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Pisier's inequality revisited

by

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Abstract. Given a Banach space X, for $n \in \mathbb{N}$ and $p \in (1,\infty)$ we investigate the smallest constant $\mathfrak{P} \in (0,\infty)$ for which every n-tuple of functions $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfies

$$\int_{\{-1,1\}^n} \Big\| \sum_{j=1}^n \partial_j f_j(\varepsilon) \Big\|^p d\mu(\varepsilon) \le \mathfrak{P}^p \int_{\{-1,1\}^n} \int_{\{-1,1\}^n} \Big\| \sum_{j=1}^n \delta_j \Delta f_j(\varepsilon) \Big\|^p d\mu(\varepsilon)\, d\mu(\delta),$$

where μ is the uniform probability measure on the discrete hypercube $\{-1,1\}^n$, and $\{\partial_j\}_{j=1}^n$ and $\Delta = \sum_{j=1}^n \partial_j$ are the hypercube partial derivatives and the hypercube Laplacian, respectively. Denoting this constant by $\mathfrak{P}_p^n(X)$, we show that

$$\mathfrak{P}_p^n(X) \le \sum_{k=1}^n \frac{1}{k}$$

for every Banach space $(X, \|\cdot\|)$. This extends the classical Pisier inequality, which corresponds to the special case $f_j = \Delta^{-1}\partial_j f$ for some $f : \{-1,1\}^n \to X$. We show that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) < \infty$ if either the dual X^* is a UMD$^+$ Banach space, or for some $\theta \in (0,1)$ we have $X = [H,Y]_\theta$, where H is a Hilbert space and Y is an arbitrary Banach space. It follows that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) < \infty$ if X is a Banach lattice of nontrivial type.

1. Introduction. Fix a Banach space $(X, \|\cdot\|)$ and $n \in \mathbb{N}$. For every $f : \{-1,1\}^n \to X$ and $j \in \{1,\ldots,n\}$ the *hypercube jth partial derivative* of f, which is denoted $\partial_j f : \{-1,1\}^n \to X$, is defined as

$$\partial_j f(\varepsilon) := \frac{f(\varepsilon) - f(\varepsilon_1,\ldots,\varepsilon_{j-1},-\varepsilon_j,\varepsilon_{j+1},\ldots,\varepsilon_n)}{2}. \tag{1.1}$$

The *hypercube Laplacian* of f, denoted $\Delta f : \{-1,1\}^n \to X$, is

$$\Delta f(\varepsilon) := \sum_{j=1}^n \partial_j f(\varepsilon). \tag{1.2}$$

It is immediate to check that Δ is invertible on the space of all mean zero functions $f : \{-1,1\}^n \to X$. Below, Δ^{-1} is understood to be defined for every $f : \{-1,1\}^n \to X$ by setting $\Delta^{-1} f = \Delta^{-1}\overline{f}$. Here $\overline{f} =$

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$f-\int_{\{-1,1\}^n} f(\delta)\,d\mu(\delta)$, where μ denotes the uniform probability measure on $\{-1,1\}^n$.

The following inequality is due to Pisier [28]. Throughout the present paper the asymptotic notation $\lesssim, \gtrsim$ indicates the corresponding inequalities up to universal constant factors. We will also denote by $\asymp$ equivalence up to universal constant factors, i.e., $A \asymp B$ is the same as $(A \lesssim B) \wedge (A \gtrsim B)$.

Theorem 1.1 (Pisier's inequality). *For every Banach space $(X, \|\cdot\|)$, every $n \in \mathbb{N}$, every $p \in [1,\infty]$ and every $f : \{-1,1\}^n \to X$, we have*

$$(1.3)\quad \Big(\int_{\{-1,1\}^n} \Big\| f(\varepsilon) - \int_{\{-1,1\}^n} f(\delta)\,d\mu(\delta)\Big\|^p d\mu(\varepsilon)\Big)^{1/p} \lesssim \log n \Big(\int_{\{-1,1\}^n}\int_{\{-1,1\}^n} \Big\|\sum_{j=1}^n \delta_j \partial_j f(\varepsilon)\Big\|^p d\mu(\varepsilon)\,d\mu(\delta)\Big)^{1/p}.$$

Due to the application of Pisier's inequality to the theory of nonlinear type (see [28, 25, 12, 23]), it is of great interest to understand when (1.3) holds true with the $\log n$ term replaced by a constant that may depend on the geometry of X but is independent of n. Talagrand proved [30] that the $\log n$ term in (1.3) is asymptotically optimal for general Banach spaces X, Wagner proved [31] that the $\log n$ term in (1.3) can be replaced by a universal constant if $p = \infty$ and X is a general Banach space, and in [25] it is shown that the $\log n$ term in (1.3) can be replaced by a constant that is independent of n if X is a UMD Banach space. It remains an intriguing open question whether every Banach space of nontrivial type satisfies (1.3) with the $\log n$ term replaced by a constant that is independent of n. If true, this would resolve a 1976 question of Enflo [9] by establishing that Rademacher type p and Enflo type p coincide (see [25, 23] and Section 6 below).

Here we obtain a new class of Banach spaces that satisfies a dimension-independent Pisier inequality. Our starting point is the following extension of Pisier's inequality.

