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On a construction of majorizing measures on subsets of $\mathbb{R}^n$ with special metrics

by

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Abstract. We consider processes X_t with values in $L_p(\Omega, \mathcal{F}, P)$ and "time" index t in a subset A of the unit cube. A natural condition of boundedness of increments is assumed. We give a full characterization of the domains A for which all such processes are a.e. continuous. We use the notion of Talagrand's majorizing measure as well as geometrical Paszkiewicz-type characteristics of the set A. A majorizing measure is constructed.

Introduction. The aim of this paper is to present some conditions of almost sure convergence of series and continuity of processes with "bounded increments" in L^p spaces, for $p > 1$. For a fixed probability space we will say that a process $(X_t)_{t\in T}$ on a metric space (T, d) *has bounded increments* if

$$(\star) \qquad \forall_{s,t\in T} \quad \|X_t - X_s\|_p \le d(t, s).$$

In this study we will restrict ourselves to T being subsets of the unit cube in Euclidean spaces. More precisely, $T \subset [0, 1]^\eta$, $\eta \ge 1$, and $d(s, t) = d_q(s, t) = \sqrt[q]{\max_{1\le i\le\eta} |s_i - t_i|}$ for $s = (s_1, \ldots, s_\eta)$, $t = (t_1, \ldots, t_\eta)$ in T. It is assumed that $p, q > 1$ are arbitrary numbers. We give conditions on T which are sufficient for a.e. continuity of X satisfying $(\star)$. These conditions are also necessary. A natural interpretation for a.e. convergence of series in L^p is also described (cf. Corollary 1).

Description of a.e. convergent sequences is a problem which has been addressed for decades. It seems that Paszkiewicz ([4], [5]) was the first to give a complete characterization of a.e. convergent sequences in L^2. The important result of Patrice Assouad (see e.g. [6, Theorem 2.3]) concerns a more general class of series or processes (not necessarily determined by a sequence), but its proof is nonconstructive and the condition it formulates is

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not expressed in terms of the geometrical properties of the space T. In fact, it uses the dual to the space of continuous functions on $T \times T \setminus \{(x,x)\}_{x\in T}$.

We stress that Paszkiewicz characterized orthogonal series in L^2 which are a.e. convergent, whereas a conclusion which could be drawn from our paper is a characterization of a.e. convergent series $\sum \Phi_n$ satisfying a condition $\|\sum_{n=m}^k \Phi_n\|_p^q \le f(k,m)$ for some additive function of interval in $\mathbb{N}$. Our reasonings are considerably simpler than those in [5] (particularly constructions in [5, Sections 3–7]).

The classical paper of Talagrand [6] investigates conditions of a.s. continuity of processes with bounded increments in general Orlicz spaces by means of existence of so called majorizing measures. The concept of majorizing measure has been extensively used in the literature, most notably, to characterize continuity and boundedness of Gaussian process (e.g. [7]). For short yet exhaustive reviews on majorizing measures see [5], [6], [1] or [8].

We should mention that while the author was working on this paper, W. Bednorz [2] presented a proof of existence of majorizing measures for a wide class of metric spaces: roughly speaking, for spaces whose metric is a root of another metric. Nevertheless, we believe our reasoning is still interesting since, using Paszkiewicz-type operators, it is constructive and it clarifies, in an elementary way, the case of the space $\mathbb{R}^\eta$.

The first section of this paper presents a fundamental result (Theorem 1) which connects three quantities: the norm of the maximal function of a process with bounded increments on a finite set A of time instances, Talagrand's majorizing measure characteristic and the values of Paszkiewicz's characteristics of the set A.

Theorem 2 in Section 2 provides an explicit relation between the characteristics under study and a majorizing measure on an arbitrary closed subset of $[0,1]^\eta$. Theorem 3 provides a construction of a.e. divergent process. To avoid excessive difficulties we perform the construction in the case of $\eta = 1$; nonetheless, the general idea based on, roughly speaking, the Borel–Cantelli lemma is worth highlighting.

1. Upper bounds for processes on finite subsets of a finite-dimensional cube. In this section we will be considering finite subsets A of the η-dimensional cube $[0,1]^\eta$, for some fixed $\eta \ge 1$. We will investigate bounds for the maximal function of an arbitrary process $(X_t)_{t\in A}$ with bounded increments (see Definition 1 below).

Throughout the paper we fix $p, q > 1$ and set

$$d_q(x,y) = \sqrt[q]{\max_{1\le i\le \eta} |x_i - y_i|}, \qquad x, y \in \mathbb{R}^\eta, \tag{1}$$

with $x = (x_1, \ldots, x_\eta)$ and $y = (y_1, \ldots, y_\eta)$.

DEFINITION 1. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space. For a set $T \subset [0,1]^\eta$ we say that a process $X = (X_t)_{t\in T} \subset L^p(\Omega)$ *has bounded increments*, written $X \in \mathcal{BI}(T)$, if for all $t, s \in T$,

$$\|X_t - X_s\|_p \le d_q(t,s).$$

We will also use the notion of majorizing measure (cf. Talagrand [6]). More precisely, we will utilize the definition used in [5].

