} - i\gamma B)N_{k+2} \\
&\quad + k^3 B(\Delta^2 N_k)k(-\Delta^2 b_{k+2} - i\gamma B)N_{k+2} \\
&\quad + k^2 B(N_{k+2} - N_k)k^2 \frac{\Delta^2 b_{k+1}}{b_{k+2}} b_{k+2} N_{k+2} \\
&\quad - k^2 B(N_{k+2} - N_k)k^2(-\Delta^2 b_{k+2} - i\gamma B)(N_{k+2} - N_k) \\
&\quad + \frac{k^3 \Delta^3 b_k}{b_{k+2}} i a_k N_{k+1} b_{k+2} N_{k+2} - \frac{k^2 \Delta^2 b_k}{b_{k+2}} k^2 B(\Delta^1 N_k) b_{k+2} N_{k+2} \\
&\quad - \frac{k^2 \Delta^2 b_k}{b_k} b_k B N_k k^2 (N_{k+2} - N_k).
\end{aligned}$$

Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 3-regular sequence, it follows from Lemmas 3.7 and 4.3 that all the terms on the right side of (5.1) and (5.2) are uniformly bounded. Therefore, $\{k^3 a_k \Delta^3 N_k\}_{k\in\mathbb{Z}}$ and $\{k^4 B\Delta^3 N_k\}_{k\in\mathbb{Z}}$ are bounded. ∎

Our two principal results in this section are Theorems 5.3 and 5.4 below.

Theorem 5.3. *Let $1 \le p, q \le \infty$, and $s > 0$, and let A and B be closed linear operators defined on a Banach space X. The following assertions are equivalent:*

(i) $\{kBN_k\}_{k\in\mathbb{Z}}$ *and* $\{M_k\}_{k\in\mathbb{Z}}$ *are bounded.*
(ii) $\{kBN_k\}_{k\in\mathbb{Z}}$ *and* $\{M_k\}_{k\in\mathbb{Z}}$ *are* $F^s_{p,q}$*-multipliers.*

Proof. (i)$\Rightarrow$(ii). Theorem 4.4 shows that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathcal{M}$-bounded of order 2. Moreover,

$$\begin{aligned}k^3\Delta^3 M_k = k^3 a_k \Delta^3 N_k + k^3(a_{k+3} - a_k)\Delta^2 N_{k+1} + k^3(\Delta^2 a_{k+1})(\Delta^1 N_{k+1})\\ - 2k^3(\Delta^2 a_k)(\Delta^1 N_{k+1}) + (\Delta^3 a_k) N_{k+2},\end{aligned}$$

and

$$k^3\Delta^3(kBN_k) = k^4 B\Delta^3 N_k + 3k^3 B\Delta^2 N_{k+1}.$$

It follows from Lemmas 3.7, 4.3 and 5.2 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}$ are $\mathcal{M}$-bounded of order 3. Condition (ii) now follows from Theorem 5.1.

(ii)$\Rightarrow$(i). The proof follows the same lines as that of Theorem 4.4. ∎

Theorem 5.4. *Let $1 \le p, q \le \infty$. If $s > 0$ and X is a Banach space, then the following assertions are equivalent:*

(i) *Equation* (1.4) *has* $F^s_{p,q}$*-maximal regularity.*
(ii) $\sigma(A,B) = \emptyset$, *and the families* $\{M_k\}_{k\in\mathbb{Z}}$ *and* $\{kBN_k\}_{k\in\mathbb{Z}}$ *are bounded.*

Proof. The proof is similar to that of Theorem 4.5. ∎

In our next corollaries, we use the notations $\{S_k\}_{k\in\mathbb{Z}}$, $\{c_k\}_{k\in\mathbb{Z}}$ and $\{d_k\}_{k\in\mathbb{Z}}$, introduced in Section 3. The proofs are similar to the corresponding ones of Section 3, so we omit them.