Definition 1.2 (Pisier constant of X). The *n-dimensional Pisier constant of X (with exponent p)*, denoted $\mathfrak{P}_p^n(X)$, is the infimum over those $\mathfrak{P} \in (0,\infty)$ such that every $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfies

$$(1.4)\quad \Big(\int_{\{-1,1\}^n} \Big\|\sum_{j=1}^n \partial_j f_j(\varepsilon)\Big\|^p d\mu(\varepsilon)\Big)^{1/p} \le \mathfrak{P}\Big(\int_{\{-1,1\}^n}\int_{\{-1,1\}^n} \Big\|\sum_{j=1}^n \delta_j \Delta f_j(\varepsilon)\Big\|^p d\mu(\varepsilon)\,d\mu(\delta)\Big)^{1/p}.$$

We also set

$$\mathfrak{P}_p(X) := \sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X).$$

Inequality (1.4) reduces to Pisier's inequality if we choose $f_j = \Delta^{-1}\partial_j f$ for some $f : \{-1,1\}^n \to X$. The generalized inequality (1.4) has the advantage of being well-behaved under duality, as explained in Section 2. The following theorem yields a logarithmic bound on $\mathfrak{P}_p^n(X)$, thus extending Pisier's inequality.

THEOREM 1.3. *For every Banach space X, every $p \in [1,\infty]$ and every $n \in \mathbb{N}$,*

$$\mathfrak{P}_p^n(X) \le \sum_{k=1}^{n} \frac{1}{k}.$$

Our approach yields a quantitative improvement over Pisier's inequality only in lower order terms: an optimization of Pisier's argument (as carried out in [23]) shows that the $O(\log n)$ term in (1.3) can be taken to be at most $\log n + O(\log\log n)$, while Theorem 1.3 shows that this term can be taken to be $\log n + O(1)$.

In [25] it was shown that the logarithmic term in (1.3) can be replaced by a constant that is independent of n if X is a UMD Banach space. Recall that X is a *UMD Banach space* if for every $p \in (1,\infty)$ there exists a constant $\beta \in (0,\infty)$ such that if $\{M_j\}_{j=0}^n$ is a p-integrable X-valued martingale defined on some probability space $(\Omega,\mathbb{P})$, then for every $\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}$ we have

$$\int_{\Omega} \Big\| M_0 + \sum_{j=1}^{n} \varepsilon_j (M_j - M_{j-1}) \Big\|^p \, d\mathbb{P} \le \beta^p \int_{\Omega} \|M_n\|^p \, d\mathbb{P}. \tag{1.5}$$

The infimum over those $\beta \in (0,\infty)$ for which (1.5) holds true is denoted $\beta_p(X)$. It can be shown (see [7]) that $\beta_p(X) \lesssim \frac{p^2}{p-1}\beta_2(X)$, so in order to define the UMD property it suffices to require the validity of (1.5) for $p = 2$. UMD Banach spaces are known to be superreflexive [20, 1], and one also has $\beta_q(X^*) = \beta_p(X)$, where $q = p/(p-1)$ (see e.g. [7]).

In [10] Garling investigated the natural weakening of (1.5) in which the desired inequality is required to hold true in expectation over $\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}$ rather than for every $\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}$. Specifically, say that X is a *UMD$^+$ Banach space* if for every $p \in (1,\infty)$ there exists a constant $\beta \in (0,\infty)$ such that if $\{M_j\}_{j=0}^n$ is a p-integrable X-valued martingale defined on some probability space $(\Omega,\mathbb{P})$ then

$$\int_{\{-1,1\}^n} \int_{\Omega} \Big\| M_0 + \sum_{j=1}^{n} \varepsilon_j (M_j - M_{j-1}) \Big\|^p \, d\mathbb{P}\, d\mu(\varepsilon) \le \beta^p \int_{\Omega} \|M_n\|^p \, d\mathbb{P}. \tag{1.6}$$

The infimum over those β for which (1.6) holds true is denoted $\beta_p^+(X)$.

THEOREM 1.4. *Let X be a Banach space such that X^* is UMD$^+$. Fix $p \in (1,\infty)$ and $n \in \mathbb{N}$. For every function $F : \{-1,1\}^n \times \{-1,1\}^n \to X$ and $j \in \{1,\ldots,n\}$ define*

$$F_j(\varepsilon) := \int_{\{-1,1\}^n} \delta_j F(\varepsilon,\delta)\, d\mu(\delta).$$

Then

$$\Big(\int_{\{-1,1\}^n} \Big\| \sum_{j=1}^n \Delta^{-1}\partial_j F_j(\varepsilon)\Big\|^p d\mu(\varepsilon)\Big)^{1/p} \le \beta_q^+(X^*)\Big(\int_{\{-1,1\}^n}\int_{\{-1,1\}^n} \|F(\varepsilon,\delta)\|^p\, d\mu(\varepsilon)\, d\mu(\delta)\Big)^{1/p}, \tag{1.7}$$

where $q = p/(p-1)$.

For every $f_1,\ldots,f_n : \{-1,1\}^n \to X$, an application of Theorem 1.4 to the function $F(\varepsilon,\delta) = \sum_{j=1}^n \delta_j f_j(\varepsilon)$ yields the following estimate on the Pisier constant of a UMD$^+$ Banach space.