DEFINITION 2. A finite Borel measure m concentrated on a set $T \subset [0,1]^\eta$ is a *majorizing measure on* T if

$$\sup_{t\in T} \int_0^{\operatorname{diam} T} \frac{d\varepsilon}{\sqrt[p]{m(B(t,\varepsilon))}} \le 1,$$

where $B(t,\varepsilon)$ is an open ball and diam is diameter with respect to the metric d_q.

For a finite set $A \subset [0,1]^\eta$ we define

$$\mathcal{S}_p = \sup_{X\in\mathcal{BI}(A)} \big\| \max_{s,t\in A} |X_t - X_s| \big\|_p, \qquad \mathcal{S} = \sup_{X\in\mathcal{BI}(A)} \big\| \max_{s,t\in A} |X_t - X_s| \big\|_1, \tag{2}$$

with $\mathcal{BI}(A)$ as in Definition 1, and

$$\mathcal{M}_p = \inf\{ \sqrt[p]{m(A)} \colon m \text{ is a majorizing measure on } A\}. \tag{3}$$

The aim of this section is to provide a comparison of the following characteristics of a finite set $A \subset [0,1]^\eta$:

$$\mathcal{S}, \quad \mathcal{S}_p, \quad \mathcal{M}_p \quad \text{and} \quad \mathcal{V}_p,$$

where

$$\mathcal{V}_p = \|V_0 V_1 \ldots V_{i_A} 0\|_p; \tag{4}$$

the integer i_A and the sequence of operators $V_i \colon L^p([0,1]^\eta) \to L^p([0,1]^\eta)$, $i \ge 0$ (to be defined later in this section, cf. (9)) depend on the set A.

It is clear that all the above mentioned characteristics also depend on the choice of the metric (cf. (1)), thus on the number q. For ease of notation the additional index is omitted.

Once we obtain the desired comparison of characteristics, we will be able to extend a result of Paszkiewicz to the case of the space L^p, $p > 1$.

For any $i \ge 0$ and $0 \le n < 2^i - 1$ let $P_n^i = [n2^{-i}, (n+1)2^{-i})$ and $P_{2^i-1}^i = [1 - 2^{-i}, 1]$. We write $\mathbf{n} = (n^1, \ldots, n^\eta)$ for multiindices $\mathbf{n} \in \mathbb{N}^\eta$. For any $\mathbf{n} \in \{0, \ldots, 2^i - 1\}^\eta$ denote

$$\delta_{\mathbf{n}}^i = \prod_{k=1}^{\eta} P_{n^k}^i. \tag{5}$$

Then we can define

$$\Delta_i = \Delta_i^A = \bigcup_{\mathbf{n}\colon \delta_{\mathbf{n}}^i \cap A \neq \emptyset} \delta_{\mathbf{n}}^i. \tag{6}$$

Moreover, we will consider the σ-fields

$$\mathcal{F}_i = \sigma(\delta_{\mathbf{n}}^i \colon \mathbf{n} \in \{0, \ldots, 2^i - 1\}^\eta). \tag{7}$$

We also introduce the constant

$$\tau = 2^{\eta/p - 1/q}. \tag{8}$$

For any $h \in L^p([0,1]^\eta)$ we will use the conditional L^p-norm defined by

$$\|h\|_{p,i} = (\mathbb{E}(|h|^p \mid \mathcal{F}_i))^{1/p}.$$

This notation may seem unusual but it will prove convenient. For any integer $i \geq 0$ let us define operators

$$V_i^A h = \tau^i \mathbb{I}_{\Delta_i} + \|h\|_{p,i} \quad \text{and} \quad W_i h = \frac{\|h\|_{p,i} + \tau^i}{\|h\|_{p,i}} \cdot h \tag{9}$$

for $h \in L^p([0,1]^\eta)$, with an agreement that $1/0 := 0$. We omit the superscript A whenever it does not lead to misunderstanding.

Another crucial definition assigns to a finite set $A \subset [0,1]^\eta$ the smallest integer i_A for which the family of sets $\mathcal{F}_{i_A}$ separates points of A. Namely, for a finite set A we define (cf. (1), (5), (6))

$$i_A = \min\{k \geq 0 \colon \forall_{\delta_{\mathbf{n}}^k, \delta_{\mathbf{m}}^k \subset \Delta_k} d_q(\delta_{\mathbf{n}}^k, \delta_{\mathbf{m}}^k) > 0\}. \tag{10}$$

Similarly to (4) we define another characteristic of a finite set A, related to the operations W_i, $i \geq 0$, i.e.

$$\mathcal{W}_p = \|W_0 \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_p. \tag{11}$$

THEOREM 1. *For any finite set $A \subset [0,1]^\eta$ and any probability space $(\Omega, \mathcal{T}, \mathbb{P})$,*

$$\mathcal{S} = \mathcal{S}_p \leq 2^{\eta+1}\mathcal{V}_p, \quad 2^{-p/q}\mathcal{M}_p \leq \mathcal{V}_p = \mathcal{W}_p, \quad \mathcal{V}_p \leq \gamma\mathcal{S}_p + \gamma, \quad \mathcal{S}_p \leq 12\sqrt{3}\mathcal{M}_p,$$

where $\gamma = 3^\eta 2^{(q+1)/q}\{(\sqrt[q]{2} - 1)^{-1} + 4(2 - \sqrt[q]{2})^{-1}\}$.

