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Estimates for projections in Banach spaces and existence of direct complements

by

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Abstract. Let W and L be complementary subspaces of a Banach space X and let $P(W,L)$ denote the projection on W along L. We obtain a sufficient condition for a subspace M of X to be complementary to W and we derive estimates for the norm of $P(W,L) - P(W,M)$.

1. Introduction. The starting point of our investigation is the following result which combines Theorem 5.2 of Berkson [1] with a characterization of minimal angles of Gurariĭ [2, p. 200].

THEOREM 1.1. *Let X be a Banach space and let W, L, and M be closed subspaces of X. Assume*

$$X = W \oplus L$$

and $L \neq 0$. Let $P(W,L)$ be the projection on W along L and let $\theta(L,M)$ denote the gap between L and M. If

$$\max\{\|P(L,W)\|, \|P(W,L)\|\}\theta(L,M) < 1 \tag{1.1}$$

then M is also complementary to W, i.e., $X = W \oplus M$, and

$$\|P(L,W) - P(M,W)\| \leq \frac{\|P(L,W)\|\theta(L,M)}{1 - \|P(L,W)\|\theta(L,M)} \cdot \|P(W,L)\|. \tag{1.2}$$

In this note we want to prove a result which contains Theorem 1.1 as a special case. We shall obtain a sufficient condition for M to be complementary to W that is weaker than (1.1) and our estimate for $\|P(L,W) - P(M,W)\|$ will be sharper than (1.2).

We shall use the following notation. Let U and V be closed nonzero subspaces of X and define

$$\delta(U,V) = \sup\{\operatorname{dist}(u,V);\ u \in U,\ \|u\| = 1\}. \tag{1.3}$$

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Then $\theta(U,V) = \max\{\delta(U,V), \delta(V,U)\}$ is the *gap* between U and V (see e.g. [3, p. 197]). The range of a linear operator T will be denoted by $\mathcal{R}(T)$.

2. Auxiliary results. In this section X is a real or complex Banach space with a direct sum decomposition $X = W \oplus L$, and W, $L \neq 0$, and M are closed subspaces of X. For our purposes the map $P(L,W)_{|M} : M \to L$ will be important.

LEMMA 2.1. (a) *We have*

$$X = W \oplus M \tag{2.1}$$

if and only if the map $P(L,W)_{|M} : M \to L$ has an inverse. Suppose

$$Q = (P(L,W)_{|M})^{-1}$$

exists. Then Q is bounded and

$$Q = P(M,W)_{|L}. \tag{2.2}$$

(b) *Assume*

$$\mu := \|P(W,L)_{|M}\| < 1. \tag{2.3}$$

Then the map $P(L,W)_{|M} : M \to L$ is one-to-one and its range is closed.

(c) *If $X = W \oplus M$ and $\mu < 1$ then*

$$\|Q\| < \frac{1}{1-\mu}. \tag{2.4}$$

Proof. (a) Suppose $Q : L \to M$ satisfies

$$QP(L,W)_{|M} = I_M \quad \text{and} \quad P(L,W)_{|M}Q = I_L. \tag{2.5}$$

By the Open Mapping Theorem, Q is continuous, and

$$(Q \cdot P(L,W))^2 = Q \cdot \underbrace{P(L,W) \cdot Q}_{I_L} \cdot P(L,W) = Q \cdot P(L,W)$$

shows that $Q\,P(L,W)$ is a projection on M along W, which proves (2.1) and (2.2). Conversely, if (2.1) holds then $Q = P(M,W)_{|L}$ is well defined and satisfies (2.5).

(b) From (2.3) we obtain

$$\|P(L,W)_{|M}x\| = \|x - P(W,L)_{|M}x\| \geq (1-\mu)\|x\| \quad \text{for all } x \in M. \tag{2.6}$$

Hence the restriction $P(L,W)_{|M}$ is bounded from below, which implies injectivity and closed range.

(c) We consider (2.6) with $x = Qy$, $y \in L$. Then $\|y\| \geq (1-\mu)\|Qy\|$, which yields (2.4). ■

The decomposition $X = W \oplus L$ implies $X^* = W^\perp \oplus L^\perp$. Thus, in the next lemma, the projection $P(L^\perp, W^\perp)$ is well defined.