COROLLARY 1.5. $\mathfrak{P}_p(X) \le \beta_q^+(X^*)$.

It is unknown if a UMD$^+$ Banach space must also be a UMD Banach space, though it seems reasonable to conjecture that there are UMD$^+$ spaces that are not UMD. Regardless of this, Theorem 1.4 and Corollary 1.5 are conceptually different from the result of [25], which relies on the full force of the UMD condition, i.e. it requires the validity of (1.5) for every choice of signs $\varepsilon_1,\ldots,\varepsilon_n$, while our argument here needs such estimates to hold true only for an average choice of signs. We also have a quantitative improvement: in [25] it was shown that Pisier's inequality holds true with the $O(\log n)$ term in (1.3) replaced by $\beta_p(X) = \beta_q(X^*)$, while here we obtain the same estimate with the $O(\log n)$ term in (1.3) replaced by $\beta_q^+(X^*) \le \beta_q(X^*)$. Geiss proved [11] that for every $\eta \in (0,1)$ there is $C_\eta \in (0,\infty)$ such that for every $M > 1$ there is a Banach space X that satisfies

$$\infty > \beta_q(X^*) \ge C_\eta \beta_q^+(X^*)^{2-\eta} \ge M.$$

REMARK 1.1. Inequality (1.7) is an extension of the generalized Pisier inequality (1.4), but for general Banach spaces it behaves very differently: unlike the logarithmic behavior of Theorem 1.3, the best constant appearing on the right hand side of (1.7) for a general Banach space X must be at least a constant multiple of $\sqrt{n}$, as exhibited by the case $X = L_1((\{-1,1\}^n,\mu),\mathbb{R})$ and $F : \{-1,1\}^n \times \{-1,1\}^n \to X$ given by $F(\varepsilon,\delta)(\eta) = \prod_{i=1}^n (1+\varepsilon_i\delta_i\eta_i)$.

Suppose that $\theta \in (0,1)$ and $X = [H,Y]_\theta$, where H is a Hilbert space and Y is an arbitrary Banach space. Here $[\cdot,\cdot]_\theta$ denotes complex interpolation (see [3]). Theorem 1.6 below shows that in this case $\mathfrak{P}_p(X) < \infty$, and therefore Pisier's inequality holds true with the $\log n$ term in (1.3) replaced by a constant that is independent of n. Pisier proved [27] that every Banach lattice of nontrivial type (see [19]) is of the form $[H,Y]_\theta$ for some $\theta \in (0,1)$, so we thus obtain the desired dimension independence in Pisier's inequality for Banach lattices of nontrivial type. This result does not follow from previously known cases in which a dimension-independent Pisier inequality has been proved, since, as shown by Bourgain [4, 5], there exist Banach lattices of nontrivial type which are not UMD. Note, however, that we are still far from proving the conjectured dimension-independent Pisier inequality for Banach spaces with nontrivial type: any space of the form $[H,Y]_\theta$ admits an equivalent norm whose modulus of smoothness has power type $2/(1+\theta)$ (see [27, 8]), while there exist Banach spaces with nontrivial type that do not admit such an equivalent norm (see [14, 16, 15, 29]).

Theorem 1.6. *Let X, Y be Banach spaces and let H be a Hilbert space. Suppose that for some $\theta \in (0,1)$ we have $X = [H,Y]_\theta$. Then for every $p \in (1,\infty)$,*

$$\mathfrak{P}_p(X) \le \frac{2\max\{p, p/(p-1)\}}{1-\theta}.$$

Remark 1.2. If $r \in (2,\infty)$ then the $O(\log n)$ term in Pisier's inequality (1.3), when $p=2$ and $X=\ell_r$, can be replaced by $O(r)$, due to the fact that $\beta_2^+(\ell_r) \asymp r$ (which follows from Hitczenko's work [13], as explained to us by Mark Veraar). This bound also follows from Theorem 1.6. At the same time, an inspection of Talagrand's example in [30] shows that this term must be at least a constant multiple of $\log r$. Determining the correct order of magnitude as $r \to \infty$ of the constant in Pisier's inequality when $X = \ell_r$ remains an interesting open problem.

2. Duality. The dimension $n \in \mathbb{N}$ will be fixed from now on. For $p \in [1,\infty]$ and a Banach space X, let $L_p(X)$ denote the vector-valued Lebesgue space $L_p((\{-1,1\}^n,\mu),X)$. Thus $L_p(L_p(X))$ can be naturally identified with the space $L_p((\{-1,1\}^n \times \{-1,1\}^n, \mu\times\mu), X)$.