Proof. The relation $\mathcal{S} = \mathcal{S}_p$ and the last inequality are known in the general setting (cf. [1, Proposition 2.1]). The rest of the proof will be accomplished in the following four steps.

STEP 1. We show that

$$\mathcal{S}_p \leq 2^{\eta+1}\mathcal{V}_p. \tag{12}$$

Let $(X_t)_{t \in A}$ be a process with bounded increments. For $i \leq i_A$ and $\mathbf{n} \in \{0, \ldots, 2^i - 1\}^\eta$ such that $\delta_{\mathbf{n}}^i \cap A \neq \emptyset$ fix an element $t_{\mathbf{n}}^i \in \delta_{\mathbf{n}}^i \cap A$. Then, for any $\mathbf{n} \in \{0, \ldots, 2^i - 1\}^\eta$, set (cf. (5))

$$M_{\mathbf{n}}^i = \max_{t \in \delta_{\mathbf{n}}^i \cap A} |X_t - X_{t_{\mathbf{n}}^i}| \quad \text{if } \delta_{\mathbf{n}}^i \cap A \neq \emptyset; \quad M_{\mathbf{n}}^i = 0 \quad \text{otherwise.}$$

By (10), $\|M_{\mathbf{n}}^{i_A}\|_p = 0$ for all $\mathbf{n} \in \{0, \ldots, 2^{i_A} - 1\}^\eta$. Let us assume that for $\mathbf{m} \in \{0, \ldots, 2^{i+1} - 1\}^\eta$,

$$\|M_{\mathbf{m}}^{i+1}\|_p \le 2^\eta \cdot \|\mathbb{I}_{\delta_{\mathbf{m}}^{i+1}} V_{i+1} \ldots V_{i_A} 0\|_p.$$

By Definition 1 and the inductive assumption, for any $\mathbf{n} \in \{0, \ldots, 2^i - 1\}^\eta$ we have

$$\begin{aligned}
\|M_{\mathbf{n}}^i\|_p &\le \Big\| \max_{2\mathbf{n} \le \mathbf{m} \le 2\mathbf{n}+1} |X_{t_{\mathbf{m}}^{i+1}} - X_{t_{\mathbf{n}}^i}| \Big\|_p + \| \max_{2\mathbf{n} \le \mathbf{m} \le 2\mathbf{n}+1} M_{\mathbf{m}}^{i+1} \|_p \\
&\le 2^\eta \operatorname{diam}(\delta_{\mathbf{n}}^i) + \Big(\sum_{2\mathbf{n} \le \mathbf{m} \le 2\mathbf{n}+1} \|M_{\mathbf{m}}^{i+1}\|_p^p \Big)^{1/p} \\
&\le 2^\eta 2^{-i/q} + 2^\eta \cdot \Big(\sum_{2\mathbf{n} \le \mathbf{m} \le 2\mathbf{n}+1} \|\mathbb{I}_{\delta_{\mathbf{m}}^{i+1}} V_{i+1} \ldots V_{i_A}\|_p^p \Big)^{1/p} \\
&= 2^\eta (2^{-i\eta/p} \tau^i + \|\mathbb{I}_{\delta_{\mathbf{n}}^i} V_{i+1} \ldots V_{i_A}\|_p) = 2^\eta \|\mathbb{I}_{\delta_{\mathbf{n}}^i} V_i \ldots V_{i_A} 0\|_p,
\end{aligned}$$

where we put $|X_{t_{\mathbf{m}}^{i+1}} - X_{t_{\mathbf{n}}^i}| = 0$ whenever $t_{\mathbf{m}}^{i+1}$ is not defined.

Since $\max_{s,t \in A} |X_t - X_s| \le 2 \cdot M_{\mathbf{0}}^0 \le 2^{\eta+1} \mathcal{V}_p$, we have the inequality (12).

Step 2. To show that $\mathcal{W}_p = \mathcal{V}_p$ we will use a simple downward induction. Let us assume that for an $i < i_A$ (cf. (10)) we have

$$\|W_{i+1} \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i+1} = V_{i+1} \ldots V_{i_A} 0, \tag{13}$$

and that

$$\operatorname{supp} \|W_{i+1} \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i+1} = \Delta_{i+1}, \tag{14}$$

which is true for $i + 1 = i_A$ since $\tau^{i_A} \mathbb{I}_{\Delta_{i_A}} = V_{i_A} 0$. Then the inductive step

$$\begin{aligned}
V_i V_{i+1} \ldots V_{i_A} 0 &= \tau^i \mathbb{I}_{\Delta_i} + \|V_{i+1} \ldots V_{i_A} 0\|_{p,i} \\
&= \tau^i \mathbb{I}_{\Delta_i} + \|W_{i+1} \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i} = \|W_i \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i}
\end{aligned}$$

follows from (13). Moreover, by (14),

$$\operatorname{supp} \|W_i \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i} = \operatorname{supp} \|W_{i+1} \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_{p,i} = \Delta_i.$$

By induction, (13) is valid for $i + 1 = 0$.