LEMMA 2.2. *The map* $P(L^\perp, W^\perp)_{|M^\perp} : M^\perp \to L^\perp$ *is one-to-one if and only if*

$$\overline{\mathcal{R}(P(L,W)_{|M})} = L. \tag{2.7}$$

Proof. Injectivity of the map $P(L^\perp, W^\perp)_{|M^\perp}$ is clearly equivalent to

$$(M+W)^\perp = M^\perp \cap W^\perp = 0.$$

On the other hand we have $\mathcal{R}(P(L,W)_{|M}) = (M+W)\cap L$. Therefore (2.7) is equivalent to

$$\overline{(M+W)\cap L} = L. \tag{2.8}$$

Let us show that (2.8) holds if and only if

$$\overline{M+W} = X. \tag{2.9}$$

Suppose (2.8) holds. Consider $x \in X$ with $x = l + w$, $l \in L$, $w \in W$. Because of (2.8) we have $l = \lim s_\nu$, $s_\nu \in L$, and

$$s_\nu = m_\nu + w_\nu, \quad m_\nu \in M,\ w_\nu \in W. \tag{2.10}$$

Set $x_\nu = s_\nu + w$. Then $x = \lim x_\nu$ and $x_\nu \in M + W$. Conversely, assume now (2.9). For $l \in L$ this implies $l = \lim s_\nu$ with s_ν as in (2.10). We also have $s_\nu = l_\nu + \widetilde{w}_\nu$, $l_\nu \in L$, $\widetilde{w}_\nu \in W$. Hence, by continuity of $P(W,L)$, we have $\lim \widetilde{w}_\nu = 0$. Therefore $\lim l_\nu = l$, and $l_\nu \in (M+W)\cap L$. As (2.9) is equivalent to $(M+W)^\perp = 0$ the proof is complete. ∎

LEMMA 2.3. *Let* δ *be as defined in* (1.3). *Then*

$$\|P(W,L)_{|M}\| \le \|P(W,L)\|\delta(M,L) \tag{2.11}$$

and

$$\|P(W^\perp, L^\perp)_{|M^\perp}\| \le \|P(L,W)\|\delta(L,M). \tag{2.12}$$

Proof. If $x \in M$, $\|x\| = 1$, and $\varepsilon > 0$ then there exists a $y \in L$ such that $\|x - y\| < \delta(M,L) + \varepsilon$. Then $P(W,L)_{|M}x = P(W,L)(x-y)$ yields

$$\|P(W,L)_{|M}x\| < \|P(W,L)\|(\delta(M,L)+\varepsilon),$$

which implies (2.11). Note that $\|P(W^\perp, L^\perp)\| = \|P(L,W)^*\| = \|P(L,W)\|$ and $\delta(M^\perp, L^\perp) = \delta(L,M)$. Hence (2.12) follows from (2.11). ∎

The following example deals with Berkson's condition (1.1) in Theorem 1.1. It shows that the conditions $\mu = \|P(W,L)_{|M}\| < 1$ and $\mu^* = \|P(W^\perp, L^\perp)_{|M^\perp}\| < 1$ in part (b) of Theorem 3.1 need not imply (1.1).

Consider the Euclidean space $\mathbb{R}^4$. Set

$$\text{(2.13)}\quad L = \operatorname{Im}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad W = \operatorname{Im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \operatorname{Im}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \hline \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$

To compute μ, μ^*, and the quantities appearing in (1.1) we use the set-up of [4]. If $X = \mathbb{R}^n$, and

$$L = \operatorname{Im}\begin{pmatrix} I_s \\ 0 \end{pmatrix}, \qquad W = \operatorname{Im}\begin{pmatrix} W_{12} \\ I_{n-s} \end{pmatrix},$$

and

$$M = \operatorname{Im}\begin{pmatrix} M_1 \\ M_{21} \end{pmatrix}, \qquad M_1^T M_1 + M_{21}^T M_{21} = I_s,$$

$$M^\perp = \operatorname{Im}\begin{pmatrix} M_{12} \\ M_2 \end{pmatrix}, \qquad M_{12}^T M_{12} + M_2^T M_2 = I_{n-s},$$

then $\theta(L, M) = \|M_{21}\| = \|M_{12}\|$, and

$$\|P(W, L)\|^2 = \|P(L, W)\|^2 = \|I + W_{12}^T W_{12}\|$$

and

$$\mu^2 = \|M_{21}^T (I + W_{12}^T W_{12}) M_{21}\|, \qquad \mu^{*2} = \|M_{12}^T (I + W_{12} W_{12}^T) M_{12}\|.$$

In (2.13) we have

$$W_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{21} = -M_{12} = \begin{pmatrix} \frac{\tau_1}{\sqrt{1+\tau_1^2}} & 0 \\ 0 & \frac{\tau_2}{\sqrt{1+\tau_2^2}} \end{pmatrix}.$$

Hence

$$\theta(L, M)^2 = \max\left\{ \frac{\tau_1^2}{1+\tau_1^2}, \frac{\tau_2^2}{1+\tau_2^2} \right\}, \qquad \|P(W, L)\|^2 = 2,$$

and

$$\mu^2 = \mu^{*2} = \max\left\{ \frac{2\tau_1^2}{1+\tau_1^2}, \frac{\tau_2^2}{1+\tau_2^2} \right\}.$$