For $f \in L_p(X)$ we denote its Fourier expansion by

$$f = \sum_{A\subseteq\{1,\dots,n\}} \widehat{f}(A) W_A,$$

where the *Walsh function* $W_A : \{-1,1\}^n \to \{-1,1\}$ corresponding to $A \subseteq \{1,\dots,n\}$ is given by $W_A(\varepsilon_1,\dots,\varepsilon_n) = \prod_{i\in A}\varepsilon_i$, and the Fourier coefficient $\widehat{f}(A) \in X$ is given by $\widehat{f}(A) = \int_{\{-1,1\}^n} f(x)W_A(x)\,d\mu(x)$. Using this (standard) notation, we have

$$\forall i \in \{1,\dots,n\},\ \forall f \in L_p(X), \quad \partial_i f = \sum_{\substack{A \subseteq \{1,\dots,n\} \\ i \in A}} \widehat{f}(A) W_A,$$

$$\forall f \in L_p(X), \quad \Delta f = \sum_{A \subseteq \{1,\dots,n\}} |A| \widehat{f}(A) W_A,$$

$$\forall f \in L_p(X), \quad \Delta^{-1} f := \sum_{\substack{A \subseteq \{1,\dots,n\} \\ A \neq \emptyset}} \frac{1}{|A|} \widehat{f}(A) W_A.$$

The Rademacher projection of $f \in L_p(X)$ is defined as usual by

$$\mathbf{Rad}(f) := \sum_{i=1}^{n} \widehat{f}(\{i\}) W_{\{i\}}.$$

We write below $\mathbf{Rad}_X := \mathbf{Rad}(L_p(X))$ and $\mathbf{Rad}_X^{\perp} := (I - \mathbf{Rad})(L_p(X))$. The dual of $(\mathbf{Rad}_X, \|\cdot\|_{L_p(X)})$ is naturally identified with the quotient $L_q(X^*)/\mathbf{Rad}_{X^*}^{\perp}$, where $q = p/(p-1)$.

Define an operator $S : L_p(L_p(X)) \to L_p(X)$ by

$$\forall F \in L_p(L_p(X)), \quad S(F) := \sum_{j=1}^{n} \Delta^{-1} \partial_j \widehat{F}(\{j\}). \tag{2.1}$$

Using this notation, Theorem 1.4 is nothing more than the following operator norm bound:

$$\|S\|_{L_p(L_p(X)) \to L_p(X)} \le \beta_q^{+}(X^*).$$

The adjoint operator $S^* : L_q(X^*) \to L_q(L_q(X^*))$ is given by

$$\forall g \in L_q(X^*),\ \forall \delta \in \{-1,1\}^n, \quad S^*(g)(\delta) = \sum_{j=1}^{n} \delta_j \Delta^{-1} \partial_j g.$$

Therefore Theorem 1.4 has the following equivalent dual formulation.

THEOREM 2.1 (Dual formulation of Theorem 1.4). *Let Z be a UMD$^+$ Banach space. Then for every $q \in (1,\infty)$ and every $g \in L_q(Z)$ we have*

$$\Big(\int_{\{-1,1\}^n} \Big\| \sum_{j=1}^{n} \delta_j \Delta^{-1} \partial_j g \Big\|_{L_q(Z)}^q d\mu(\delta) \Big)^{1/q} \le \beta_q^{+}(Z) \|g\|_{L_q(Z)}.$$

Theorem 2.1, and consequently also Theorem 1.4, will be proven in Section 3.

Let T be the restriction of S to $\mathbf{Rad}_{L_p(X)}$. Thus

$$\mathfrak{P}_p^n(X) = \|T\|_{\mathbf{Rad}_{L_p(X)} \to L_p(X)} = \|T^*\|_{L_q(X^*) \to L_q(L_q(X^*))/\mathbf{Rad}_{L_q(X^*)}^{\perp}}.$$

The adjoint $T^* : L_q(X^*) \to L_q(L_q(X^*))/\mathbf{Rad}_{L_q(X^*)}^{\perp}$ is given by

$$\forall g \in L_q(X^*),\ \forall \delta \in \{-1,1\}^n, \quad T^*(g) = \sum_{j=1}^n \delta_j \Delta^{-1} \partial_j g + \mathbf{Rad}^{\perp}_{L_q(X^*)}.$$

Therefore Theorem 1.3 has the following equivalent dual formulation.

THEOREM 2.2 (Dual formulation of Theorem 1.3). *Let Z be a Banach space and $q \in [1,\infty]$. Then for every $g \in L_q(Z)$ we have*

$$\inf_{\Phi \in \mathbf{Rad}^{\perp}_{L_q(Z)}} \Big(\int_{\{-1,1\}^n} \Big\| \Phi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1} \partial_j g \Big\|^q_{L_q(Z)} d\mu(\delta) \Big)^{1/q} \le \Big(\sum_{k=1}^n \frac{1}{k} \Big) \|g\|_{L_q(Z)}.$$

Theorem 2.2, and consequently also Theorem 1.3, will be proven in Section 4. Since $[H,Y]^*_\theta = [H,Y^*]_\theta$ (see [3]), we also have the following equivalent dual formulation of Theorem 1.6.

THEOREM 2.3 (Dual formulation of Theorem 1.6). *Let H be a Hilbert space, W a Banach space, and $\theta \in (0,1)$. Set $Z = [H,W]_\theta$. Then for every $q \in (1,\infty)$ and $g \in L_q(Z)$,*

$$\inf_{\Psi \in \mathbf{Rad}^{\perp}_{L_q(Z)}} \Big(\int_{\{-1,1\}^n} \Big\| \Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1} \partial_j g \Big\|^q_{L_q(Z)} d\mu(\delta) \Big)^{1/q} \le \frac{2\max\{q, q/(q-1)\}}{1-\theta} \|g\|_{L_q(Z)}.$$

Theorem 2.3, and consequently also Theorem 1.6, will be proven in Section 5.