Corollary 5.5. *Let $1 \le p, q \le \infty$, $s > 0$, and X a Banach space. Suppose that the families $\{a_k S_k\}_{k\in\mathbb{Z}}$ and $\{ik(\gamma/\beta)BS_k\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}} \|a_k S_k\| < 1$, then equation* (1.4) *has $F^s_{p,q}$-maximal regularity.*

Corollary 5.6. *Let $1 \le p, q \le \infty$, $s > 0$ and X a Banach space. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv 0$ has $F^s_{p,q}$-maximal regularity.*
(ii) $\{c_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{a_k(c_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is bounded.*

Corollary 5.7. *Let $1 \le p, q \le \infty$, $s > 0$, and X a Banach space. The following assertions are equivalent:*

(i) *Equation* (1.4) *with $B \equiv A$ has $F^s_{p,q}$-maximal regularity.*
(ii) $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is bounded.*

6. Examples. In this section, we apply our results to some interesting examples.

EXAMPLE 6.1. Let $\alpha, \beta, \gamma \in \mathbb{R}_+$. Let $1 \le p, q \le \infty$, and $s > 0$. Consider the abstract equation

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A u'(t) + f(t) \quad \text{for } t \in [0, 2\pi] \tag{6.1}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, and A a positive selfadjoint operator defined on a Hilbert space X such that $\inf_{\lambda \in \sigma(A)}\{\lambda\} \neq 0$. If $f \in L^2(\mathbb{T}; X)$ (resp. $B^s_{p,q}(\mathbb{T}; X)$ and $F^s_{p,q}(\mathbb{T}; X)$), then equation (6.1) has L^2-maximal regularity (resp. $B^s_{p,q}$-maximal and $F^s_{p,q}$-maximal regularity).

Proof. We have

$$d_k = \frac{-(\alpha\gamma k^4 + \beta k^2)}{(\gamma k)^2 + \beta^2} + i\frac{(\gamma - \alpha\beta)k^3}{(\gamma k)^2 + \beta^2}.$$

Since A is positive selfadjoint such that $\inf_{\lambda \in \sigma(A)} \|\lambda\| \neq 0$, we know that $\sigma(A) \subseteq [\varepsilon, \infty)$ with some $\varepsilon > 0$. This implies that $d_k \in \rho(A)$ for all $k \in \mathbb{Z}$. Moreover, by [38, Chapter 5, Section 3.5],

$$\|(d_k - A)^{-1}\| = \frac{1}{\operatorname{dist}(d_k, \sigma(A))}.$$

Therefore, $\sup_{k \in \mathbb{Z}} \|d_k(d_k - A)^{-1}\| < \infty$. It follows from Corollary 3.12 that equation (6.1) has L^2-maximal regularity. According to Corollaries 4.8 and 5.7, equation (6.1) has, respectively, $B^s_{p,q}$-maximal regularity and $F^s_{p,q}$-maximal regularity. ∎

For the next example we need to introduce some preliminaries on sectorial operators. Denote by $\Sigma_\phi \subseteq \mathbb{C}$ the open sector

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}.$$

We denote

$$\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$$

and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

$\mathcal{H}^\infty(\Sigma_\phi)$ is endowed with the norm

$$\|f\|^\phi_\infty = \sup_{|\arg(\lambda)| < \phi} |f(\lambda)|.$$

We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows:

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \{f \in \mathcal{H}(\Sigma_\phi) : \|f\|^\infty_{\alpha,\beta} < \infty\}$$

where

$$\|f\|_{\alpha,\beta}^{\infty} = \sup_{|\lambda|\le 1} |\lambda^{\alpha} f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)|.$$

DEFINITION 6.2. A closed linear operator A in X is called *sectorial* if the following two conditions hold:

(i) $\overline{D(A)} = X$, $\overline{R(A)} = X$, and $(-\infty, 0) \subseteq \rho(A)$.
(ii) $\sup_{t>0} \|t(t+A)^{-1}\| \le M$ for some $M > 0$.

A is called *R-sectorial* if the set $\{t(t+A)^{-1}\}_{t>0}$ is R-bounded. We denote the class of sectorial operators (resp. R-sectorial operators) in X by $\mathcal{S}(X)$ (resp. $\mathcal{RS}(X)$).