Step 3. In order to show that $\mathcal{M}_p \le \sqrt[q]{2} \cdot \mathcal{W}_p$ we will construct a majorizing measure with total mass $2^{p/q} \|W_0 \ldots W_{i_A - 1}(\tau^{i_A} \mathbb{I}_{\Delta_{i_A}})\|_p^p$.

More precisely, let

$$d\tilde{m}_{k,j} = [W_k \ldots W_{j-1}(\tau^j \mathbb{I}_{\Delta_j})]^p d\lambda \tag{15}$$

with λ being the Lebesgue measure, $0 \le k \le j$, $\Delta_j = \Delta_j^A$ and (cf. (6))

$$m_{k,j}(E) = \tilde{m}_{k,j}(\Delta_j^E) \quad \text{for } E \subset A. \tag{16}$$

By (10), $m_{k,j}$ is a well defined measure for $j \geq i_A$. In particular, $m_{j,j}(E) = 2^{-j\eta}\tau^{jp}\,\mathrm{card}(E)$ for any $E \subset A$. Each measure $2^{p/q} \cdot m_{0,j}$ for $j \geq i_A$ turns out to be a majorizing measure on A.

First we show an auxiliary relation between $m_{k,j}$ and $m_{k+1,j}$. Taking an arbitrary $E \subset \delta^k_{\mathbf{n}}$ for some $k < j$, $j \geq i_A$, $\mathbf{n} \in \{0, \ldots, 2^k - 1\}^\eta$ (cf. (10)), and denoting

$$g = W_{k+1} \ldots W_{j-1}(\tau^j \mathbb{I}_{\Delta_j})$$

for brevity, we have $\Delta^E_j \subset \delta^k_{\mathbf{n}}$ and (cf. (9), (15), (16))

$$\begin{aligned} (17) \qquad m_{k,j}(E) &= \|W_k g \cdot \mathbb{I}_{\Delta^E_j}\|^p_p = \left\| \frac{\|g\|_{p,k} + \tau^k}{\|g\|_{p,k}} g \mathbb{I}_{\Delta^E_j} \right\|^p_p \\ &= \frac{(\|\mathbb{I}_{\delta^k_{\mathbf{n}}} g\|_p + 2^{-k\eta/p}\tau^k)^p}{\|\mathbb{I}_{\delta^k_{\mathbf{n}}} g\|^p_p} \|g \mathbb{I}_{\Delta^E_j}\|^p_p = \frac{(\sqrt[p]{m_{k+1,j}(\delta^k_{\mathbf{n}})} + 2^{-k/q})^p}{m_{k+1,j}(\delta^k_{\mathbf{n}})} m_{k+1,j}(E). \end{aligned}$$

To prove that $2^{p/q} \cdot m_{0,j}$ for $j \geq i_A$ is a majorizing measure on A, it is enough to show that

$$(18) \qquad \forall_{j \geq i_A} \forall_{1 \leq k \leq j} \qquad \int_0^{\sqrt[q]{2}\,\mathrm{diam}\,\delta^k_{\mathbf{r}}} [m_{k,j}(B(t,\varepsilon) \cap \delta^k_{\mathbf{r}})]^{-1/p}\, d\varepsilon \leq 2^{1/q}$$

for $t \in \delta^k_{\mathbf{r}} \cap A$. Notice that for $j \geq i_A$, $t \in \delta^j_{\mathbf{r}} \cap A$, $\mathbf{r} \in \{0, \ldots, 2^j - 1\}^\eta$,

$$\int_0^{\sqrt[q]{2}\,\mathrm{diam}\,\delta^j_{\mathbf{r}}} [m_{j,j}(B(t,\varepsilon) \cap \delta^j_{\mathbf{r}})]^{-1/p}\, d\varepsilon = \sqrt[q]{2} \cdot 2^{-j/q} \cdot 2^{j\eta/p}\tau^{-j} = 2^{1/q},$$

thus (18) is valid for $k = j$. Furthermore, assuming (18) to hold for some $j \geq k \geq 1$, for $t \in \delta^k_{\mathbf{r}} \cap A \subset \delta^{k-1}_{\mathbf{n}}$ we have by (17), (18),

$$\begin{aligned} &\int_0^{\sqrt[q]{2}\,\mathrm{diam}\,\delta^{k-1}_{\mathbf{n}}} [m_{k-1,j}(B(t,\varepsilon) \cap \delta^{k-1}_{\mathbf{n}})]^{-1/p}\, d\varepsilon \\ &\qquad \leq \int_0^{\sqrt[q]{2}\,\mathrm{diam}\,\delta^k_{\mathbf{r}}} [m_{k-1,j}(B(t,\varepsilon) \cap \delta^k_{\mathbf{r}})]^{-1/p}\, d\varepsilon \\ &\qquad\quad + \int_{\mathrm{diam}\,\delta^{k-1}_{\mathbf{n}}}^{\sqrt[q]{2}\,\mathrm{diam}\,\delta^{k-1}_{\mathbf{n}}} [m_{k-1,j}(\delta^{k-1}_{\mathbf{n}})]^{-1/p}\, d\varepsilon \end{aligned}$$