3. Proof of Theorem 2.1. Fix $q \in (1,\infty)$ and $g \in L_q(Z)$. Let S_n denote the symmetric group on $\{1,\ldots,n\}$. For $\sigma \in S_n$ and $k \in \{0,\ldots,n\}$ define $g_k^\sigma \in L_q(Z)$ by

$$(3.1)\quad g_k^\sigma(\varepsilon) := \sum_{A \subseteq \{\sigma^{-1}(1),\ldots,\sigma^{-1}(k)\}} \widehat{g}(A) W_A(\varepsilon)$$
$$= \frac{1}{2^{n-k}} \sum_{\delta_{\sigma^{-1}(k+1)},\ldots,\delta_{\sigma^{-1}(n)} \in \{-1,1\}} g\Big(\sum_{i=1}^k \varepsilon_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} + \sum_{i=k+1}^n \delta_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} \Big);$$

here, and in what follows, $e_1,\ldots,e_n$ denotes the standard basis of $\mathbb{R}^n$. Then $\{g_k^\sigma\}_{k=0}^n$ is a Z-valued martingale with $g_n^\sigma = g$ and $g_0^\sigma = \widehat{g}(\emptyset)$, implying that

$$(3.2)\quad \Big(\int_{\{-1,1\}^n} \Big\| \sum_{k=1}^n \delta_k (g_k^\sigma - g_{k-1}^\sigma) \Big\|^q_{L_q(Z)} d\mu(\delta) \Big)^{1/q} \le \beta_q^+(Z) \|g\|_{L_q(Z)}.$$

In (3.2) we may replace $\{\delta_k\}_{k=1}^n$ by $\{\delta_{\sigma^{-1}(k)}\}_{k=1}^n$, since these two sequences of signs have the same joint distribution. Then we make the change of variable $j=\sigma^{-1}(k)$, so that $k=\sigma(j)$. Averaging the resulting inequality over $\sigma\in S_n$, and using the convexity of the norm, we see that

$$\Big(\int_{\{-1,1\}^n}\Big\|\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{j=1}^n \delta_j(g^\sigma_{\sigma(j)}-g^\sigma_{\sigma(j)-1})\Big\|^q_{L_q(Z)}d\mu(\delta)\Big)^{1/q} \le \beta_q^+(Z)\|g\|_{L_q(Z)}. \tag{3.3}$$

It remains to note that for each $\delta\in\{-1,1\}^n$ we have

$$\begin{aligned}\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{j=1}^n \delta_j(g^\sigma_{\sigma(j)}-g^\sigma_{\sigma(j)-1}) &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{j=1}^n \delta_j \sum_{\substack{\emptyset\subsetneq A\subseteq\{1,\ldots,n\}\\ \max\sigma(A)=\sigma(j)}}\widehat{g}(A)W_A\\ &= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}\sum_{j\in A}\delta_j\frac{|\{\sigma\in S_n:\max\sigma(A)=\sigma(j)\}|}{n!}\widehat{g}(A)W_A\\ &= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}\frac{\sum_{j\in A}\delta_j}{|A|}\widehat{g}(A)W_A=\sum_{j=1}^n\delta_j\Delta^{-1}\partial_j g.\end{aligned}\tag{3.4}$$

Due to (3.3) and (3.4) the proof of Theorem 2.1 is complete. ∎

4. Proof of Theorem 2.2. The following lemma introduces an auxiliary function which is a variant of a similar function that was used by Pisier in [28].

Lemma 4.1. *Let Z be a Banach space. Fix $n\in\mathbb{N}$, $q\in[1,\infty]$ and $t\in(0,1)$. For $g\in L_q(Z)$ define $G_t\in L_q(L_q(Z))$ by*

$$G_t(\delta):=\frac{1}{1-t}\sum_{A\subseteq\{1,\ldots,n\}}\widehat{g}(A)W_A\prod_{i\in A}(t+(1-t)\delta_i)-\frac{t^n}{1-t}g. \tag{4.1}$$

Then

$$\mathbf{Rad}(G_t)(\delta)=\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}t^{|A|-1}\sum_{j\in A}\delta_j\widehat{g}(A)W_A \tag{4.2}$$

and

$$\|G_t\|_{L_q(L_q(Z))}\le\frac{1-t^n}{1-t}\|g\|_{L_q(Z)}. \tag{4.3}$$

Proof. Identity (4.2) follows from (4.1) since, for every $A \subseteq \{1,\dots,n\}$,

$$\mathbf{Rad}\Big(\prod_{i\in A}(t+(1-t)\delta_i)\Big) = t^{|A|-1}(1-t)\sum_{j\in A}\delta_j.$$