If $A \in \mathcal{S}(X)$, then $\Sigma_{\phi} \subseteq \rho(-A)$ for some $\phi > 0$ and

$$\sup_{|\arg(\lambda)|<\phi} \|\lambda(\lambda+A)^{-1}\| < \infty.$$

We denote the *spectral angle* of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\Big\{\phi : \Sigma_{\pi-\phi} \subseteq \rho(-A),\ \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda+A)^{-1}\| < \infty\Big\}.$$

DEFINITION 6.3. Let A be a sectorial operator. If there exist $\phi > \phi_A$ and a constant $K_{\phi} > 0$ such that

$$\|f(A)\| \le K_{\phi} \|f\|_{\infty}^{\phi} \quad \text{for all } f \in \mathcal{H}_0(\Sigma_{\phi}) \tag{6.2}$$

then we say that a sectorial operator A *admits a bounded $\mathcal{H}^{\infty}$-calculus.*

We denote the class of sectorial operators A which admit a bounded $\mathcal{H}^{\infty}$-calculus by $\mathcal{H}^{\infty}(X)$. Moreover, the *$\mathcal{H}^{\infty}$-angle* is defined by

$$\phi_A^{\infty} = \inf\{\phi > \phi_A : (6.2) \text{ holds for some } K_{\phi}\}.$$

REMARK 6.4. Let A be a sectorial operator which admits a bounded $\mathcal{H}^{\infty}$-calculus. If the set

$$\{h(A) : h \in \mathcal{H}^{\infty}(\Sigma_{\theta}),\ \|h\|_{\infty}^{\theta} < 1\}$$

is R-bounded for some $\theta > 0$, then we say that A *admits an R-bounded $\mathcal{H}^{\infty}$-calculus.* We denote the class of such operators by $\mathcal{RH}^{\infty}(X)$. The $\mathcal{RH}^{\infty}$-angle is defined analogously to the $\mathcal{H}^{\infty}$-angle, and is denoted $\theta_A^{R_\infty}$. For further information about sectorial and R-sectorial operators, see [37].

To prove Lemma 6.6 below, we need, the following proposition from functional calculus theory (cf. [20]).

PROPOSITION 6.5. *Let $A \in \mathcal{RH}^{\infty}(X)$ and suppose that $\{h_{\lambda}\}_{\lambda\in\Lambda} \subseteq \mathcal{H}^{\infty}(\Sigma_{\theta})$ is uniformly bounded for some $\theta > \theta_A^{R_\infty}$, where Λ is an arbitrary index set. Then the set $\{h_{\lambda}(A)\}_{\lambda\in\Lambda}$ is R-bounded.*

LEMMA 6.6. *Let $\alpha, \beta \in \mathbb{R}_+$ and X be a UMD-space. If $A \in \mathcal{RH}^\infty(X)$ with $\theta_A^{R_\infty} < \pi/3$, then the families of operators*

$$\left\{ ik^3\left(-\frac{i\alpha k^3 + k^2}{\beta} - A\right)^{-1}\right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \left\{ ikA^{1/2}\left(-\frac{i\alpha k^3 + k^2}{\beta} - A\right)^{-1}\right\}_{k\in\mathbb{Z}}$$

are R-bounded.

Proof. For every $k \in \mathbb{Z}$ we define $F_k^1 : \Sigma_{\pi/3} \to \mathbb{C}$ and $F_k^2 : \Sigma_{\pi/3} \to \mathbb{C}$ by

$$F_k^1(z) = \frac{i\beta k^3}{-(i\alpha k^3 + k^2 + \beta z)} \quad \text{and} \quad F_k^2(z) = \frac{i\beta k z^{1/2}}{-(i\alpha k^3 + k^2 + \beta z)},$$

where $z^{1/2}$ is defined in $\mathbb{C} \setminus \{0\}$ and it is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Furthermore, for all $k \in \mathbb{Z}$ and $z \in \Sigma_{\pi/3}$ we have $i\alpha k^3 + k^2 + \beta z \neq 0$. Therefore, for all $k \in \mathbb{Z}$ the functions F_k^1 and F_k^2 are holomorphic in $\Sigma_{\pi/3}$.