$$\leq \frac{\sqrt[p]{m_{k,j}(\delta_{\mathbf{n}}^{k-1})}}{\sqrt[p]{m_{k,j}(\delta_{\mathbf{n}}^{k-1})} + 2^{(-k+1)/q}} \int\limits_0^{\sqrt[q]{2}\,\mathrm{diam}\,\delta_{\mathbf{r}}^k} [m_{k,j}(B(t,\varepsilon)\cap\delta_{\mathbf{r}}^k)]^{-1/p}\,d\varepsilon$$
$$+ \sqrt[q]{2}\cdot(2^{(-k+1)/q} - 2^{-k/q})\frac{1}{\sqrt[p]{m_{k,j}(\delta_{\mathbf{n}}^{k-1})} + 2^{(-k+1)/q}} \leq 2^{1/q}.$$

Thus, the desired relation (18) is proved.

STEP 4. The proof will be complete if we show that $\mathcal{W}_p \leq \gamma\mathcal{S}_p + \gamma$ for some $\gamma > 0$.

Let $A \subset [0,1]^\eta$ be a finite set. Let us introduce some additional notation. For any $t \in A$ and $k \geq 0$ let (cf. (6))

$$\delta^k(t) = \varDelta_k^{\{t\}},$$

which is the $\mathcal{F}_k$-measurable atom containing t. Moreover, for an atom δ in $\mathcal{F}_k$ let

$$\mathcal{N}(\delta) = \{\varDelta_k^{\{t\}} : t \in A \wedge d_q(\delta, \varDelta_k^{\{t\}}) = 0\},$$

which is the set of all $\mathcal{F}_k$-atoms adjacent to δ (including δ).

For $t \in A$, $j \geq 0$ and $\omega \in [0,1]^\eta$ let

$$X_t^j(\omega) = \sum_{k=j}^{i_A} \sum_{\delta\in\mathcal{N}(\delta^k(t))} \left(1 - \frac{d_q^q(t,\delta)}{2^{-k}}\right) \frac{\tau^k W_{k+1}\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\varDelta_{i_A}})(\omega)}{\|W_{k+1}\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\varDelta_{i_A}})\|_{p,k}}\, \mathbb{I}_\delta(\omega);$$

moreover, let $X_t = X_t^0$. An easy computation shows that for any $j \geq 0$ and $t \in A$ we have

$$\|X_t^j\|_p \leq \sum_{k=j}^{i_A} 3^\eta \|\tau^k \mathbb{I}_{\delta^k(t)}\|_p \leq 3^\eta \sum_{k=j}^{i_A} 2^{-k/q} = \frac{3^\eta\sqrt[q]{2}}{\sqrt[q]{2}-1}\cdot 2^{-j/q}. \tag{19}$$

To show that $\gamma^{-1}X_t$ has bounded increments for a suitably large constant $\gamma > 0$ fix $s,t \in A$ and set

$$l = \min\{k \in \mathbb{N} : d_q(\delta^k(s), \delta^k(t)) > 0\}.$$

Notice that $2^{-l/q} \leq d_q(s,t) \leq 4\cdot 2^{-l/q}$. Thus it is enough to show that $\|X_t - X_s\|_p$ is also of the order $2^{-l/q}$.

We have

$$\|X_t - X_s\|_p \leq \|X_t - X_t^l + X_s^l - X_s\|_p + \|X_t^l\|_p + \|X_s^l\|_p.$$

We can also compute that, with

$$\xi_k = \frac{(2\tau)^k W_{k+1}\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\varDelta_{i_A}})}{\|W_{k+1}\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\varDelta_{i_A}})\|_{p,k}}, \qquad k \geq 1,$$

we have

$$
\begin{aligned}
&\|X_t - X_t^l + X_s^l - X_s\|_p \\
&\le \Big\|\sum_{k=0}^{l-1} \sum_{\delta\in\mathcal{N}(\delta^k(t))\cup\mathcal{N}(\delta^k(s))} |(d_q(t,\delta)\wedge 2^{-k/q})^q - (d_q(s,\delta)\wedge 2^{-k/q})^q|\cdot \xi_k \mathbb{I}_\delta\Big\|_p \\
&\le \sum_{k=0}^{l-1} 2\cdot 3^\eta (d_q(t,s))^q \cdot 2^k \|\tau^k\cdot \mathbb{I}_{\delta^k_{\mathbf{0}}}\|_p \le 8\cdot 3^\eta \sum_{k=0}^{l-1} 2^{-l}2^k 2^{-k/q} = \frac{8\cdot 3^\eta \sqrt[q]{2}}{2-\sqrt[q]{2}}\cdot 2^{-l/q}.
\end{aligned}
$$

This, together with (19), implies that we can take

$$\gamma = 3^\eta 2^{(q+1)/q}\{(\sqrt[q]{2}-1)^{-1} + 4(2-\sqrt[q]{2})^{-1}\}.$$

Now, a simple induction will show that

$$\max_{t\in A} X_t^0 \ge W_0 \ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\Delta_{i_A}}).$$

