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Distinguished subspaces of L_p of maximal dimension

by

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Abstract. Let (Ω, Σ, μ) be a measure space and $1 < p < \infty$. We show that, under quite general conditions, the set $L_p(\Omega) - \bigcup_{1 \le q < p} L_q(\Omega)$ is maximal spaceable, that is, it contains (except for the null vector) a closed subspace F of $L_p(\Omega)$ such that $\dim(F) = \dim(L_p(\Omega))$. This result is so general that we had to develop a hybridization technique for measure spaces in order to construct a space such that the set $L_p(\Omega) - L_q(\Omega)$, $1 \le q < p$, fails to be maximal spaceable. In proving these results we have computed the dimension of $L_p(\Omega)$ for arbitrary measure spaces (Ω, Σ, μ). The aim of the results presented here is, among others, to generalize all the previous work (since the 1960's) related to the linear structure of the sets $L_p(\Omega) - L_q(\Omega)$ with $q < p$ and $L_p(\Omega) - \bigcup_{1 \le q < p} L_q(\Omega)$.

1. Introduction and preliminaries. This paper is devoted to the search for what are often large linear spaces of functions enjoying certain *special* properties. Let E be a topological vector space and let us consider such a special property $\mathcal{P}$. We say that the subset M of E formed by all vectors in E which satisfy $\mathcal{P}$ is *spaceable* if $M \cup \{0\}$ contains a *closed* infinite-dimensional subspace. The set M will be called *lineable* if $M \cup \{0\}$ contains an infinite-dimensional linear (not necessarily closed) space.

The terms "lineability" and "spaceability" were originally coined by V. Gurariy and they first appeared in [4, 48]. After the first appearance of this notion, many authors became interested in this topic: see, for instance, the recent works by R. Aron (e.g. [1, 2, 4–6]), P. Enflo [21], V. Gurariy [4, 21, 34] or G. Godefroy [9], just to cite some. It is important to recall that, prior to the publication of [4, 48], some authors (when working with infinite-dimensional spaces) already found large linear structures enjoying these type of "special" properties (even though they did not explicitly used terms like lineability or spaceability). Probably the very first result illustrating this was due to B. Levine and D. Milman (1940, [41]):

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Theorem 1.1. *The subset of $\mathcal{C}[0,1]$ of all functions of bounded variation is not spaceable.*

Later, the following analogue of this result was proved by V. Gurariy (1966, [32]):

Theorem 1.2. *The set of everywhere differentiable functions on* $[0,1]$ *is not spaceable.*

On the other hand (see also [32]):

Theorem 1.3. *There exist closed infinite-dimensional subspaces of* $\mathcal{C}[0,1]$ *all of whose members are differentiable on* $(0,1)$.

Within the context of subsets of continuous functions, in 1966 V. Gurariy [33] showed that the set of continuous nowhere differentiable functions on $[0,1]$ is lineable. Soon after, V. Fonf, V. Gurariy and M. Kadets [23] showed that the set of continuous nowhere differentiable functions on $[0,1]$ is spaceable in $\mathcal{C}[0,1]$. Actually, much more is known about this set. L. Rodríguez-Piazza [45] showed that the space constructed in [23] can be chosen to be isometrically isomorphic to any separable Banach space. More recently, S. Hencl [36] showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder. Another set that has also attracted the attention of several authors is the set of differentiable nowhere monotone functions on $\mathbb{R}$, which was proved to be lineable (see, e.g., [4, 26]). We refer the interested reader to [1, 3, 5, 6, 10, 16–18, 24, 25, 27–29, 38, 44] for recent results on lineability and spaceability, where many more examples can be found and techniques are developed in several different frameworks.

This paper deals with standard L_p-spaces and canonical concepts of linear algebra (such as subspaces or dimension), thus it is addressed to a wide general audience. More particularly, we shall focus on sets of the form $L_p(\Omega) - \bigcup_{1\le q<p} L_q(\Omega)$. The study of structural properties of subspaces of L_p spaces is a classical topic in Banach space theory, dating back to the early days of the theory (see, e.g., [7, 8]) and developed up to the present days (see, e.g., [14, 17, 35]).

First of all, let us provide a clear summary and chronological overview of the spaceability results in this direction throughout the years.

1. H. Rosenthal (1968, [47]) showed that c_0 is *quasi-complemented* in ℓ_∞ (a closed subspace Y of a Banach space X is *quasi-complemented* if there is a closed subspace Z of X such that $Y\cap Z=\{0\}$ and $Y+Z$ is dense in X); this clearly implies that $\ell_\infty - c_0$ is spaceable.
2. Later, García-Pacheco, Martín and Seoane-Sepúlveda proved (2009, [30]) that $\ell_\infty(\Gamma) - c_0(\Gamma)$ is spaceable for every infinite set Γ. In view of the previous point, it is interesting to recall that J. Lindenstrauss

(1968, [42]) proved that, if Γ is uncountable, then $c_0(\Gamma)$ is not quasi-complemented in $\ell_\infty(\Gamma)$.

3. In (2008, [43]), Muñoz-Fernández, Palmberg, Puglisi and Seoane-Sepúlveda proved that if I is a bounded interval and $q > p \geq 1$, then $L_p(I) - L_q(I)$ is $\mathfrak{c}$-lineable. In this same paper it is proved that both $\ell_p - \ell_q$ and $L_p(J) - L_q(J)$ are $\mathfrak{c}$-lineable for any unbounded interval J and for $p > q \geq 1$.
4. One year later (2009, [2]), Aron, García-Pacheco, Pérez-García and Seoane-Sepúlveda showed that the linear subspaces constructed in [43] can be chosen to be dense.
5. Bernal-González (2010, [13]) provided a series of conditions from which one can obtain (maximal) lineability (and dense-lineability) of the set of functions in $L_p(X, \mu)$ that are not in $L_q(X, \mu)$, where $1 \leq q \neq p < \infty$ and μ denotes a regular Borel measure on a topological space X.
6. In [31, Theorem 2.6] García-Pacheco, Pérez-Eslava and Seoane-Sepúlveda proved that if (Ω, Σ, μ) is a measure space such that there exists $\varepsilon > 0$ and an infinite family $(A_n)_{n\in\mathbb{N}} \subset \Sigma$ of pairwise disjoint measurable sets with $\mu(A_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, then
$$\bigcap_{p=1}^{\infty} (L_\infty(\Omega, \Sigma, \mu) - L_p(\Omega, \Sigma, \mu)) \quad \text{is spaceable in } L_\infty(\Omega, \Sigma, \mu).$$
7. The results above, somehow, kept *evolving* and, in ([15], 2011), Botelho, Diniz, Fávaro and Pellegrino proved (for any Banach space X) that for large classes of Banach (and even quasi-Banach) spaces E of X-valued sequences, the sets $E - \bigcup_{q\in\Gamma} \ell_q(X)$ (where $\Gamma \subset [0, \infty)$), and $E - c_0(X)$ are both spaceable in E.
8. Next, and as a consequence of a lecture delivered by V. Fávaro at an international conference held in Valencia (Spain) in 2010, R. Aron asked whether the result above ([15, Corollary 1.7]) would hold for L_p-spaces. This question was answered in the positive (and independently) in [14, 17]. More precisely, in [14] Bernal-González and Ordóñez Cabrera provided a series of conditions on a measure space $(X, \mathcal{M}, \mu)$ to ensure the spaceability of the sets
$$L_p(\mu, X) - \bigcup_{q\in[1,p)} L_q(\mu, X), \quad L_p(\mu, X) - \bigcup_{q\in[p,\infty)} L_q(\mu, X),$$
$$L_p(\mu, X) - \bigcup_{q\in[1,\infty)-\{p\}} L_q(\mu, X)$$
(for $p \geq 1$); whereas in [17] Botelho, Fávaro, Pellegrino and Seoane-Sepúlveda obtained a quasi-Banach version of this result by proving that $L_p[0, 1] - \bigcup_{q>p} L_q[0, 1]$ is spaceable for every $p > 0$.