We claim that for $j \in \{1, 2\}$ there exists a constant $M \geq 0$ such that

$$\sup_{k\in\mathbb{Z}} \|F_k^j\|_\infty^{\pi/3} \leq M.$$

Indeed, note that for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$-(i\alpha k^3 + k^2 + \beta z) = -(i\alpha k^3 + k^2)\left(1 + \frac{\beta z}{i\alpha k^3 + k^2}\right).$$

Since for all $k \in \mathbb{Z} \setminus \{0\}$ and $z \in \Sigma_{\pi/3}$ we have $\frac{\beta z}{i\alpha k^3 + k^2} \in \Sigma_{\pi/3+\pi/2}$ and the distance of -1 to this sector is positive, we have

$$\sup_{k\in\mathbb{Z}\setminus\{0\}} \|F_k^1\|_\infty^{\pi/3} \leq M_1 \quad \text{for some } M_1 \geq 0.$$

Note also that for all $k \in \mathbb{Z} \setminus \{0\}$,

$$-(i\alpha k^3 + k^2 + \beta z) = -\sqrt{i\alpha k^3 + k^2}\, z^{1/2}\left(1 + \frac{i\beta^{1/2} z^{1/2}}{\sqrt{i\alpha k^3 + k^2}}\right)\left(\frac{\sqrt{i\alpha k^3 + k^2}}{z^{1/2}} - i\beta^{1/2}\right).$$

For all $k \in \mathbb{Z} \setminus \{0\}$ and $z \in \Sigma_{\pi/3}$ we have

$$\frac{i\beta^{1/2} z^{1/2}}{\sqrt{i\alpha k^3 + k^2}} \in \Sigma_{\pi/2+\pi/6+\pi/4} \quad \text{and} \quad \frac{\sqrt{i\alpha k^3 + k^2}}{z^{1/2}} \in \Sigma_{\pi/6+\pi/4}.$$

Since the distance of -1 to $\Sigma_{11\pi/12}$ is positive and the distance of i to $\Sigma_{5\pi/12}$ is also positive, we see that $\sup_{k\in\mathbb{Z}\setminus\{0\}} \|F_k^2\|_\infty^{\pi/3} \leq M_2$ for some $M_2 \geq 0$.

In addition, for all $z \in \Sigma_{\pi/3}$ the functions $F_0^1(z) = 0 = F_0^2(z)$. Therefore, there exists $M \geq 0$ such that $\sup_{k\in\mathbb{Z}} \|F_k^j\|_\infty^{\pi/3} \leq M$ for $j = 1, 2$. With a direct computation for all $k \in \mathbb{Z}$ and $z \in \Sigma_{\pi/3}$ we have

$$F_k^1(z) = \frac{ik^3}{-\frac{i\alpha k^3 + k^2}{\beta} - z} \quad \text{and} \quad F_k^2(z) = \frac{ikz^{1/2}}{\frac{-(i\alpha k^3 + k^2)}{\beta} - z}.$$

Since $A \in \mathcal{RH}^\infty(X)$, for all $k \in \mathbb{Z}$ the operators

$$F_k^1(A) = ik^3\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} \quad \text{and} \quad F_k^2(A) = ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}$$

exists. It follows from Proposition 6.5 that the families of operators $\{F_k^1(A)\}_{k\in\mathbb{Z}}$ and $\{F_k^2(A)\}_{k\in\mathbb{Z}}$ are R-bounded. ∎

EXAMPLE 6.7. Let X be a UMD-space, and let $p \in (1,\infty)$. Suppose $A \in \mathcal{RH}^\infty(X)$ with $\theta_A^{R_\infty} < \pi/3$. Consider the family of operators

$$\mathcal{F} = \Big\{ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} : k \in \mathbb{Z}\Big\}$$

with $\alpha, \beta > 0$. If $\gamma > 0$ is such that $(\gamma/\beta)\mathcal{R}_p(\mathcal{F}) < 1$, then the equation

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A^{1/2} u'(t) + f(t) \quad \text{for } t \in [0, 2\pi] \tag{6.3}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$ has L^p-maximal regularity.

Proof. According to Lemma 6.6, the families of operators

$$\Big\{ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}\Big\}_{k\in\mathbb{Z}} \quad \text{and} \quad \Big\{ik^3\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}\Big\}_{k\in\mathbb{Z}}$$

are R-bounded. Since $(\gamma/\beta)\mathcal{R}_p(\mathcal{F}) < 1$, it follows from Corollary 3.10 that equation (6.3) has L^p-maximal regularity. ∎

Acknowledgements. This research was partially supported by FONDECYT Grant # 1110090 and the second author was partially supported by MECESUP PUC 0711.