For any $k\ge 0$ and $\delta^k_{\mathbf{n}} \subset \Delta_k$ we have

$$\mathbb{I}_{\delta^k_{\mathbf{n}}} \max_{t\in A\cap\delta^k_{\mathbf{n}}} X_t^k \ge \mathbb{I}_{\delta^k_{\mathbf{n}}} W_k \ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\Delta_{i_A}}). \tag{20}$$

Indeed, this is true for $k = i_A$. If we assume that (20) holds for a $k\le i_A$, then for $\delta^{k-1}_{\mathbf{m}} \subset \Delta_{k-1}$ we have

$$
\begin{aligned}
\mathbb{I}_{\delta^{k-1}_{\mathbf{m}}} \max_{t\in A\cap\delta^{k-1}_{\mathbf{m}}} X_t^{k-1} &\ge \sum_{\delta^k_{\mathbf{n}}\subset\Delta_k\cap\delta^{k-1}_{\mathbf{m}}} \mathbb{I}_{\delta^k_{\mathbf{n}}} \max_{t\in A\cap\delta^k_{\mathbf{n}}} X_t^{k-1} \\
&\ge \sum_{\delta^k_{\mathbf{n}}\subset\Delta_k\cap\delta^{k-1}_{\mathbf{m}}} \mathbb{I}_{\delta^k_{\mathbf{n}}} \max_{t\in A\cap\delta^k_{\mathbf{n}}} \Big[X_t^k + \frac{\tau^k W_k\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\Delta_{i_A}})}{\|W_k\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\Delta_{i_A}})\|_{p,k-1}}\mathbb{I}_{\delta^{k-1}(t)}\Big] \\
&\ge \mathbb{I}_{\delta^{k-1}_{\mathbf{m}}} W_{k-1}\ldots W_{i_A-1}(\tau^{i_A}\mathbb{I}_{\Delta_{i_A}}).
\end{aligned}
$$

Finally, we have

$$\mathcal{S}_p(A) \ge \gamma^{-1}\|\max_{t\in A} X_t\|_p - \gamma^{-1}\|X_{\min A}\|_p \ge \gamma^{-1}\mathcal{W}_p - 1,$$

which completes the proof of Step 4, and the proof of Theorem 1. ∎

2. Processes on infinite sets. In this section we will present a selection of corollaries to Theorem 1. Let us notice that for an arbitrary set $A\subset[0,1]^\eta$ the finiteness of the quantity

$$\bar{\mathcal{V}} = \bar{\mathcal{V}}(A) = \lim_{n\to\infty}\|V_0^A\ldots V_n^A 0\|_p \tag{21}$$

determines whether the set A has the property that every process with bounded increments on A has an a.s. continuous modification.

The first result below shows how the existence of a majorizing measure on an arbitrary closed set can be obtained from Theorem 1. The second gives an example of a construction of an a.e. discontinuous process on an infinite set.

THEOREM 2. *Let $A \subset [0,1]^\eta$ be a closed set for which $\bar{\mathcal{V}} < \infty$. The sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ defined by*

$$d\mu_n = [W_0 \ldots W_n \mathbb{I}_{\Delta_{n+1}}]^p d\lambda,$$

with $\Delta_n = \Delta_n^A$, is weakly convergent and $2^{p/q} \lim_{n\to\infty} \mu_n$ is a majorizing measure on A with total mass $2^{p/q}\bar{\mathcal{V}}^p$.

Proof. Since the family (μ_n) is tight, by Prokhorov's theorem there exists a measure μ which is its cluster point, i.e. $\mu = \lim_{i\to\infty} \mu_{n_i}$ for some $(n_i)_{i\in\mathbb{N}}$.

The measure μ is finite since $\mu([0,1]^\eta) = \bar{\mathcal{V}}^p < \infty$, and it is concentrated on $\bigcap_{n\in\mathbb{N}} \overline{\Delta_n} = A$, with the bar denoting closure. Moreover, by [3, Theorem 29.1] for any $k \in \mathbb{N}$ there exist integers $a_k \ge b_k \ge k$ for which

$$\mu_{a_k}(Z) - k^{-1} \le \limsup_{i\to\infty} \mu_{n_i}(\overline{Z}) \le \mu(\overline{Z}) \tag{22}$$

for every $Z \in \mathcal{F}_{b_k}$ (cf. (7)). The sequence $(b_k)_{k\in\mathbb{N}}$ can be easily chosen to be increasing.

Let t be a point in A. For every natural number n we define the function

$$\varrho_n(\varepsilon) = \mu\Big(\overline{\Delta_n^{\overline{B}(t,\varepsilon)\cap A}} \setminus \overline{B}(t,\varepsilon)\Big),$$

where $\Delta_n^{\overline{B}(t,\varepsilon)\cap A}$ is defined in (6) with $\overline{B}(t,\varepsilon)\cap A$ substituted for A, and $\overline{B}$ denotes the closed ball in $[0,1]^\eta$ with the metric d_q. Obviously $\varrho_n \searrow 0$ pointwise.