9. In this direction it is also crucial to mention a recent paper [39], where Kitson and Timoney provided a general result from which some of the above ones (for the normed case) can be inferred.

At this point, and after all the effort invested in looking for the "optimal" results on the spaceability of sets of the form $L_p(\Omega) - L_q(\Omega)$ with $p > q$ and $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$, we now continue with this ongoing work and provide rather conclusive contributions in the form of what it is called *maximal*-spaceability. In other words, given a measure space (Ω, Σ, μ),

- *When does* $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$ *contain, except for the null vector, a closed subspace* F *of* $L_p(\Omega)$ *such that* $\dim(F) = \dim(L_p(\Omega))$*?*

Of course, for the above problem to be well-posed we should have $\mu(\Omega) = +\infty$. In order to decide whether a subspace of $L_p(\Omega)$ has maximal dimension or not, it is of course crucial to know the dimension of $L_p(\Omega)$. So, as a preparation for the forthcoming results—but with interest on its own—in Section 2 we compute the dimension of $L_p(\Omega)$ for arbitrary measure spaces (Ω, Σ, μ). In Section 3 we shall benefit from this computation to provide quite general sufficient conditions for $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$ to be maximal spaceable. Although the results of Section 3 cover most cases, including all common $L_p(\Omega)$ spaces and some cases never studied before, there might be a (rather exotic) infinite measure space such that $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$ fails to be maximal spaceable. In Section 4 we develop a hybridization technique which, with the help of the results of Sections 2 and 3, provides an example of a measure space such that even the larger set $L_p(\Omega) - L_q(\Omega)$ with $q < p$ fails to be maximal spaceable (of course the conditions given in Section 3 are not fulfilled by this space). By doing this we provide an ultimate answer to the question of spaceability of the sets of the form $L_p - \bigcup_{1 \leq q < p} L_q$ for all measure spaces we are aware of.

Many recent results concern spaceability/maximal spaceability of complements of subspaces of topological vector spaces (sometimes complements of dense subspaces). For example, [39] provides quite strong results in this line. So it is important to mention that our results on the maximal spaceability of $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$ do not require $L_q(\Omega)$ to be a subspace of $L_p(\Omega)$ for $q < p$.

The proofs of the following results use techniques that, to the best of our knowledge, have never been used before (at least in the context of lineability/spaceability): Lemma 2.2, Theorem 2.3, Lemma 3.1, Theorem 3.4, Theorem 4.4.

Throughout this paper, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$, $\#A$ denotes the cardinality of the set A, $\aleph_0 = \#\mathbb{N}$ and $\mathfrak{c} = \#\mathbb{R}$, the continuum. The rest of the notation will be rather usual.

2. Computing the dimension of $L_p(\Omega)$. In this section we compute the dimension of $L_p(\Omega)$, for arbitrary measures spaces (Ω, Σ, μ), in terms that will reveal useful in the investigation of the maximal spaceability of $L_p(\Omega) - \bigcup_{1\le q<p} L_q(\Omega)$.

In this section (Ω, Σ, μ) will denote a measure space and $0 < p < \infty$.

Definition 2.1.

(i) $\Sigma_{\mathrm{fin}} := \{A \in \Sigma : \mu(A) < \infty\}$.

(ii) Two sets $A, B \in \Sigma_{\mathrm{fin}}$ are *equivalent*, denoted $A \sim B$, if $\mu((A - B) \cup (B - A)) = 0$. The elements of $\Sigma_{\mathrm{fin}}/{\sim}$ are denoted by $[B]$, for $B \in \Sigma_{\mathrm{fin}}$.

(iii) The cardinal number $\#\Sigma_{\mathrm{fin}}/{\sim}$ is called the *entropy* of the measure space (Ω, Σ, μ) and is denoted by $\mathrm{ent}(\Omega)$.

(iv) Given a cardinal number ζ, we say that the measure space (Ω, Σ, μ) is *ζ-bounded* if, for every $A \in \Sigma_{\mathrm{fin}}$ with $\mu(A) > 0$, there are at most ζ subsets of A with positive measure belonging to different classes of $\Sigma_{\mathrm{fin}}/{\sim}$.

(v) A set $A \in \Sigma$ is an *atom* if $\mu(A) > 0$ and there is no $B \in \Sigma$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$.

Lemma 2.2. *If* $\mathrm{ent}(\Omega) \ge \aleph_0$, *then there are sets* $(B_i)_{i\in\mathbb{N}}$ *in* Σ_{fin} *such that* $\mu(B_i) > 0$ *for every* $i \in \mathbb{N}$ *and* $\mu(B_i \cap B_j) = 0$ *whenever* $i \ne j$.

Proof. Assume first that there is a set $A_1 \in \Sigma_{\mathrm{fin}}$ with $\mu(A_1) > 0$ and containing no atoms. Therefore A_1 is not an atom and hence there is a set $A_2 \subset A_1$ such that $0 < \mu(A_2) < \mu(A_1)$. By the assumption on the existence of such A_1, we find that A_2 is not an atom either. Repeating this argument we obtain $A_1 \supset A_2 \supset A_3 \supset \cdots$ with $0 < \mu(A_{i+1}) < \mu(A_i)$ for every i. Defining $B_i = A_i - A_{i+1}$ we obtain $\mu(B_i) > 0$ for every $i \in \mathbb{N}$ and $\mu(B_i \cap B_j) = 0$ for $i \ne j$.

To complete the proof, suppose now that every $B \in \Sigma_{\mathrm{fin}}$ with $\mu(B) > 0$ contains an atom. Let $B_1 \in \Sigma_{\mathrm{fin}}$ be an atom. Suppose that we have defined pairwise disjoint atoms $B_1, \ldots, B_k \in \Sigma_{\mathrm{fin}}$ and let us prove that there is a measurable set $B \in \Sigma_{\mathrm{fin}}$ such that

$$\mu\Big(B - \bigcup_{i=1}^{k} (B \cap B_i)\Big) > 0.$$

If we suppose that $\mu(B) = \mu(\bigcup_{i=1}^{k}(B\cap B_i))$ for every measurable set B, then $[B] = [\bigcup_{i=1}^{k}(B\cap B_i)]$ for every measurable set B. In other words, every class in $\Sigma_{\mathrm{fin}}/{\sim}$ contains a subset of $\bigcup_{i=1}^{k} B_i$ as a representative. Since $B_1, \ldots, B_k$ are atoms, the only subsets of $\bigcup_{i=1}^{k} B_i$ that belong to different equivalence classes are equivalent to either $B_1, \ldots, B_k$ or unions of some of them. In this case we have $\mathrm{ent}(\Omega) = 2^k$, which is absurd. Hence there is a measurable B

such that $\mu(B) > \mu(\bigcup_{i=1}^{k}(B\cap B_i))$, that is, $\mu(B-\bigcup_{i=1}^{k}(B\cap B_i)) > 0$. So there is an atom $B_{k+1} \subset B - \bigcup_{i=1}^{k}(B \cap B_i)$. Therefore the sets $B_1, \ldots, B_k, B_{k+1} \in \Sigma_{\mathrm{fin}}$ are pairwise disjoint, and, in particular, $\mu(B_i \cap B_j) = 0$ for $i \neq j$. ∎

Since the continuum hypothesis is not required in what follows, we would rather prefer not to assume it.