To prove (4.3) observe that for every $\varepsilon,\delta\in\{-1,1\}^n$,

$$\begin{aligned}
&(1-t)G_t(\delta)(\varepsilon)\\
(4.4)\quad &= \sum_{A\subseteq\{1,\dots,n\}} \widehat{g}(A)W_A(\varepsilon)\prod_{i=1}^{n}(t+(1-t)\delta_i^{\mathbf{1}_A(i)}) - t^n g(\varepsilon)\\
&= \sum_{A\subseteq\{1,\dots,n\}} \widehat{g}(A)W_A(\varepsilon)\sum_{B\subseteq\{1,\dots,n\}} t^{|B|}(1-t)^{n-|B|}W_{A\smallsetminus B}(\delta) - t^n g(\varepsilon)\\
&= \sum_{B\subsetneq\{1,\dots,n\}} t^{|B|}(1-t)^{n-|B|}\sum_{A\subseteq\{1,\dots,n\}} \widehat{g}(A)W_{A\cap B}(\varepsilon)W_{A\smallsetminus B}(\varepsilon\delta)\\
(4.5)\quad &= \sum_{B\subsetneq\{1,\dots,n\}} t^{|B|}(1-t)^{n-|B|}g_B(\varepsilon,\delta),
\end{aligned}$$

where in (4.4) we use (4.1) and in (4.5) for every $B\subseteq\{1,\dots,n\}$ we set

$$g_B(\varepsilon,\delta) := g\Big(\sum_{j\in B}\varepsilon_j e_j + \sum_{j\in\{1,\dots,n\}\smallsetminus B}\varepsilon_j\delta_j e_j\Big).$$

Since g_B is equidistributed with g, it follows from (4.5) that

$$\frac{\|G_t\|_{L_q(L_q(Z))}}{\|g\|_{L_q(Z)}} \le \frac{1}{1-t}\sum_{B\subsetneq\{1,\dots,n\}} t^{|B|}(1-t)^{n-|B|} = \frac{1-t^n}{1-t}. \ \blacksquare$$

Proof of Theorem 2.2. Observe that for every $\delta\in\{-1,1\}^n$ we have

$$\begin{aligned}
(4.6)\qquad \sum_{j=1}^{n}\delta_j\Delta^{-1}\partial_j g &= \sum_{\substack{A\subseteq\{1,\dots,n\}\\ A\neq\emptyset}}\frac{1}{|A|}\sum_{j\in A}\delta_j\widehat{g}(A)W_A\\
&= \sum_{\substack{A\subseteq\{1,\dots,n\}\\ A\neq\emptyset}}\Big(\int_0^1 t^{|A|-1}\,dt\Big)\sum_{j\in A}\delta_j\widehat{g}(A)W_A\\
&\overset{(4.2)}{=} \mathbf{Rad}\Big(\int_0^1 G_t(\delta)\,dt\Big).
\end{aligned}$$

It follows that if we set

$$(4.7)\qquad \Phi := \int_0^1 G_t\,dt - \mathbf{Rad}\Big(\int_0^1 G_t\,dt\Big),$$

then $\Phi \in \mathbf{Rad}_{L_q(Z)}^{\perp}$ and

$$\Big(\int_{\{-1,1\}^n} \Big\|\Phi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g\Big\|_{L_q(Z)}^q d\mu(\delta)\Big)^{1/q} \overset{(4.6)}{=} \Big\|\int_0^1 G_t\, dt\Big\|_{L_q(L_q(Z))} \overset{(4.3)}{\le} \Big(\int_0^1 \frac{1-t^n}{1-t}\, dt\Big)\|g\|_{L_q(Z)}.$$

It remains to note that

$$\int_0^1 \frac{1-t^n}{1-t} dt = \sum_{k=0}^{n-1}\int_0^1 t^k dt = \sum_{k=1}^n \frac{1}{k}. \ \blacksquare$$

5. Proof of Theorem 2.3. For $t \in (0,1)$ define a linear operator $V_t : L_q(Z) \to L_q(L_q(Z))$ by

$$V_t(g)(\delta) := G_t(\delta) - \widehat{G_t}(\emptyset) \overset{(4.1)}{=} \frac{1}{1-t}\sum_{A\subseteq\{1,\dots,n\}} \widehat{g}(A) W_A\Big(\prod_{i\in A}(t+(1-t)\delta_i) - t^{|A|}\Big). \tag{5.1}$$

Lemma 5.1. *Let H be a Hilbert space. Then for every $t \in (0,1)$,*

$$\|V_t\|_{L_2(H)\to L_2(L_2(H))} \le \frac{1}{\sqrt{1-t^2}} \le \frac{1}{\sqrt{1-t}}. \tag{5.2}$$

Proof. Observe that for every $A \subseteq \{1,\dots,n\}$ we have

$$\int_{\{-1,1\}^n} \Big(\prod_{i\in A}(t+(1-t)\delta_i) - t^{|A|}\Big)^2 d\mu(\delta) = \sum_{B\subsetneq A} t^{2|B|}(1-t)^{2(|A|-|B|)} = (t^2+(1-t)^2)^{|A|} - t^{2|A|}. \tag{5.3}$$

It follows from (5.1), (5.3), and the orthogonality of $\{W_A\}_{A\subseteq\{1,\dots,n\}}$ that

$$\|V_t\|_{L_2(H)\to L_2(L_2(H))} = \max_{a\in\{1,\dots,n\}} \frac{\sqrt{(t^2+(1-t)^2)^a - t^{2a}}}{1-t}. \tag{5.4}$$