Taking (22) into account, for any $k \in \mathbb{N}$ we have the estimate

$$\begin{aligned}
&\int_0^1 \frac{d\varepsilon}{\sqrt[p]{\mu(\overline{B}(t,\varepsilon)\cap A) + \varrho_{b_k}(\varepsilon) + k^{-1}}} \\
&\quad = \int_0^1 \bigg[\mu\Big(\overline{\Delta_{b_k}^{\overline{B}(t,\varepsilon)\cap A}}\Big) + \frac{1}{k}\bigg]^{-1/p} d\varepsilon \le \int_0^1 \mu_{a_k}(\Delta_{b_k}^{\overline{B}(t,\varepsilon)\cap A})^{-1/p}\, d\varepsilon \\
&\quad = \int_0^1 \frac{d\varepsilon}{\|\mathbb{I}_{\Delta_{a_k}^{\overline{B}(t,\varepsilon)\cap A}} \cdot W_0 \ldots W_{a_k-1}\tau^{a_k}(\mathbb{I}_{\Delta_{a_k}^A}\mathbb{I}_{\Delta_{a_k}^E})\|_p} = \int_0^1 \frac{d\varepsilon}{\tilde{m}_{0,a_k}(\Delta_{a_k}^{\overline{B}(t,\varepsilon)\cap A})^{1/p}} \\
&\quad \le \int_0^1 \tilde{m}_{0,a_k}(\Delta_{a_k}^{\overline{B}(t,\varepsilon)\cap E})^{-1/p}\, d\varepsilon \le \int_0^1 m_{0,a_k}(\overline{B}(t,\varepsilon)\cap E)^{-1/p}\, d\varepsilon \le \sqrt[q]{2}.
\end{aligned}$$

Here the set E can be chosen to be any finite subset of A such that $\Delta_{a_k}^A = \Delta_{a_k}^E$, $i_E = a_k$ (cf. (10)). The measures $\tilde{m}_{0,j}$, $m_{0,j}$, $j \ge 0$, are defined as in the proof of Theorem 1, Step 3, on the stipulation that $m_{0,j}$, $j \ge 0$, is concentrated on E, instead of A. The last estimate results from (18).

By the monotone convergence theorem and an easy observation that $\mu_n(A) = \|V_0 \dots V_n 0\|_p^p$, we can take $2^{p/q}\mu$ for a majorizing measure on A.

It remains to justify that $(\mu_n)_{n\in\mathbb{N}}$ has a weak limit. Observe that for every $k \in \mathbb{N}$, $Z \in \mathcal{F}_k$ and $n > k$ we have (cf. (13))

$$\begin{aligned}\mu_n(Z) &= \|\mathbb{I}_Z \cdot W_0 \dots W_n \mathbb{I}_{\Delta_{n+1}}\|_p^p \\ &= \|\mathbb{I}_Z \cdot W_0 \dots W_k \|W_{k+1} \dots W_n \mathbb{I}_{\Delta_{n+1}}\|_{p,k}\|_p^p \\ &= \|\mathbb{I}_Z \cdot W_0 \dots W_k \|V_{k+1} \dots V_{n+1} 0\|_{p,k}\|_p^p.\end{aligned}$$

Since the operator $W_0 \dots W_k$ is continuous and $\|V_{k+1} \dots V_{n+1} 0\|_{p,k}$ increases with n, the limit $\lim_{n\to\infty} \mu_n(Z)$ exists. Moreover, any (uniformly) continuous function f on $[0,1]^\eta$ lies in the L^∞ closure of the set of all $\bigcup_{i\in\mathbb{N}} \mathcal{F}_i$-measurable simple functions, thus by a straightforward argument,

$$\int_{[0,1]^\eta} f\, d\mu_n \to \int_{[0,1]^\eta} f\, d\mu. \ \blacksquare$$

THEOREM 3. *Let $\mathcal{A}$ be an infinite subset of $[0,1]$ with 0 being its only cluster point. If $\bar{\mathcal{V}}(\mathcal{A}) = \infty$ (cf. (21)) then there exists a process on $\mathcal{A} \cup \{0\}$ with bounded increments which is almost surely discontinuous at 0.*

Proof. Basically, the idea is to use Theorem 1 to obtain a sequence of (independent) processes, say X^n, on some finite sets A^n, which have large upper bounds and the sets A^n are (exponentially) close to 0 at the same time. Then we apply the Borel–Cantelli lemma.