THEOREM 2.3.

(a) *If* $\mathrm{ent}(\Omega) > \mathfrak{c}$, *then* $\dim(L_p(\Omega)) = \mathrm{ent}(\Omega)$.
(b) *If* $\aleph_0 \leq \mathrm{ent}(\Omega) \leq \mathfrak{c}$, *then* $\dim(L_p(\Omega)) = \mathfrak{c}$.
(c) *If* $\mathrm{ent}(\Omega) \in \mathbb{N}$, *then there is* $k \in \mathbb{N}$ *such that* $\mathrm{ent}(\Omega) = 2^k$ *and* $\dim(L_p(\Omega)) = k$.

Proof. By χ_A we denote the characteristic function of the set $A \in \Sigma$. Let

$$W := \Big\{\sum_{i=1}^{n} a_i\chi_{A_i} : n \in \mathbb{N},\ a_i \in \mathbb{K} \text{ and } A_i \text{ is a representative of a class in } \Sigma_{\mathrm{fin}}/{\sim}\Big\}.$$

By [22, Proposition 6.7] we know that $L_p(\Omega) = \overline{W}$. Therefore

$$\#L_p(\Omega) = \#\overline{W} \leq \#\{\text{Cauchy sequences in } W\} \leq \#W^{\mathbb{N}}.$$

Assume that $\mathrm{ent}(\Omega) \geq \mathfrak{c}$. On the one hand, $\#W = \mathrm{ent}(\Omega)$, hence

$$\#L_p(\Omega) \leq \#(\Sigma_{\mathrm{fin}}/{\sim})^{\mathbb{N}} = \mathrm{ent}(\Omega).$$

On the other hand, if $A, B \in \Sigma$ are not equivalent in Σ_{fin}, then $\chi_A \neq \chi_B$ in $L_p(\Omega)$. So $\mathrm{ent}(\Omega) \leq \#L_p(\Omega)$. Therefore $\#L_p(\Omega) = \mathrm{ent}(\Omega)$.

(a) Since $\mathrm{ent}(\Omega) > \mathfrak{c}$, we have $\#L_p(\Omega) = \mathrm{ent}(\Omega) > \mathfrak{c}$. And since the cardinality of this vector space is greater than the cardinality of the scalar field, its cardinality and dimension coincide.

(b) Since $\mathrm{ent}(\Omega) \leq \mathfrak{c}$, again we obtain $\#L_p(\Omega) = \#W \leq \mathfrak{c}$, therefore $\dim(L_p(\Omega)) \leq \mathfrak{c}$. On the other hand, since $\mathrm{ent}(\Omega) \geq \aleph_0$, by Lemma 2.2 there are countably many sets $B_1, B_2, \ldots$ such that $\mu(B_i\cap B_j) = 0$ whenever $i \neq j$, all of them of positive measure. Choose a sequence $(a_j)_{j=1}^{\infty} \in \ell_p$ with $a_j > 0$ for every j and define

$$f\colon \Omega \to \mathbb{K}, \qquad f(x) = \sum_{i=1}^{\infty} \frac{a_j}{\mu(B_j)^{1/p}}\chi_{B_j}(x).$$

Notice that $\int_{\Omega} |f|^p\, d\mu = \sum_{i=1}^{\infty} |a_j|^p$, thus $f \in L_p(\Omega)$. Now let $\mathcal{F}$ be a totally ordered (with respect to the inclusion) family of subsets of $\mathbb{N}$ such that $\#\mathcal{F} = \mathfrak{c}$. For example, identify $\mathbb{N}$ with $\mathbb{Q}$ and consider the family $\mathcal{F} = \{(-\infty, r) \cap \mathbb{Q} : r \in \mathbb{R}\}$. Given $S \in \mathcal{F}$, define

$$\chi_S\colon \Omega \to \mathbb{K}, \qquad \chi_S(x) = \begin{cases} 1 & \text{if } x \in B_j \text{ with } j \in S,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that $\{f\chi_S : S \in \mathcal{F}\}$ is a linearly independent subset of $L_p(\Omega)$. Therefore

$$\dim(L_p(\Omega)) \geq \#\{f\chi_S : S \in \mathcal{F}\} = \#\mathcal{F} = \mathfrak{c}.$$

It follows from the Cantor–Bernstein–Schröder Theorem that $\dim(L_p(\Omega)) = \mathfrak{c}$.

(c) Firstly let us see that, under the assumption $\operatorname{ent}(\Omega) \in \mathbb{N}$, every measurable set of positive measure contains an atom. In fact, otherwise we could build a sequence $A_1 \supset A_2 \supset \cdots$ in Σ with $\mu(A_1) > \mu(A_2) > \cdots$. In this case, A_i and A_j belong to different classes whenever $i \neq j$. This is a contradiction because there are only finitely many equivalence classes.

Let $\mathcal{S}$ be the family of all subsets of Σ_{fin} whose elements are pairwise disjoint atoms. Consider the partial order in $\mathcal{S}$ given by the natural inclusion, that is, for $S_1, S_2 \in \mathcal{S}$,

$$S_1 \leq S_2 \iff S_1 \subset S_2.$$

Consider a subfamily $\mathcal{S}' = \{S_i : i \in I\} \subset \mathcal{S}$ totally ordered by inclusion, where I is an index set. Hence $S = \bigcup_{i\in I} S_i \in \mathcal{S}$ and $S_i \subset S$ for every $i \in I$. Then S is an upper bound for $\mathcal{S}'$. Therefore, by Zorn's Lemma there is a maximal set $U \in \mathcal{S}$ with respect to inclusion. Since the elements of U are pairwise disjoint atoms, they are in different equivalence classes. But $\operatorname{ent}(\Omega) < \infty$, so $\#U < \infty$, say $U = \{A_i : i = 1, \ldots, k\}$ where $k \in \mathbb{N}$. Let $B \in \Sigma_{\mathrm{fin}}$ be given. Of course B can be written as the union of the following two disjoint sets:

$$B = \Big(B - \bigcup_{i=1}^{k} A_i\Big) \cup \Big(\bigcup_{i=1}^{k} (B \cap A_i)\Big).$$

Suppose that $\mu(B - \bigcup_{i=1}^k A_i) > 0$. In this case there is an atom A_{k+1} contained in $B - \bigcup_{i=1}^k A_i$ such that $A_{k+1} \cap A_i = \emptyset$ for every $i = 1, \ldots, k$. Thus $\{A_{k+1}\} \cup U > U$, which contradicts the maximality of U. Hence $\mu(B - \bigcup_{i=1}^k A_i) = 0$ and so $[B] = [\bigcup_{i=1}^k (B \cap A_i)]$. For each $i \in \{1, \ldots, k\}$ such that $[B \cap A_i] \neq [\emptyset]$, we have $\mu(B \cap A_i) > 0$, and, since $B \cap A_i \subset A_i$ and A_i is an atom, we obtain $\mu(B \cap A_i) = \mu(A_i)$, that is, $[B \cap A_i] = [A_i]$. Denoting by J_B the set of all $i \in \{1, \ldots, k\}$ such that $[B \cap A_i] \neq [\emptyset]$, it follows that

$$\mu\Big(\bigcup_{i\in J_B} A_i\Big) = \sum_{i\in J_B} \mu(A_i) = \sum_{i\in J_B} \mu(B \cap A_i) = \mu\Big(\bigcup_{i\in J_B} (B \cap A_i)\Big) = \mu(B).$$

So every set $B \in \Sigma_{\mathrm{fin}}$ satisfies

$$[B] = \Big[\bigcup_{i\in J_B} A_i\Big], \tag{2.1}$$

where $J_B = \{i \in \{1, \ldots, k\} : [B \cap A_i] \neq [\emptyset]\}$. This proves that $\operatorname{ent}(\Omega) = 2^k$.