Now, for every $a \in \{1,\dots,n\}$ and $t \in (0,1)$ we have

$$(t^2+(1-t)^2)^a - t^{2a} = (1-t)^2\sum_{k=0}^{a-1}(t^2+(1-t)^2)^{a-1-k}t^{2k} \le (1-t)^2\sum_{k=0}^{a-1} t^{2k} = (1-t)^2\frac{1-t^{2a}}{1-t^2} \le \frac{1-t}{1+t}, \tag{5.5}$$

where in the first inequality we used the estimate $t^2 + (1-t)^2 \le 1$, which holds for every $t \in [0,1]$. The desired estimate (5.2) now follows from a substitution of (5.5) into (5.4). ∎

LEMMA 5.2. *Let H be a Hilbert space and let W be a Banach space. Fix $\theta \in (0,1)$ and $q \in (1,\infty)$. Set $Z = [H,W]_\theta$. Then for every $t \in (0,1)$ we have*

$$\|V_t\|_{L_q(Z)\to L_q(L_q(Z))} \le \frac{2}{(1-t)^{1-(1-\theta)\min\{1/q,1-1/q\}}}. \tag{5.6}$$

Proof. For every $r \in [1,\infty]$ we have

$$\|V_t(g)\|_{L_r(L_r(W))} \overset{(5.1)}{=} \|G_t - \widehat{G_t}(\emptyset)\|_{L_r(L_r(W))} \le 2\|G_t\|_{L_r(L_r(W))} \overset{(4.3)}{\le} \frac{2\|g\|_{L_r(W)}}{1-t}.$$

Consequently,

$$\forall r \in [1,\infty], \quad \|V_t\|_{L_r(W)\to L_r(L_r(W))} \le \frac{2}{1-t}. \tag{5.7}$$

If $q \in [2,\infty)$ then we interpolate (see [3]) between (5.2) and (5.7) with $W = H$ and $r = \infty$. If $q \in (1,2]$ then we interpolate between (5.2) and (5.7) with $W = H$ and $r = 1$. The norm bound thus obtained implies the estimate

$$\forall q \in (1,\infty), \quad \|V_t\|_{L_q(H)\to L_q(L_q(H))} \le \frac{2}{(1-t)^{\max\{1/q,1-1/q\}}}. \tag{5.8}$$

Finally, interpolation between (5.8) and (5.7) with $r = q$ gives the desired norm bound (5.6). ∎

Proof of Theorem 2.3. By (5.1) we have $\mathbf{Rad}(V_t(g)) = \mathbf{Rad}(G_t)$. Therefore, analogously to (4.7), if we set

$$\Psi := \int_0^1 V_t(g)\,dt - \mathbf{Rad}\Big(\int_0^1 G_t\,dt\Big) = \int_0^1 V_t(g)\,dt - \mathbf{Rad}\Big(\int_0^1 V_t(g)\,dt\Big),$$

then $\Psi \in \mathbf{Rad}^\perp_{L_q(Z)}$ and by (4.6) for every $\delta \in \{-1,1\}^n$ we have

$$\Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g = \int_0^1 V_t(g)(\delta)\,dt. \tag{5.9}$$

Hence,

$$\Big(\int_{\{-1,1\}^n} \Big\|\Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g\Big\|^q_{L_q(Z)}\,d\mu(\delta)\Big)^{1/q} \overset{(5.9)\wedge(5.6)}{\le} \int_0^1 \frac{2\|g\|_{L_q(Z)}}{(1-t)^{1-(1-\theta)\min\{1/q,1-1/q\}}}\,dt = \frac{2\|g\|_{L_q(Z)}}{(1-\theta)\min\{1/q,1-1/q\}}.$$

This is precisely the assertion of Theorem 2.3. ∎

6. Enflo type in uniformly smooth Banach spaces. A Banach space X has *Rademacher type* $p \in [1,2]$ (see e.g. [21]) if there exists $T_R \in (0,\infty)$ such that, for all $n \in \mathbb{N}$ and all $x_1,\ldots,x_n \in X$,

$$\int_{\{-1,1\}^n} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|^p \, d\mu(\varepsilon) \le T_R^p \sum_{j=1}^n \|x_j\|^p. \tag{6.1}$$

Furthermore X has *Enflo type* p (see [9, 6, 28, 25]) if there exists $T_E \in (0,\infty)$ such that, for all $n \in \mathbb{N}$ and all $f : \{-1,1\}^n \to X$,

$$\int_{\{-1,1\}^n} \frac{\|f(\varepsilon) - f(-\varepsilon)\|^p}{2^p} \, d\mu(\varepsilon) \le T_E^p \sum_{j=1}^n \|\partial_j f\|_{L_p(X)}^p. \tag{6.2}$$

By considering the function $f(\varepsilon) = \sum_{j=1}^n \varepsilon_j x_j$ one sees that (6.1) is a special case of (6.2). It is a long-standing open problem [9] whether, conversely, (6.1) implies (6.2). A crucial feature of (6.2) is that it is a purely metric condition (thus one can define when a *metric space* has Enflo type p), while (6.1) is a linear condition. See [22] for a purely metric condition (which is more complicated than, but inspired by, Enflo type) that is known to be equivalent to Rademacher type.

