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Matrix subspaces of L_1

by

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Abstract. If $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in L_1 with E r-concave and F p-convex, for some $1 \le r < p \le 2$, then the space of matrices $\{a_{i,j}\}$ with norm $\|\{a_{i,j}\}\|_{E(F)} = \big\|\sum_k \|\sum_l a_{k,l} f_l\| e_k\big\|$ embeds into L_1. This generalizes a recent result of Prochno and Schütt.

1. Introduction. Recall that a basis $E = \{e_i\}_{i=1}^N$ of a finite ($N < \infty$) or infinite ($N = \infty$) dimensional real or complex Banach space is said to be K-*unconditional* if $\|\sum_i a_i e_i\| \le K\|\sum_i b_i e_i\|$ whenever $|a_i| = |b_i|$ for all i. Given a finite or an infinite 1-unconditional basis, $E = \{e_i\}_{i=1}^N$, and a sequence $\{X_i\}_{i=1}^N$ of Banach spaces, denote by $(\sum \bigoplus X_i)_E$ the space of sequences $x = (x_1, x_2, \dots)$, $x_i \in X_i$, for which the norm $\|x\| = \big\|\sum_i \|x_i\| e_i\big\|$ is finite.

If X has a 1-unconditional basis $F = \{f_j\}$ then $(\sum \bigoplus X)_E$ can be represented as a space of matrices $A = \{a_{i,j}\}$, denoted $E(F)$, with norm

$$\|A\|_{E(F)} = \Big\|\sum_i \Big\|\sum_j a_{i,j} f_j\Big\| e_i\Big\|.$$

In [PS], Prochno and Schütt gave a sufficient condition for bases E and F of two Orlicz sequence spaces which ensures that $E(F)$ embeds into L_1. Here we generalize this result by giving a sufficient condition on two unconditional bases E, F which ensures that $E(F)$ embeds into L_1. As we shall see, this condition is also "almost" necessary.

Recall that an unconditional basis $\{e_i\}$ is said to be p-*convex*, resp. r-*concave*, with constant K provided that for all n and all $x_1, \dots, x_n$ in the span of $\{e_i\}$,

$$\Big\|\sum_{i=1}^n (|x_i|^p)^{1/p}\Big\| \le K\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p},$$

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resp.

$$\Big(\sum_{i=1}^n \|x_i\|^r\Big)^{1/r} \le K\Big\|\sum_{i=1}^n (|x_i|^r)^{1/r}\Big\|.$$

Here, for $x = \sum x(j)e_j$ and a positive α, $|x|^\alpha = \sum |x(j)|^\alpha e_j$.

In what follows, L_p will denote $L_p([0,1],\lambda)$, λ being the Lebesgue measure. As is known and quite easy to prove, any 1-unconditional basic sequence in L_p, $1 \le p \le 2$ (resp. $2 \le p < \infty$), is p-convex (resp. p-concave) with constant depending only on p. It is also worthwhile to remind the reader that any K-unconditional basic sequence in L_p is equivalent, with a constant depending only on K, to a 1-unconditional basic sequence in L_p. Indeed, if $\{x_i\}_{i=1}^\infty$ is a K-unconditional basic sequence in L_p and $\{r_i\}_{i=1}^\infty$ is the Rademacher sequence then clearly the sequence $\{x_i(s)r_i(t)\}_{i=1}^\infty$ in $L_p([0,1]^2)$ (which is isometric to L_p) is equivalent to $\{x_i\}_{i=1}^\infty$, with a constant depending only on K. This sequence is clearly also 1-unconditional.

It is due to Maurey [Ma] (see also [Wo, III.H.10]) that, for every $1 \le r < p \le 2$, the span of every p-convex 1-unconditional basic sequence in L_1 embeds into L_p and also embeds into L_r after a change of density; that is, there exists a probability measure μ on $[0,1]$ so that this span is isomorphic (with constants depending on r, p and the p-convexity constant only) to a subspace of $L_r([0,1],\mu)$ on which the $L_r(\mu)$ and the $L_1(\mu)$ norms are equivalent.