Now, we know that $L_p(X)$ is the closure of

$$W = \Big\{\sum_{i=1}^{n} b_i\chi_{B_i} : n \in \mathbb{N},\ b_i \in \mathbb{K},\ B_i \in \Sigma_{\mathrm{fin}}\Big\}.$$

By (2.1), each $\sum_{i=1}^{n} b_i\chi_{B_i} \in W$ is μ-almost everywhere equal to an element of $\{\sum_{i=1}^{k} a_i\chi_{A_i} : a_i \in \mathbb{K}\}$. Thus

$$\begin{aligned} L_p(X) = \overline{W} &= \overline{\Big\{\sum_{i=1}^{n} b_i\chi_{B_i} : n \in \mathbb{N},\ b_i \in \mathbb{K},\ B_i \in \Sigma_{\mathrm{fin}}\Big\}} \\ &= \overline{\Big\{\sum_{i=1}^{k} a_i\chi_{A_i} : a_i \in \mathbb{K}\Big\}}. \end{aligned}$$

Since any finite-dimensional subspace of a topological vector space is closed, it follows that $\dim(L_p(\Omega)) = \dim \overline{W} = \dim\{\sum_{i=1}^{k} a_i\chi_{A_i} : a_i \in \mathbb{K}\} = k$, as required. ∎

Let us state, for future reference, a fact proved in the proof above:

Corollary 2.4. *If* $\mathrm{ent}(\Omega) \geq \mathfrak{c}$, *then* $\#L_p(\Omega) = \mathrm{ent}(\Omega) = \dim(L_p(\Omega))$.

Remark 2.5. Let us recall that the standard proof of the fact that the dimension of every infinite-dimensional Banach space is, at least, $\mathfrak{c}$ (via Baire's Theorem) depends on the Continuum Hypothesis (CH). As a by-product, we shall now see that Theorem 2.3, whose proof does not depend on CH, can be used to give a CH-free proof of this fact: Let E be an infinite-dimensional Banach space and let $(x_n)_{n=1}^{\infty}$ be a normalized basic sequence in E (Mazur's classical proof of the existence of such a sequence does not depend on CH; see [19, Corollary 5.3]). The operator

$$(a_n)_{n=1}^{\infty} \in \ell_1 \mapsto \sum_{n=1}^{\infty} a_n x_n \in E$$

is well-defined because the series is absolutely convergent (and its linearity is obvious). The uniqueness of the representation of a vector in E as a (possibly infinite) linear combination of vectors of the basic sequence guarantees the injectivity of this linear operator. Then $\dim(E) \geq \dim(\ell_1)$. By Theorem 2.3 we know that $\dim(\ell_1) \geq \mathfrak{c}$, and thus $\dim(E) \geq \mathfrak{c}$.

3. $L_p(\Omega) - \bigcup_{q<p} L_q(\Omega)$ is "usually" maximal spaceable. In this section we give quite general conditions under which $L_p(\Omega) - \bigcup_{1\leq q<p} L_q(\Omega)$ is maximal spaceable. These conditions are so general that only highly exotic spaces may not fulfill them. It is worth mentioning once again that, unlike several results on lineability/spaceability of complements of subspaces or

unions of subspaces (see, e.g., [16–18, 39]), we are not assuming that $L_q(\Omega) \subset L_p(\Omega)$.

Of course we need $L_p(\Omega) - \bigcup_{q<p} L_q(\Omega) \neq \emptyset$, thus throughout this section (Ω, Σ, μ) is an infinite measure space.

LEMMA 3.1. *Let $\mathcal{X}$ be the set of all subsets F of Σ_{fin} satisfying the following conditions:*

1. *$\mu(A) > 0$ for every $A \in F$.*
2. *If $A, B \in F$ are distinct, then $\mu(A \cap B) = 0$.*

If the measure space (Ω, Σ, μ) is ζ-bounded for some cardinal number ζ with $\mathfrak{c} \leq \zeta < \operatorname{ent}(\Omega)$, then there exists a set $G \in \mathcal{X}$ with $\#G = \operatorname{ent}(\Omega)$.

Proof. Consider the partial order in $\mathcal{X}$ given by inclusion, that is, for $F_1, F_2 \in \mathcal{X}$,

$$F_1 \leq F_2 \;\Leftrightarrow\; F_1 \subset F_2.$$

Given a totally ordered subset $\mathcal{Y}$ of $\mathcal{X}$, define F as the union of all elements of $\mathcal{Y}$. Since $F \in \mathcal{X}$, F is an upper bound for $\mathcal{Y}$. Thus, by Zorn's Lemma there is a maximal element $G \in \mathcal{X}$. By assumption, each element of G has at most ζ subsets with positive measure belonging to different classes of $\Sigma_{\text{fin}}/\sim$, hence the number of subsets of elements of G that represent different classes in $\Sigma_{\text{fin}}/\sim$ is at most $(\#G) \cdot \zeta$.

Now fix $A \in \Sigma_{\text{fin}}$ with $\mu(A) > 0$ and define

$$H = \{B \in G : \mu(A \cap B) > 0\}.$$

Clearly $H \neq \emptyset$, because otherwise we would have $G \cup \{A\} > G$, which contradicts the maximality of G. Let us prove that $\#H$ is at most $\aleph_0$. Suppose, for contradiction, that H is uncountable, and note that, for each $B \in H$, the positive real number $\mu(A \cap B)$ belongs to, at least, one of the sets $(1/n, \infty)$ $(n \in \mathbb{N})$. There are countably many sets $(1/n, \infty)$, so it follows from the Infinite Pigeonhole Principle that there is $n_0 \in \mathbb{N}$ such that

$$\mu(A \cap B) > 1/n_0$$

for uncountably many sets $B \in H$. In particular, there are distinct $(C_i)_{i \in \mathbb{N}}$ in H such that

$$\mu(A \cap C_m) > 1/n_0$$

for every $m \in \mathbb{N}$. By Condition 2 we have $\mu(C_i \cap C_j) = 0$ whenever $i \neq j$, so

$$\mu(A) \geq \mu\Big(A \cap \Big(\bigcup_{m=1}^{\infty} C_m\Big)\Big) = \sum_{m=1}^{\infty} \mu(A \cap C_m) = \infty,$$

a contradiction that proves that $\#H \leq \aleph_0$.