Observe that if (6.1) holds then it follows from (1.4) that, for every $f_1,\ldots,f_n : \{-1,1\}^n \to X$,

$$\Big\| \sum_{j=1}^n \Delta^{-1} \partial_j f_j \Big\|_{L_p(X)} \le T_R \mathfrak{P}_p^n(X) \Big(\sum_{j=1}^n \|f_j\|_{L_p(X)}^p \Big)^{1/p}. \tag{6.3}$$

The special case $f_j = \partial_j f$ shows that (6.3) implies (6.2) with

$$T_E \le T_R \mathfrak{P}_p^n(X).$$

For this reason it is worthwhile to investigate (6.3) in its own right.

Let $\mathfrak{Q}_p^n(X)$ be the infimum over those $\mathfrak{Q} \in (0,\infty)$ such that every $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfies

$$\Big\| \sum_{j=1}^n \Delta^{-1} \partial_j f_j \Big\|_{L_p(X)} \le \mathfrak{Q} \Big(\sum_{j=1}^n \|f_j\|_{L_p(X)}^p \Big)^{1/p}. \tag{6.4}$$

We also set

$$\mathfrak{Q}_p(X) := \sup_{n\in\mathbb{N}} \mathfrak{Q}_p^n(X).$$

By duality, $\mathfrak{Q}_p^n(X)$ equals the infimum over those $\mathfrak{Q} \in (0,\infty)$ for which every $g \in L_q(X^*)$ satisfies

$$\Big(\sum_{j=1}^n \|\Delta^{-1} \partial_j g\|_{L_q(X^*)}^q \Big)^{1/q} \le \mathfrak{Q} \|g\|_{L_q(X^*)}. \tag{6.5}$$

Letting $S_X = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X, recall that the *modulus of uniform convexity* of X is defined for $\varepsilon \in [0,2]$ as

$$\delta_X(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X,\ \|x-y\| = \varepsilon\right\}.$$

The *modulus of uniform smoothness* of X is defined for $\tau \in (0,\infty)$ as

$$\rho_X(\tau) := \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S_X\right\}.$$

These moduli relate to each other via the following classical duality formula of Lindenstrauss [18]:

$$\delta_{X^*}(\varepsilon) = \sup\{\tau\varepsilon/2 - \rho_X(\tau) : \tau \in [0,1]\}. \tag{6.6}$$

THEOREM 6.1. *For every $K, p \in (1,\infty)$ there exists $C(K,p) \in (0,\infty)$ such that if X is a Banach space that satisfies $\rho_X(\tau) \le K\tau^p$ for all $\tau \in (0,\infty)$, then $\mathfrak{Q}_p(X) \le C(K,p)$.*

Proof. We shall use here the notation introduced in the proof of Theorem 2.1 (Section 3). It follows from (6.6) that $\delta_{X^*}(\varepsilon) \gtrsim_{K,p} \varepsilon^q$ for every $\varepsilon \in [0,2]$ (here, and it what follows, the notation $\lesssim_{K,p}$ suppresses constant factors that may depend only on K and p). Hence, for $g \in L_q(X^*)$ and $\sigma \in S_n$, since $\{g_k^\sigma\}_{k=0}^n$, as defined in (3.1), is an X^*-valued martingale, it follows from Pisier's martingale inequality [26] that

$$\Big(\sum_{k=1}^n \|g_k^\sigma - g_{k-1}^\sigma\|_{L_q(X^*)}^q\Big)^{1/q} \lesssim_{K,p} \|g\|_{L_q(X^*)}. \tag{6.7}$$

By reindexing (6.7) with $k = \sigma(j)$, averaging over $\sigma \in S_n$, and using the convexity of the norm, we obtain the estimate

$$\bigg(\sum_{j=1}^n \bigg\|\frac{1}{n!}\sum_{\sigma\in S_n}(g_{\sigma(j)}^\sigma - g_{\sigma(j)-1}^\sigma)\bigg\|_{L_q(X^*)}^q\bigg)^{1/q} \lesssim_{K,p} \|g\|_{L_q(X^*)}. \tag{6.8}$$

Arguing as in (3.4), for every $j \in \{1,\dots,n\}$ we have the identity

$$\begin{aligned}\frac{1}{n!}\sum_{\sigma\in S_n}(g_{\sigma(j)}^\sigma - g_{\sigma(j)-1}^\sigma) &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{\substack{\emptyset\subsetneq A\subseteq\{1,\dots,n\}\\ \max\sigma(A)=\sigma(j)}} \widehat{g}(A)W_A \\ &= \sum_{\substack{A\subseteq\{1,\dots,n\}\\ j\in A}} \frac{|\{\sigma\in S_n : \max\sigma(A) = \sigma(j)\}|}{n!}\widehat{g}(A)W_A = \Delta^{-1}\partial_j g.\end{aligned} \tag{6.9}$$

Consequently, (6.8) combined with (6.9) implies that (6.5) holds true with $\mathfrak{Q} \lesssim_{K,p} 1$. This concludes the proof of Theorem 6.1. ∎

REMARK 6.1. It follows from [17, Sec. 6] that a Banach space X satisfying the assumption of Theorem 6.1 has Enflo type p. Theorem 6.1 can be viewed as a generalization of this fact to yield the inequality (6.4). In [24] it was shown that any Banach space satisfying the assumption of Theorem 6.1 actually has K. Ball's Markov type p property [2], a property which is a useful strengthening of Enflo type p.

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