Let $\alpha_0 = \beta_0 = 1$ and $\theta > 0$. If α_n, β_n are defined then $\alpha_{n+1}, \beta_{n+1}$ are chosen so that $\alpha_{n+1} < \beta_{n+1} < \frac{1}{2}\alpha_n$ and for $A^{n+1} := [\alpha_{n+1}, \beta_{n+1}] \cap \mathcal{A}$ we have (cf. (10))

$$\|V_0^{A^{n+1}} \dots V_{i_{A^{n+1}}}^{A^{n+1}} 0\|_p \geq \theta.$$

This can be done since for any $\mathcal{B} \subset [0,1]$ and $\mathcal{B}_l := \mathcal{B} \cap [l^{-1}, 1]$ with $l \geq (\max \mathcal{B})^{-1}$ the condition $\lim_{k\to\infty} \|V_1^{\mathcal{B}} \dots V_k^{\mathcal{B}} 0\|_p = \infty$ implies both

$$\lim_{m\to\infty} \|V_0^{\mathcal{B}_m} \dots V_{i_{\mathcal{B}_m}}^{\mathcal{B}_m} 0\|_p = \infty \quad \text{and} \quad \lim_{k\to\infty} \|V_0^{\mathcal{B}\setminus\mathcal{B}_l} \dots V_k^{\mathcal{B}\setminus\mathcal{B}_l} 0\|_p = \infty.$$

Indeed, we have the inequalities

$$\lim_{k\to\infty} \|V_0^{\mathcal{B}} \dots V_k^{\mathcal{B}} 0\|_p \leq \lim_{k\to\infty} \|V_0^{\mathcal{B}_l} \dots V_k^{\mathcal{B}_l} 0\|_p + \lim_{k\to\infty} \|V_0^{\mathcal{B}\setminus\mathcal{B}_l} \dots V_k^{\mathcal{B}\setminus\mathcal{B}_l} 0\|_p,$$

$$\lim_{k\to\infty} \|V_0^{\mathcal{B}_l} \dots V_{i_{\mathcal{B}_l}}^{\mathcal{B}_l} 0\|_p < \infty,$$

$$\begin{aligned}\lim_{k\to\infty} \|V_0^{\mathcal{B}} \dots V_k^{\mathcal{B}} 0\|_p &= \lim_{k\to\infty} \lim_{m\to\infty} \|V_0^{\mathcal{B}_m} \dots V_k^{\mathcal{B}_m} 0\|_p \\ &= \lim_{m\to\infty} \lim_{k\to\infty} \|V_0^{\mathcal{B}_m} \dots V_k^{\mathcal{B}_m} 0\|_p \\ &\leq \lim_{m\to\infty} \lim_{k\to\infty} (\|V_0^{\mathcal{B}_m} \dots V_{i_{\mathcal{B}_m}}^{\mathcal{B}_m} 0\|_p + \|V_{i_{\mathcal{B}_m}+1}^{\mathcal{B}_m} \dots V_k^{\mathcal{B}_m} 0\|_p)\end{aligned}$$

and for any finite set E with $i_E < k$,

$$\|V^E_{i_E+1} \dots V^E_k 0\|_p \le \|\tau^{i_E+1}\mathbb{I}_{\Delta^E_{i_E+1}}\|_p + \dots + \|\tau^k \mathbb{I}_{\Delta^E_k}\|_p$$
$$< 1 + 2^{-1/q} + 2^{-2/q} + \dots < \infty.$$

Theorem 1 implies that if we take θ sufficiently large, for every $n \in \mathbb{N}$ there exists a process X^n with bounded increments on the set A^n for which $\|\max_{s,t\in A^n} |X_t - X_s|\,\|_p \ge 2$ and $X^n_{\min A^n} = 0$. Moreover, by applying a fairly standard argument we can choose the process so that

$$\mathrm{Prob}(\max_{s,t\in A^n} |X_t - X_s| \ge 1) \ge \frac{1}{2}.$$

Furthermore, $(X^n)_{n\in\mathbb{N}}$ can be chosen so that $X^1, X^2, \dots$ are independent.

Let $X_t = X^n_t$ for $t \in A^n$, $X_0 = 0$. By the Borel–Cantelli lemma X_t diverges almost surely as $t \to 0$. It is a simple exercise to show that $(c\cdot X_t)_{t\in\bigcup_n A^n}$ with $c = 2^{(1-q)/q}$ has bounded increments. Namely, it is enough to notice that for $t, s \in \bigcup_{n\in\mathbb{N}} A^n$, $t \in [\alpha_m, \beta_m]$, $s \in [\alpha_k, \beta_k]$, $k > m \ge 1$, we have

$$\|c\cdot X_t - c\cdot X_s\|^q_p = \|c\cdot X^m_t - c\cdot X^k_s\|^q_p \le \|c\cdot X^m_t\|^q_p + \|c\cdot X^k_s\|^q_p$$
$$\le t - \min A^m + s - \min A^k \le t - \alpha_m + \beta_k \le t - \frac{1}{2}\alpha_m \le t - \beta_k \le t - s. \ \blacksquare$$

Corollary 1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^\infty a_n = 1$. The series $\sum_{n=1}^\infty \phi_n$ converges a.e. for any sequence of functions $(\phi_n)_{n\in\mathbb{N}} \subset L^p(0,1)$ satisfying $\|\sum_{n=n_1}^{n_2} \phi_n\|_p \le \sqrt[q]{\sum_{n=n_1}^{n_2} a_n}$ for all $n_1, n_2 \in \mathbb{N}$ if and only if for the set $A = \{\sum_{n=k}^\infty a_n : k \in \mathbb{N}\cup\{\infty\}\}$ we have*

$$\lim_{k\to\infty} \|V^A_0 \dots V^A_k 0\|_p < \infty.$$

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