Now, note that $\bigcup_{B\in H}(A\cap B)\in \Sigma_{\mathrm{fin}}$ because $\bigcup_{B\in H}(A\cap B)\subset A$. Let us prove that $[A]=[\bigcup_{B\in H}(A\cap B)]$. Assuming that $[A]\neq[\bigcup_{B\in H}(A\cap B)]$ in $\Sigma_{\mathrm{fin}}/{\sim}$, we have

$$\mu\Big(A-\bigcup_{B\in H}(A\cap B)\Big)>0.$$

Let $C\in G$ be given and assume that

$$\mu\Big(\Big(A-\bigcup_{B\in H}(A\cap B)\Big)\cap C\Big)>0.$$

In this case,

$$\mu(A\cap C)\geq \mu\Big(\Big(A-\bigcup_{B\in H}(A\cap B)\Big)\cap C\Big)>0,$$

which implies that $C\in H$. Hence $A-\bigcup_{B\in H}(A\cap B)\subset A-(A\cap C)$. Since $(A-(A\cap C))\cap C=\emptyset$, by the inclusion above we obtain

$$\mu\Big(\Big(A-\bigcup_{B\in H}(A\cap B)\Big)\cap C\Big)\leq \mu\big((A-(A\cap C))\cap C\big)=0.$$

This contradiction proves that the intersection of $A-\bigcup_{B\in H}(A\cap B)$ with each element of G has null measure. Therefore

$$G\cup\Big\{A-\bigcup_{B\in H}(A\cap B)\Big\}\in\mathcal{X}\quad\text{and}\quad G\cup\Big\{A-\bigcup_{B\in H}(A\cap B)\Big\}>G,$$

which contradicts the maximality of G. Therefore $[A]=[\bigcup_{B\in H}(A\cap B)]$. Since A is an arbitrary set in Σ_{fin} with positive measure, we have just proved that each class in $\Sigma_{\mathrm{fin}}/{\sim}$ can be represented by a union of countably many subsets of elements of G. Combining this with the fact that the number of subsets of elements of G that represent different classes in $\Sigma_{\mathrm{fin}}/{\sim}$ is at most $(\#G)\cdot\zeta$, we conclude that $\mathrm{ent}(\Omega)\leq(\#G)\cdot\zeta$. By assumption we have $\mathrm{ent}(\Omega)>\zeta\geq\mathfrak{c}$, so $\mathrm{ent}(\Omega)\leq\#G$.

On the other hand, we know that distinct elements of G determine different classes in $\Sigma_{\mathrm{fin}}/{\sim}$. Thus $\#G\leq\mathrm{ent}(\Omega)$. Hence $\mathrm{ent}(\Omega)=\#G$. ∎

Lemma 3.2. *Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable sets in a measure space (Ω,Σ,μ) with $0<\mu(B_i)<\infty$ for every $i\in\mathbb{N}$. Then:*

(a) *$\Sigma':=\{\bigcup_{j\in J}B_j : J\subset\mathbb{N}\}$ is a σ-algebra of subsets of $\Omega':=\bigcup_{i=1}^{\infty}B_i$.*
(b) *The restriction of μ to Σ' is a measure.*
(c) *For every $r\geq 1$,*

$$L_r(\Omega')=\Big\{\sum_{i=1}^{\infty}a_i\chi_{B_i} : \sum_{i=1}^{\infty}|a_i|^r\mu(B_i)<\infty\Big\}. \tag{3.1}$$

Proof. (a) and (b) are straightforward. Let us prove (c). It is easy to see that any simple function having support of finite measure can be written as $\sum_{j=1}^{\infty} a_j \chi_{B_j}$, where only finitely many a_j's are nonzero. So given $f \in L_r(\Omega')$ there is a sequence $(f_n)_{n=1}^{\infty}$ with $f_n = \sum_{j=1}^{\infty} a_n^j \chi_{B_j}$ such that only finitely many a_n^j's are nonzero for every $n \in \mathbb{N}$ and $f = \lim_{n\to\infty} f_n$ in $L_r(\Omega')$. Fix $j \in \mathbb{N}$ for a moment. Trivially, we have $f\chi_{B_j} = \lim_{n\to\infty} f_n \chi_{B_j}$ in $L_r(\Omega')$. On the other hand, $f_n\chi_{B_j}(x) = a_n^j \chi_{B_j}(x)$ for every $x \in \Omega'$ and every n; so $f\chi_{B_j} = \lim_{n\to\infty} a_n^j \chi_{B_j}$ in $L_r(\Omega')$. Hence $(a_n^j \chi_{B_j})_{n=1}^{\infty}$ is a Cauchy sequence in $L_r(\Omega')$, and since $0 < \mu(B_j) < \infty$ we infer that $(a_n^j)_{n=1}^{\infty}$ is a Cauchy scalar sequence, say $a_j = \lim_{n\to\infty} a_n^j$. It follows easily that $a_j\chi_{B_j} = \lim_{n\to\infty} a_n^j \chi_{B_j}$ in $L_r(\Omega')$. The uniqueness of the limit in $L_r(\Omega')$ implies that $a_j\chi_{B_j} = f\chi_{B_j}$ in $L_r(\Omega')$. Observing that B_j contains strictly no nonvoid measurable subset it follows that $f\chi_{B_j}(x) = a_i\chi_{B_j}(x)$ for every $x \in \Omega'$. In particular, $f(x) = a_j$ for every $x \in B_j$. This holds for every $j \in \mathbb{N}$, so $f(x) = \sum_{j=1}^{\infty} a_j\chi_{B_j}(x)$ for every $x \in \Omega'$. Since $|\sum_{j=1}^{k} a_j\chi_{B_j}(x)| \le |f(x)|$ for every $x \in \Omega'$ and every $k \in \mathbb{N}$, by a standard application of the Dominated Convergence Theorem (see, e.g., [11, Theorem 7.2]), we conclude that $f = \sum_{j=1}^{\infty} a_j\chi_{B_j}$ in $L_r(\Omega')$. Now it is immediate that $\sum_{j=1}^{\infty} |a_j|^r \mu(B_j) = \|f\|_r^r < \infty$. ∎

We shall need the following result due to Subramanian [49] and Romero [46] (see also [14, Theorem 3.1]):

THEOREM 3.3. *Let (Ω, Σ, μ) be a measure space and $p > q \ge 1$. Then:*

(a) *$L_p(\Omega) \supset L_q(\Omega)$ if and only if* $\inf\{\mu(A) : A \in \Sigma_{\mathrm{fin}}, \mu(A) > 0\} > 0$.
(b) *$L_q(\Omega) \supset L_p(\Omega)$ if and only if* $\sup\{\mu(A) : A \in \Sigma_{\mathrm{fin}}\} < \infty$.

As to the maximal spaceability of $L_p(\Omega) - \bigcup_{q<p} L_q(\Omega)$, there is nothing to do if $\mathrm{ent}(\Omega) \in \mathbb{N}$, because in this case we know, by Theorem 2.3(c), that $L_p(\Omega)$ is finite-dimensional. So we restrict ourselves, without loss of generality, to the case $\mathrm{ent}(\Omega) \ge \aleph_0$.

THEOREM 3.4. *Let $p > 1$. The set $L_p(\Omega) - \bigcup_{1\le q<p} L_q(\Omega)$ is maximal spaceable if either*

(a) *$L_p(\Omega) - L_r(\Omega) \neq \emptyset$ for some $1 \le r < p$ and $\aleph_0 \le \mathrm{ent}(\Omega) \le \mathfrak{c}$, or*
(b) *the measure space (Ω, Σ, μ) is ζ-bounded for some cardinal number ζ with $\mathfrak{c} \le \zeta < \mathrm{ent}(\Omega)$.*

Proof. (a) Since $\aleph_0 \le \mathrm{ent}(\Omega) \le \mathfrak{c}$, by Theorem 2.3(b) we know that $\dim(L_p(\Omega)) = \mathfrak{c}$. Therefore we only need to prove that $L_p(\Omega) - \bigcup_{1\le q<p} L_q(\Omega)$ is spaceable.

Since $L_p(\Omega) - L_r(\Omega) \neq \emptyset$ for some $1 \le r < p$, by Theorem 3.3(b) we have

$$\sup\{\mu(A) : A \in \Sigma_{\mathrm{fin}}\} = \infty. \tag{3.2}$$

In this case we can choose pairwise disjoint measurable sets $(B_i)_{i\in\mathbb{N}}$ such that $0 < \mu(B_1) < \mu(B_i)$ for every $i \geq 2$. Indeed, choose $B_1 \in \Sigma_{\mathrm{fin}}$ with $\mu(B_1) > 0$ and proceed inductively in the following way: if $B_1, \dots, B_k$ have been chosen satisfying those conditions, by (3.2) there is $A_{k+1} \in \Sigma_{\mathrm{fin}}$ such that $\mu(A_{k+1}) > 2\mu(B_1 \cup \dots \cup B_k)$. Choose $B_{k+1} = A_{k+1} - (B_1 \cup \dots \cup B_k)$.