We also recall the fact, used in [PS] and due to Bretagnolle and Dacunha-Castelle [BD], that if M is an Orlicz function then the Orlicz space ℓ_M embeds into L_p, $1 \le p \le 2$, if and only if $M(t)/t^p$ is equivalent to an increasing function and $M(t)/t^2$ is equivalent to a decreasing function. This happens if and only if the natural basis of ℓ_M is p-convex and 2-concave.

Theorem 2.1 below states in particular that if E and F are two 1-unconditional basic sequences in L_1 with E r-concave and F p-convex for some $1 \le r < p \le 2$ then $E(F)$ embeds into L_1. When specializing to Orlicz spaces, this implies the main result of [PS].

2. The main result

THEOREM 2.1. *Let $E = \{e_i\}$ be a 1-unconditional basic sequence in L_1 with $\{e_i\}$ r-concave with constant K_1, and let X be a subspace of $L_1([0,1],\mu)$ for some probability measure μ satisfying $\|x\|_{L_r([0,1],\mu)} \le K_2\|x\|_{L_1([0,1],\mu)}$ for some constant K_2 and all $x \in X$. Then $(\sum \bigoplus X)_E$ embeds into L_1 with a constant depending on K_1, K_2 and r only.*

Consequently, if $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in L_1 with E r-concave with constant K_1 and F p-convex with constant K_2, for some $1 \le r < p \le 2$, then the space of matrices $A = \{a_{k,l}\}$

with norm

$$\|A\|_{E(F)} = \Big\| \sum_k \Big\| \sum_l a_{k,l} f_l \Big\| e_k \Big\|$$

embeds into L_1 *with a constant depending only on* r, p, K_1 *and* K_2.

Proof. The p-convexity of $\{f_i\}$ implies that after a change of density the L_1 and L_r norms are equivalent on the span of $\{f_i\}$ (see [Ma]). That is, there exists a probability measure μ on $[0,1]$ and a constant K_3, depending only on r, p and K_2, such that $\|\sum a_j \tilde{f}_j\|_{L_r([0,1],\mu)} \le K_3 \|\sum a_j \tilde{f}_j\|_{L_1([0,1],\mu)}$ for some sequence $\{\tilde{f}_j\}$ 1-equivalent, in the relevant L_1 norm, to $\{f_j\}$, and for all coefficients $\{a_i\}$. Therefore, the second part of the theorem follows from the first part.

To prove the first part, in $L_1([0,1] \times [0,1], \lambda \times \mu)$ consider the tensor product of the span of $\{e_i\}$ and of X, that is, the space of all functions of the form $\sum_i e_i \otimes x_i$, $x_i \in X$ for all i, where $e_i \otimes x_i(s,t) = e_i(s) x_i(t)$. Then, by the 1-unconditionality of $\{e_i\}$ and the triangle inequality,

$$\begin{aligned}\Big\| \sum_i e_i \otimes x_i \Big\|_1 &= \int \Big\| \sum_i |x_i(t)| e_i \Big\|_{L_1([0,1],\lambda)} \, d\mu(t) \\ &\ge \Big\| \sum_i \Big(\int |x_i(t)| \, d\mu(t) \Big) e_i \Big\|_{L_1([0,1],\lambda)} = \Big\| \sum_i \|x_i\| e_i \Big\|.\end{aligned}$$