If we choose τ_1, τ_2 such that $0 < \tau_1 < 1 < \tau_2$ then

$$\frac{2\tau_1^2}{1+\tau_1^2} < 1 < \frac{2\tau_2^2}{1+\tau_2^2},$$

which implies $\mu = \mu^* < 1$ and

$$\max\{\|P(W, L)\|, \|P(W, L)\|\}\theta(L, M) = \|P(W, L)\|\theta(L, M) = \frac{\sqrt{2}\,\tau_2}{\sqrt{1+\tau_2^2}} > 1.$$

3. The main result

THEOREM 3.1. *Let W, L, and M be closed subspaces of a Banach space X. Assume $X = W \oplus L$ and $L \neq 0$. Define*

$$\mu = \|P(W,L)_{|M}\| \quad \text{and} \quad \mu^* = \|P(W^\perp, L^\perp)_{|M^\perp}\|.$$

(a) *If $X = W \oplus M$, then $\mu < 1$ implies*

$$\|P(W,L) - P(W,M)\| \le \frac{\mu}{1-\mu}\,\|P(L,W)\|, \tag{3.1}$$

and $\mu^* < 1$ implies

$$\|P(L,W) - P(M,W)\| \le \frac{\mu^*}{1-\mu^*}\,\|P(W,L)\|. \tag{3.2}$$

(b) *If $\mu < 1$ and $\mu^* < 1$, then $X = W \oplus M$.*

Proof. (a) Since both L and M are complementary to W we can use Q in (2.2). We have $P(M,W) = QP(L,W)$ and

$$\begin{aligned} P(W,L) - P(W,M) &= P(W,L) - P(W,L)P(W,M) = P(W,L)P(M,W) \\ &= P(W,L)_{|M} QP(L,W), \end{aligned}$$

and the estimate (3.1) follows from (2.4). To prove (3.2) observe that $X^* = W^\perp \oplus L^\perp$ and $X^* = W^\perp \oplus M^\perp$ imply an inequality corresponding to (3.1), namely

$$\|P(W^\perp, L^\perp) - P(W^\perp, M^\perp)\| \le \frac{\mu^*}{1-\mu^*}\,\|P(L^\perp, W^\perp)\|,$$

which is equivalent to (3.2).

(b) If $\mu^* < 1$ then Lemma 2.1(b) shows that the map $P(L^\perp, W^\perp)_{|M^\perp} : M^\perp \to L^\perp$ is one-to-one. According to Lemma 2.2 this is equivalent to $\overline{\mathcal{R}(P(L,W)_{|M})} = L$. Similarly, $\mu < 1$ implies that $P(L,W)_{|M}$ is one-to-one and has closed range. Hence we have $\mathcal{R}(P(L,W)_{|M}) = L$, and $P(L,W)_{|M} : M \to L$ is a bijection. Thus Lemma 2.1(a) yields $X = W \oplus M$. ∎

COROLLARY 3.2. *Let W, L, M be as in Theorem 3.1. Assume $\mu < 1$ and*

$$\dim M = \dim L < \infty. \tag{3.3}$$

Then M and W are complementary subspaces and (3.1) holds.

Proof. According to Lemma 2.1(b) the map $P(L,W)_{|M} : M \to L$ is one-to-one, and it follows from (3.3) that it has an inverse. Thus, again by Lemma 2.1, we obtain $X = W \oplus M$. ∎

We remark that we cannot discard the condition $\mu^* < 1$ from Theorem 3.1(b). If a subspace M is topologically isomorphic to L, but only the condition $\mu < 1$ is satisfied, then M need not be complementary to W. Consider the following example. Take $X = \ell_2$, $W = \langle e_1 \rangle$, $L = \langle e_1 \rangle^\perp =$

$\{(x_n) \in \ell_2;\, x_1 = 0\}$. Choose $M = \{(x_n) \in \ell_2;\, x_1 = x_2 = 0\}$. Then $M \subseteq L$, $P(W,L)_{|M} = 0$, and M is topologically isomorphic to L, but $X \neq W \oplus M$.

The following proof shows that Theorem 1.1 is an immediate consequence of Theorem 3.1.

Proof of Theorem 1.1. Condition (1.1) implies

$$\|P(W,L)\|\delta(M,L) < 1 \quad \text{and} \quad \|P(L,W)\|\delta(L,M) < 1. \tag{3.4}$$

Because of Lemma 2.3 the inequalities (3.4) yield $\mu < 1$ and $\mu^* < 1$. Hence by Theorem 3.1 the subspace M is complementary to W. Moreover, since the function $f(t) = t(1-t)^{-1}$ is increasing on $[0,1)$ we obtain the estimate (1.2) from (2.12) and (3.2). ■

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