Consider now the measure space (Ω', Σ', μ), where Ω' and Σ' are defined as in Lemma 3.2. Let us prove that $L_p(\Omega') - \bigcup_{1\leq q<p} L_q(\Omega')$ is spaceable in $L_p(\Omega')$. First note that

$$\inf\{\mu(A) : A \in \Sigma'_{\mathrm{fin}} \text{ and } \mu(A) > 0\} = \mu(B_1) > 0.$$

From Theorem 3.3(a) it follows that $L_p(\Omega') \supset L_q(\Omega')$ for every $1 \leq q < p$. Applying (3.1) for $r = q$ we know that every function in $L_q(\Omega')$ can be written as $\sum_{i=1}^{\infty} a_i \chi_{B_i}$ with $\sum_{i=1}^{\infty} |a_i|^q \mu(B_i) < \infty$. Note that if $\| \sum_{i=1}^{\infty} a_i \chi_{B_i} \|_q < \mu(B_1)^{1/q}$, then $|a_i| < 1$ for every $i \in \mathbb{N}$. Since $p > q \geq 1$ and $|a_i| < 1$ for every $i \in \mathbb{N}$, we have

$$\Big\|\sum_{i=1}^{\infty} a_i \chi_{B_i}\Big\|_q > \Big\|\sum_{i=1}^{\infty} a_i \chi_{B_i}\Big\|_p.$$

Given $\varepsilon > 0$, choose $\delta = \min\{\varepsilon, \mu(B_1)^{1/q}\} > 0$. If $\| \sum_{i=1}^{\infty} a_i \chi_{B_i} \|_q < \delta$, then

$$\varepsilon > \Big\|\sum_{i=1}^{\infty} a_i \chi_{B_i}\Big\|_q > \Big\|\sum_{i=1}^{\infty} a_i \chi_{B_i}\Big\|_p.$$

This shows that the inclusion $L_q(\Omega') \hookrightarrow L_p(\Omega')$ is continuous for every $1 \leq q < p$. Choosing a sequence $(a_j)_{j=1}^{\infty} \in \ell_p - \bigcup_{q<p} \ell_q$, it is clear that the function

$$f = \sum_{j=1}^{\infty} \frac{a_j}{\mu(B_j)^{1/p}} \chi_{B_j}$$

belongs to $L_p(\Omega')$. Using that $\mu(B_j) \geq \mu(B_1)$ for every j, it follows that $f \notin L_q(\Omega')$ for every $1 \leq q < p$. So $L_p(\Omega') \neq \bigcup_{1\leq q<p} L_q(\Omega')$. And since

$$W := \Big\{\sum_{i=1}^{n} a_i \chi_{A_i} : n \in \mathbb{N}, a_i \in \mathbb{K} \text{ and } A_i \in \Sigma'_{\mathrm{fin}}\Big\} \subset \bigcup_{1\leq q<p} L_q(\Omega') \subset L_p(\Omega')$$

and W is dense in $L_p(\Omega')$, it follows that $\bigcup_{1\leq q<p} L_q(\Omega')$ is dense in $L_p(\Omega')$ as well. So $\bigcup_{1\leq q<p} L_q(\Omega')$ is not closed in $L_p(\Omega')$ because $L_p(\Omega') \neq \bigcup_{1\leq q<p} L_q(\Omega')$. Choose a sequence $(q_j)_{j=1}^{\infty}$ such that $1 \leq q_j < q_{j+1}$ for every j and $q_j \to p$. Theorem 3.3(a) ensures that $L_{q_j} \subset L_{q_{j+1}}$ for every j, hence

$$\bigcup_{1\leq q<p} L_q(\Omega') = \bigcup_{j=1}^{\infty} L_{q_j}(\Omega').$$

The spaceability of $L_p(\Omega') - \bigcup_{1\le q<p} L_q(\Omega')$ in $L_p(\Omega')$ follows now from [39, Theorem 3.3].

A function f defined on Ω' will be identified with a function defined on Ω by putting $f(x) = 0$ for every $x \in \Omega - \Omega'$. Since $\|f\|_{L_p(\Omega')} = \|f\|_{L_p(\Omega)}$ for every $f \in L_p(\Omega')$, it is plain that $L_p(\Omega')$ is a closed subspace of $L_p(\Omega)$ up to this identification.

Use that $L_p(\Omega') \supset \bigcup_{1\le q<p} L_q(\Omega')$ and apply (3.1) for $r = p$ and for $r = q < p$ to conclude that

$$L_p(\Omega') \cap \Big(\bigcup_{1\le q<p} L_q(\Omega)\Big) = \bigcup_{1\le q<p} L_q(\Omega').$$

Thus

$$L_p(\Omega') - \bigcup_{1\le q<p} L_q(\Omega') = L_p(\Omega') - \bigcup_{1\le q<p} L_q(\Omega).$$

It follows that $L_p(\Omega') - \bigcup_{1\le q<p} L_q(\Omega)$ is spaceable in the closed subspace $L_p(\Omega')$ of $L_p(\Omega)$, hence $L_p(\Omega') - \bigcup_{1\le q<p} L_q(\Omega)$ is spaceable in $L_p(\Omega)$. Therefore $L_p(\Omega) - \bigcup_{1\le q<p} L_q(\Omega)$ is spaceable in $L_p(\Omega)$.

(b) Let G be the family whose existence is guaranteed by Lemma 3.1. Since $\#G = \mathrm{ent}(\Omega) > \zeta$ and there are only $\mathfrak{c}$ possible values for the measures of the sets in G (of course $\mu(B) \in (0,\infty)$ for every $B \in G$), there is a subfamily $G' \subset G$, with the same cardinality of G, such that all members of G' have the same measure, say α (this is another application of the Infinite Pigeonhole Principle). Denote $G' = \{A_k : k \in I\}$ with $\#I = \mathrm{ent}(\Omega)$. Recall that $A_k \neq A_s$ implies $\mu(A_k \cap A_s) = 0$ but $\mu(A_k) = \alpha = \mu(A_s)$. Since the cardinality of I is greater than ζ and $\aleph_0 \cdot \zeta = \zeta$, for every $i \in I$ and every $n \in \mathbb{N}$ there is a set A_i^n so that:

(i) $A_i^j \neq A_i^k$ whenever $i \in I$ and $j \neq k$ are positive integers;
(ii) the sets $J_i := \{A_i^j : j \in \mathbb{N}\}$, $i \in I$, are pairwise disjoint;
(iii) $G' = \bigcup_{i\in I} J_i$.

Select a sequence $(b_j)_{j=1}^\infty \in \ell_p - \bigcup_{q<p} \ell_q$ with $b_j > 0$ for every j. For each $k \in I$, define $f_k := \sum_{j=1}^\infty b_j \chi_{A_k^j}$. Observe that:

1. The intersection of the supports of f_k and f_s, $k \neq s$, has measure zero. Therefore $\#\{f_k : k \in I\} = \#I$ and the functions f_k's are linearly independent.
2. Let $k \in I$ and $i \in \mathbb{N}$. Since $\mu(A_k^j \cap A_k^s) = 0$ for all positive integers $j \neq s$ and $\mu(A_k^j) = \alpha = \mu(A_k^i)$ for all $j \in \mathbb{N}$, for every $t > 0$ we have

$$\int |f_k|^t \, d\mu = \sum_{j=1}^\infty |b_j|^t \mu(A_k^j) = \Big(\sum_{j=1}^\infty |b_j|^t\Big) \cdot \alpha. \tag{3.3}$$

Therefore each f_k is in $L_p(\Omega) - \bigcup_{q<p} L_q(\Omega)$.