On the other hand, by the 1-unconditionality and the r-concavity with constant K_1 of $\{e_i\}$ (used in integral instead of summation form),

$$\begin{aligned}\Big\| \sum_i e_i \otimes x_i \Big\|_1 &= \iint \Big| \sum_i |x_i(t)| e_i(s) \Big| \, d\lambda(s) \, d\mu(t) \\ &\le \Big(\int \Big(\int \Big| \sum_i |x_i(t)| e_i(s) \Big| \, d\lambda(s) \Big)^r \, d\mu(t) \Big)^{1/r} \\ &= \Big(\int \Big\| \sum_i |x_i(t)| e_i \Big\|^r_{L_1([0,1],\lambda)} \, d\mu(t) \Big)^{1/r} \\ &\le K_1 \Big\| \sum_i \Big(\int |x_i(t)|^r \, d\mu(t) \Big)^{1/r} e_i \Big\|_{L_1([0,1],\lambda)} \\ &\le K_1 K_2 \Big\| \sum_i \int |x_i(t)| \, d\mu(t) e_i \Big\|_{L_1([0,1],\lambda)} = K_1 K_2 \Big\| \sum_i \|x_i\| e_i \Big\|. \quad \blacksquare\end{aligned}$$

As is explained in the introduction, the main result of [PS] follows as corollary.

COROLLARY 2.2. *If* M *and* N *are Orlicz functions such that* $M(t)/t^r$ *is equivalent to a decreasing function,* $N(t)/t^p$ *is equivalent to an increasing function and* $N(t)/t^2$ *is equivalent to a decreasing function, then* $\ell_M(\ell_N)$ *embeds into* L_1.

Remark. The role of L_1 in Theorem 2.1 can easily be replaced with L_s for any $1 \le s \le r$.

Remark. If the bases E and F are infinite, say, and the smallest r such that E is r-concave is larger than the largest p such that F is p-convex, then $E(F)$ does not embed into L_1. This follows from the fact that in this case it is known that the ℓ_r^n uniformly embed as blocks of E, and the ℓ_p^n uniformly embed as blocks of F, for some $r > p$, while it is known that in this case the spaces $\ell_r^n(\ell_p^n)$ do not uniformly embed into L_1.

This still leaves open the case $r = p$, which is not covered in Theorem 2.1:

- *If E and F are two 1-unconditional basic sequences in L_1 with E r-concave and F r-convex, does $E(F)$ embed into L_1?*

In the case that E is an Orlicz space the problem above has a positive solution. We only sketch it. By the factorization theorem of Maurey mentioned above ([Wo, III.H.10] is a good place to read it), and a simple compactness argument (to pass from the finite to the infinite case), it is enough to consider the case that F is the ℓ_r unit vector basis. If the basis of ℓ_M is r-concave, then the $2/r$-convexification of ℓ_M (which is the space with norm $\|\{|a_i|^{2/r}\}\|_{\ell_M}^{r/2}$) embeds into $L_{2/r}$. This is again an Orlicz space, say, $\ell_{\tilde{M}}$. Now, tensoring with the Rademacher sequence (or a standard Gaussian sequence) we find that $\ell_{\tilde{M}}(\ell_2)$ embeds into $L_{2/r}$. We now want to $2/r$ concavify back, staying in L_1, so as to ensure that $\ell_M(\ell_r)$ embeds into L_1. This is known to be possible (and is buried somewhere in [MS]): If $\{x_i\}$ is a 1-unconditional basic sequence in L_s, $1 < s \le 2$, then its s-concavification (which is the space with norm $\|\{|a_i|^{1/s}\}\|_{\ell_M}^{s}$) embeds into L_1. Indeed, let $\{f_i\}$ be a sequence of independent $2/s$ symmetric stable random variables normalized in L_1 and consider the span of the sequence $\{f_i \otimes |x_i|^s\}$ in L_1.

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References

[BD] J. Bretagnolle et D. Dacunha-Castelle, *Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p*, Ann. Sci. École Norm. Sup. (4) 2 (1969), 437–480.

[Ma] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Astérisque 11 (1974).

[MS] B. Maurey and G. Schechtman, *Some remarks on symmetric basic sequences in L_1*, Compos. Math. 38 (1979), 67–76.

[PS] J. Prochno and C. Schütt, *Combinatorial inequalities and subspaces of L_1*, Studia Math. 211 (2012), 21–39.

[Wo] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Stud. Adv. Math. 25, Cambridge Univ. Press, Cambridge, 1991.

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