References

[1] H. Amann, *Linear and Quasilinear Parabolic Problems*, Monogr. Math. 89, Birkhäuser, Basel, 1995.

[2] H. Amann, *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr. 186 (1997), 5–56.

[3] B. de Andrade and C. Lizama, *Existence of asymptotically almost periodic solutions for damped wave equations*, J. Math. Anal. Appl. 382 (2011), 761–771.

[4] W. Arendt, C. Batty and S. Bu, *Fourier multipliers for Hölder continuous functions and maximal regularity*, Studia Math. 160 (2004), 23–51.

[5] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monogr. Math. 96, Birkhäuser, Basel, 2001.

[6] W. Arendt and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. 240 (2002), 311–343.

[7] W. Arendt and S. Bu, *Operator-valued Fourier multipliers on periodic Besov spaces and applications*, Proc. Edinburgh Math. Soc. 47 (2004), 15–33.

[8] W. Arendt and S. Bu, *Tools for maximal regularity*, Math. Proc. Cambridge Philos. Soc. 134 (2003), 317–336.

[9] S. Bose and G. Gorain, *Stability of the boundary stabilised damped wave equation* $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ *in a bounded domain in* $\mathbb{R}^n$, Indian J. Math. 40 (1998), 1–15.

[10] J. Bourgain, *Some remarks on Banach spaces in which martingale differences sequences are unconditional*, Ark. Mat. 21 (1983), 163–168.

[11] S. Bu, *Mild well-posedness of vector-valued problems*, Arch. Math. (Basel) 95 (2010), 63–73.

[12] S. Bu, *Well-posedness of second order degenerate differential equations in vector-valued function spaces*, Studia Math. 214 (2013), 1–16.

[13] S. Bu and Y. Fang, *Maximal regularity of second order delay equations in Banach spaces*, Sci. China Math. 53 (2010), 51–62.

[14] S. Bu and J. Kim, *Operator-valued Fourier multipliers on periodic Triebel spaces*, Acta Math. Sinica (English Ser.) 21 (2004), 1049–1056.

[15] D. Burkholder, *A geometrical condition that implies the existence of certain singular integrals on Banach-space-valued functions*, in: Proc. Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981), Wadsworth, Belmont, CA, 1983, 270–286.

[16] D. Burkholder, *Martingales and singular integrals in Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 1, North-Holland, Amsterdam, 2001, 233–269.

[17] Ph. Clément, *On the method of sums of operators*, in: Semi-groupes d'opérateurs et calcul fonctionnel (Besançon, 1998), Publ. Math. UFR Sci. Tech. Besançon 16, Univ. Franche-Comté, Besançon, 1998, 1–30.

[18] P. Clément, B. de Pagter, F. Sukochev and H. Witvliet, *Schauder decompositions and multiplier theorems*, Studia Math. 138 (2000), 135–163.

[19] C. Cuevas and C. Lizama, *Well posedness for a class of flexible structure in Hölder spaces*, Math. Problems Engrg. 2009, art. ID 358329, 13 pp.

[20] R. Denk, M. Hieber and J. Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. 166 (2003), no. 788.

[21] C. Fernández, C. Lizama and V. Poblete, *Maximal regularity for flexible structural systems in Lebesgue spaces*, Math. Problems Engrg. 2009, art. ID196956, 15 pp.

[22] C. Fernández, C. Lizama and V. Poblete, *Regularity of solutions for a third order differential equation in Hilbert spaces*, Appl. Math. Comput. 217 (2011), 8522–8533.

[23] M. Girardi and L. Weis, *Criteria for R-boundedness of operator families*, in: Lecture Notes in Pure Appl. Math. 234, Dekker, New York, 2003, 203–221.

[24] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on Besov spaces*, Math. Nachr. 251 (2003), 34–51.

[25] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on* $L_p(X)$ *and geometry of Banach spaces*, J. Funct. Anal. 204 (2003), 320–354.

[26] G. Gorain, *Exponential energy decay estimate for the solutions of internally damped wave equation in a bounded domain*, J. Math. Anal. Appl. 216 (1997), 510–520.