3. For all $k, l \in I$,

$$\int |f_k|^p \, d\mu = \int |f_l|^p \, d\mu. \tag{3.4}$$

Let $W = \operatorname{span}\{f_k : k \in I\} \subset L_p(\Omega)$. Let $(h_n)_{n=1}^{\infty}$ be a Cauchy sequence in W (with respect to the $L_p(\Omega)$-norm). Each h_n is a finite linear combination of some f_k's, so all these functions together require only countably many f_k's in their representations as linear combinations. Let $(g_l)_{l=1}^{\infty}$ be an enumeration of these f_k's. Thus

$$h_n = \sum_{l=1}^{\infty} a_l^n g_l,$$

where, for each n, only finitely many a_l^n's are nonzero. Using that the intersection of the supports of g_k and g_s, $k \neq s$, has measure zero and (3.4) we obtain, for any fixed $j \in \mathbb{N}$,

$$\begin{aligned}\int |h_n - h_s|^p \, d\mu &= \sum_{l=1}^{\infty} |a_l^n - a_l^s|^p \cdot \int |g_l|^p \, d\mu \\ &= \Big(\sum_{l=1}^{\infty} |a_l^n - a_l^s|^p\Big) \int |g_j|^p \, d\mu.\end{aligned}$$

It follows that $((a_l^n)_{l=1}^{\infty})_{n=1}^{\infty}$ is a Cauchy sequence in ℓ_p, say $\lim_{n\to\infty} (a_l^n)_{l=1}^{\infty} = (a_l)_{l=1}^{\infty} \in \ell_p$. Define $h = \sum_{l=1}^{\infty} a_l g_l$ and notice that $h \in L_p(\Omega)$. Now

$$\int |h_n - h|^p \, d\mu = \sum_{l=1}^{\infty} |a_l^n - a_l|^p \cdot \int |g_l|^p \, d\mu.$$

Since $\int |g_l|^p \, d\mu$ does not depend on l, by (3.3) and $\lim_{n\to\infty} (a_l^n)_{l=1}^{\infty} = (a_l)_{l=1}^{\infty}$ in ℓ_p, we obtain $\lim_{n\to\infty} h_n = h$ in $L_p(\Omega)$. Finally, if $h \neq 0$ then some a_l is not zero, hence $\|h\|_q^q \geq |a_l|^q \cdot \|g_l\|_q^q = |a_l|^q \cdot \alpha \cdot \sum_{j=1}^{\infty} b_j^q = +\infty$ and so $h \notin \bigcup_{q<p} L_q(\Omega)$, as required. ∎

Remark 3.5. Observe that in case (b) of the theorem above we have actually proved that $L_p(\Omega) - \bigcup_{0<q<p} L_q(\Omega)$ is maximal spaceable for every $p > 0$. Notice that, as a particular case, from Theorem 3.3(b) one deduces that condition (3.2) is fulfilled, and thus we also obtain (independently) a result already given in [14] on the spaceability of this set.

All usual infinite measure spaces satisfy either condition (a) of Theorem 3.4 or condition (b) with $\zeta = \mathfrak{c}$ (for instance, a concrete example of an infinite measure space satisfying condition (b) is a set of cardinality greater than $\mathfrak{c}$ endowed with the counting measure).

4. $L_p(\Omega) - L_q(\Omega)$ may fail to be maximal spaceable for $p > q$. As we have proved in the previous section, $L_p(\Omega) - \bigcup_{1\leq q<p} L_q(\Omega)$ is maximal

spaceable in most cases. Nevertheless, in this section we prove that there exist (quite exotic) infinite measure spaces (Ω, Σ, μ) such that the larger set $L_p(\Omega) - L_q(\Omega)$, $q < p$, fails to be maximal spaceable. Actually we develop a hybridization technique that allows us to prove much more: given $1 \leq q < p$ and cardinal numbers $\kappa > \zeta \geq \mathfrak{c}$, we construct an infinite measure space (Ω, Σ, μ) such that:

(i) $\dim(L_p(\Omega)) = \kappa$;
(ii) ζ is the maximal dimension of a closed subspace of $L_p(\Omega)$ contained (except for the null vector) in $L_p(\Omega) - L_q(\Omega)$.

Besides its own interest, this result reinforces the role of Theorem 3.4, because now we know that some conditions should be imposed on the measure space for $L_p(\Omega) - L_q(\Omega)$ to be maximal spaceable, and Theorem 3.4 establishes quite general conditions for the smaller set $L_p(\Omega) - \bigcup_{1 \leq q < p} L_q(\Omega)$ to be maximal spaceable.

It is worth mentioning that the construction we describe in this section depends on the results of Sections 2 and 3.

Lemma 4.1. *Let ζ be a cardinal number such that $\zeta \geq \mathfrak{c}$, let X_ζ be a set such that $\#X_\zeta = \zeta$ and let the set $\mathcal{P}(X_\zeta)$ of all subsets of X_ζ endow with the counting measure. Then* $\dim(L_p(X_\zeta)) = \mathrm{ent}(X_\zeta) = \zeta$ *for every* $0 < p < \infty$.

Proof. For $A \subset X_\zeta$, it is clear that $A \in \mathcal{P}(X_\zeta)_{\mathrm{fin}}$ if and only if $\#A < \infty$, so $\#\mathcal{P}(X_\zeta)_{\mathrm{fin}} = \#X_\zeta$. It follows that $\mathrm{ent}(X_\zeta) = \#\mathcal{P}(X_\zeta)_{\mathrm{fin}}/{\sim} \leq \#X_\zeta$. On the other hand, different singletons belong to different classes in $\mathcal{P}(X_\zeta)_{\mathrm{fin}}$, therefore $\#X_\zeta \leq \mathrm{ent}(X_\zeta)$. Combining this with $\mathrm{ent}(X_\zeta) = \#X_\zeta = \zeta \geq \mathfrak{c}$, by Theorem 2.3 we have $\dim(L_p(X_\zeta)) = \mathrm{ent}(X_\zeta) = \#X_\zeta = \zeta$. ∎

The key to the proof of the following lemma was communicated to the authors by L. Bernal-González.

Lemma 4.2. *For every cardinal number $\kappa \geq \mathfrak{c}$ there exists a probability space $(T_\kappa, \Sigma_\kappa, \mu_\kappa)$ such that* $\dim(L_p(T_\kappa)) = \mathrm{ent}(T_\kappa) = \kappa$ *for every* $0<p<\infty$.

Proof. Let Γ be a set with $\#\Gamma = \kappa$. Let T_κ be the product of κ copies of $[0,1]$, that is, $T_\kappa = \prod_{\gamma \in \Gamma}[0,1]$, and let Σ_κ be the product σ-algebra of the Borel σ-algebra on $[0,1]$, that is, the σ-algebra on T_κ generated by the inverse images of Borel subsets of $[0,1]$ by the projections onto each coordinate (cf. [12, Definition 9.1], [37, Definition 22.2]). By [37, Section 22] (see also [20, p. 259]) there exists a probability measure μ_κ on Σ_κ such that if $A = \prod_{\gamma \in \Gamma} A_\gamma$, where $A_\gamma = [0,1]$ except for $\gamma = \gamma_i$, $i = 1, \ldots, n$, then $\mu_\kappa(A) = m(A_{\gamma_1}) \cdots m(A_{\gamma_n})$, where m is the Lebesgue measure. Since $\kappa \geq \mathfrak{c}$, Σ_κ is generated by $\kappa \times \mathfrak{c} = \kappa$ sets, by [40, Problem 23, Chapter 12] it follows that $\#\Sigma_\kappa = \kappa$ and, *a fortiori*, $\mathrm{ent}(T_\kappa) \leq \kappa$. On the other hand, for $i, j \in \Gamma$, $i \neq j$, setting $A_i = B_j = [0, 1/2]$, the sets $A = \prod_{t \in \Gamma} A_t$, where $A_t = [0,1]$ for every $t \neq i$, and $B = \prod_{t \in \Gamma} B_t$, where $B_t = [0,1]$ for every $t \neq j$, belong to different classes in $(\Sigma_\kappa)_{\mathrm{fin}}/\sim$. This shows that $\kappa \leq \mathrm{ent}(T_\kappa)$. By Theorem 2.3 we have $\dim(L_p(T_\kappa)) = \mathrm{ent}(T_\kappa) = \kappa$. ∎

Definition 4.3. Let $\zeta, \kappa \geq \mathfrak{c}$ be cardinal numbers. Consider the measure spaces $(X_\zeta, \mathcal{P}(X_\zeta), \nu)$ of Lemma 4.1, where ν is the counting measure, and $(T_\kappa, \Sigma_\kappa, \mu_\kappa)$ of Lemma 4.2. Choose X_ζ in such a way that $X_\zeta \cap T_\kappa = \emptyset$. Then the measure space $(Y, \mathcal{A}, \lambda)$ is defined by the following identities:

- $Y = T_\kappa \cup X_\zeta$,
- $\mathcal{A} = \{B \cup C : B \in \Sigma_\kappa \text{ and } C \in \mathcal{P}(X_\zeta)\}$, and
- $\lambda(B \cup C) = \mu_\kappa(B) + \nu(C)$ for all $B \in \Sigma_\kappa$ and $C \in \mathcal{P}(X_\zeta)$.

A subset A of a topological vector space E is *η-lineable* (*η-spaceable*, respectively), where η is a cardinal number, if $A \cup \{0\}$ contains a (closed, respectively) η-dimensional subspace of E.

Theorem 4.4. *Let ζ, κ be cardinal numbers such that $\kappa > \zeta \geq \mathfrak{c}$, let $(Y, \mathcal{A}, \lambda)$ be the measure space of Definition 4.3 and let $1 \leq q < p$. Then:*

(i) $\dim(L_p(Y)) = \kappa$*;*
(ii) $L_p(Y) - L_q(Y)$ *is ζ-spaceable but is not η-lineable for any cardinal number $\eta > \zeta$.*

In particular, $L_p(Y) - L_q(Y)$ fails to be maximal spaceable.

Proof. (i) By Lemmas 4.1 and 4.2 we have

$$\mathrm{ent}(Y) = \mathrm{ent}(T_\kappa) \times \mathrm{ent}(X_\zeta) = \kappa \times \zeta = \kappa$$

because $c \leq \zeta < \kappa$. Thus $\dim(L_p(Y)) = \mathrm{ent}(Y) = \kappa$ by Theorem 2.3.

(ii) Of course each $0 \neq f \in L_p(Y)$ can be written as $f = f \cdot \chi_{T_\kappa} + f \cdot \chi_{X_\zeta}$. Assume, for a while, that there is a subspace V of dimension greater than ζ inside $(L_p(Y) - L_q(Y)) \cup \{0\}$. In that case, consider the projection

$$\pi \colon V \to L_p(X_\zeta), \qquad \pi(f) = f|_{X_\zeta}.$$

So $V = \bigcup_{g \in \pi(V)} \pi^{-1}(\{g\})$. By Lemma 4.1 we know that $\mathrm{ent}(X_\zeta) = \zeta \geq \mathfrak{c}$, thus $\# L_p(X_\zeta) = \mathrm{ent}(X_\zeta) = \zeta$ by Corollary 2.4. The dimension of V being greater than ζ implies that the cardinality of V is also greater than ζ. But V is the union of at most ζ sets of the form $\pi^{-1}(\{g\})$ because

$$\# \pi(V) \leq \# L_p(X_\zeta) = \zeta.$$

So there is $g \in \pi(V)$ such that the set $\pi^{-1}(\{g\})$ has cardinality greater than 1. Then there are $f, h \in V$, $h \neq f$, such that $\pi(f) = g = \pi(h)$, hence $f \cdot \chi_{X_\zeta} = h \cdot \chi_{X_\zeta}$. Finally,

$$0 \neq f - h = f \cdot \chi_{T_\kappa} - h \cdot \chi_{T_\kappa} = (f - h) \cdot \chi_{T_\kappa}.$$

We know that $f - h \in L_p(Y)$, so $(f - h) \cdot \chi_{T_\kappa} \in L_p(T_\kappa)$. Since $\mu_\kappa(T_\kappa) = 1$, by Theorem 3.3(b) we have $L_p(T_\kappa) \subset L_q(T_\kappa)$. So $(f - h) \cdot \chi_{T_\kappa} \in L_q(T_\kappa)$, therefore $f - h = (f - h) \cdot \chi_{T_\kappa} \in L_q(Y)$. But V is a linear subspace, so $f - h \in V$, which is not possible because $V \subset (L_p(Y) - L_q(Y)) \cup \{0\}$. So there is no subspace V of dimension greater than ζ inside $L_p(Y) - L_q(Y)$.

Now let us prove that there is a closed ζ-dimensional subspace of $L_p(Y)$ inside $(L_p(Y) - L_q(Y)) \cup \{0\}$. If $\mathfrak{c} = \zeta$, then $\mathrm{ent}(X_\zeta) = \mathfrak{c}$, so $(L_p(X_\zeta) - L_q(X_\zeta)) \cup \{0\}$ contains a closed $\dim(L_p(X_\zeta))$-dimensional subspace V of $L_p(X_\zeta)$ by Theorem 3.4(a). And if $\mathfrak{c} < \zeta$, then $\mathrm{ent}(X_\zeta) = \zeta > \mathfrak{c}$. Since every set of finite measure in X is a finite set, we conclude that X_ζ is $\mathfrak{c}$-bounded. In this case, $(L_p(X_\zeta) - L_q(X_\zeta)) \cup \{0\}$ contains a closed $\dim(L_p(X_\zeta))$-dimensional subspace V of $L_p(X)$ by Theorem 3.4(b).

Therefore, in any case there is a closed $\dim(L_p(X_\zeta))$-dimensional subspace V of $L_p(X_\zeta)$ inside $(L_p(X_\zeta) - L_q(X_\zeta)) \cup \{0\}$. It is plain that the correspondence

$$f \in L_p(X_\zeta) \mapsto \widetilde{f} \in L_p(Y), \qquad \widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X_\zeta, \\ 0 & \text{if } x \in T_\kappa, \end{cases}$$

is a linear embedding, so $L_p(X_\zeta)$ can be regarded as a closed subspace of $L_p(Y)$. By Theorem 3.3(a) we know that $L_q(X_\zeta) \subseteq L_p(X_\zeta)$, so $L_p(X_\zeta) \cap L_q(Y) = L_q(X_\zeta)$. It follows that

$$L_p(X_\zeta) - L_q(X_\zeta) \subset L_p(Y) - L_q(Y).$$

Therefore there is a copy of V inside $(L_p(Y) - L_q(Y)) \cup \{0\}$. ∎

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