[27] G. Gorain, *Boundary stabilization of nonlinear vibrations of a flexible structure in a bounded domain in* $\mathbb{R}^n$, J. Math. Anal. Appl. 319 (2006), 635–650.

[28] G. Gorain, *Stabilization for the vibrations modeled by the standard linear models of viscoelasticity*, Proc. Indian Acad. Sci. (Math. Sci.) 120 (2010), 495–506.

[29] G. Gorain and S. Bose, *Exact controllability and boundary stabilization of flexural vibrations of an internally damped flexible space structure*, Appl. Math. Comput. 126 (2002), 341–360.

[30] H. Henriquez and V. Poblete, *Periodic solutions of neutral fractional differential equations*, submitted.

[31] T. Hytönen, *R-boundedness and Multiplier Theorems*, Helsinki University of Technology Institute of Mathematics Research Reports.

[32] P. Jordan, *Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons*, lecture, 9th International Conf. on Theoretical and Computational Acoustic (ICTCA 2009), Dresden, 2009; for abstract see J. Acoust. Soc. Amer. 124 (2008), 2491.

[33] B. Kaltenbacher and I. Lasiecka, *Well-posedness of the Westervelt and the Kuznetsov equation with nonhomogeneous Neumann boundary conditions*, Discrete Contin. Dynam. Syst. 2011, Dynamical Systems, Differential Equations and Applications, 8th AIMS Conf., Suppl. Vol. II, 763–773.

[34] B. Kaltenbacher and I. Lasiecka, *An analysis of nonhomogeneous Kuznetsov's equation: local and global well-posedness; exponential decay*, Math. Nachr. 285 (2012), 295–321.

[35] B. Kaltenbacher, I. Lasiecka and R. Marchand, *Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound*, Control Cybernet. 40 (2011), 971–988.

[36] N. Kalton and G. Lancien, *A solution to the problem of L^p-maximal regularity*, Math. Z. 235 (2000), 559–568.

[37] N. Kalton and L. Weis, *The H^∞ calculus and sums of closed operators*, Math. Ann. 321 (2001), 319–345.

[38] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren Math. Wiss. 132, Springer, New York, 1980.

[39] V. Keyantuo and C. Lizama, *Periodic solutions of second order differential equations in Banach spaces*, Math. Z. 253 (2006), 489–514.

[40] V. Keyantuo and C. Lizama, *Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces*, Studia Math. 168 (2005), 25–50.

[41] V. Keyantuo and C. Lizama, *Mild well-posedness of abstract differential equations*, in: Functional Analysis and Evolution Equations, The Günter Lumer Volume (H. Amann et al., eds.), Birkhäuser, 2008, 371–387.

[42] V. Keyantuo, C. Lizama and V. Poblete, *Periodic solutions of integro-differential equations in vector-valued function spaces*, J. Differential Equations 246 (2009), 1007–1037.

[43] C. Lizama, *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl. 324 (2006), 921–933.

[44] S. Meyer and M. Wilke, *Optimal regularity and long-time behavior of solutions for the Westervelt equation*, Appl. Math. Optim. 64 (2011), 257–271.

[45] F. Moore and W. Gibson, *Propagation of weak disturbances in a gas subject to relaxations effects*, J. Aerospace Sci. Technol. 27 (1960), 117–127.

[46] V. Poblete, *Fourier multipliers and maximal regularity for integrodifferential equations in Banach spaces*, Ph.D. thesis, USACH, 2006.

[47] V. Poblete, *Maximal regularity of second-order equations with delay*, J. Differential Equations 246 (2009), 261–276.

[48] P. Thompson, *Compressible-Fluid Dynamics*, McGraw-Hill, New York, 1972.

[49] R. Zacher, *Maximal regularity of type L_p for abstract parabolic Volterra equations*, J. Evol. Equations 5 (2005), 79–103.

Verónica Poblete
Facultad de Ciencias
Universidad de Chile
Las Palmeras 3425
Santiago, Chile
E-mail: vpoblete@uchile.cl

Juan C. Pozo
Facultad de Economía y Empresa
Universidad Diego Portales
Avda. Santa Clara 797, Huechuraba
Santiago, Chile
E-mail: jpozo@ug.uchile.cl

Received March 23, 2012
Revised version April 30, 2013 (7465)