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Perturbation theorems for local integrated semigroups

by

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Abstract. We apply the contraction mapping theorem to establish some bounded and unbounded perturbation theorems concerning nondegenerate local α-times integrated semigroups. Some unbounded perturbation results of Wang et al. [Studia Math. 170 (2005)] are also generalized. We also establish some growth properties of perturbations of local α-times integrated semigroups.

1. Introduction. Let X be a Banach space with a norm $\|\cdot\|$, and $L(X)$ the set of all bounded linear operators on X. For each $\alpha > 0$ and $0 < T_0 \le \infty$, a family $S(\cdot)$ $(= \{S(t) \mid 0 \le t < T_0\})$ in $L(X)$ is called a *local α-times integrated semigroup* on X if it is strongly continuous and satisfies

$$S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\Big[\int_0^{t+s} - \int_0^t - \int_0^s\Big](t+s-r)^{\alpha-1}S(r)x\,dr \tag{1.1}$$

for all $x \in X$ and $0 \le t, s \le t+s < T_0$ (see [12, 14, 16]). Here $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $S(\cdot)$ is

(1.2) *locally Lipschitz continuous* if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|S(t+h)-S(t)\| \le K_{t_0}h$ for all $0 \le t, h \le t+h \le t_0$;

(1.3) *exponentially bounded* if there exist $K, \omega \ge 0$ such that $\|S(t)\| \le Ke^{\omega t}$ for all $t \ge 0$;

(1.4) *exponentially Lipschitz continuous* if there exist $K, \omega \ge 0$ such that $\|S(t+h) - S(t)\| \le Khe^{\omega(t+h)}$ for all $t, h \ge 0$;

(1.5) *nondegenerate* if $x = 0$ whenever $S(t)x = 0$ for all $0 \le t < T_0$. In this case, the (integral) *generator* of $S(\cdot)$ is defined by $D(A) = \{x \in X \mid y_x \in X \text{ and } S(t)x - j_\alpha(t)x = \int_0^t S(r)y_x\,dr \text{ for all } 0 \le t < T_0\}$ and $Ax = y_x$ for each $x \in D(A)$. Here $j_\beta(t) = t^\beta/\Gamma(\beta+1)$ for $\beta > -1$ and $t > 0$.

A local α-times integrated semigroup is called an *α-times integrated semigroup* if $T_0 = \infty$ (see [1–9, 14, 26–27]). In general, an α-times inte-

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grated semigroup may not be exponentially bounded and the generator of a nondegenerate local α-times integrated semigroup may not be densely defined.

The problem of bounded perturbations of (local) α-times integrated semigroups has been extensively studied by many authors [1, 4, 5, 11, 14–15, 21, 25–27]. In particular, Xiao and Liang [25, Theorem 1.3.5] show that if A generates an exponentially bounded nondegenerate α-times integrated semigroup on X and B is a bounded linear operator on X such that $BA \subset AB$, then $A+B$ generates an exponentially bounded nondegenerate α-times integrated semigroup on X; this has been extended by the author in [11] to the case when B is only a bounded linear operator on $\overline{D(A)}$, and Li and Shaw [15] show that if B is a bounded linear operator on X which commutes with $S(\cdot)$ on X, then $A+B$ generates a nondegenerate α-times integrated semigroup on X which may not be exponentially bounded; this result is also extended to the context of local α-times integrated semigroups in [13] by another method. Recently, some unbounded perturbation theorems concerning local α-times integrated semigroups are also established in [15, 25] and some interesting applications of this topic are illustrated in [1–8, 25–26]. In particular, Wang et al. [25] show that $A+B$ generates a local α-times integrated semigroup if $\alpha \in \mathbb{N}$ and B is a bounded linear operator on $[D(A)]$ such that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and either $A+B$ is a closed linear operator or $AB = BA$ on $D(A^2)$. Here l denotes the smallest nonnegative integer that is larger than or equal to α.

The purpose of this paper is to investigate several bounded and unbounded additive perturbation theorems for local α-times integrated semigroups on X. Growth properties of perturbations are also established. In Section 2, we show that if A generates a nondegenerate local α-times integrated semigroup $S(\cdot)$ on X and if B is a bounded linear operator from $\overline{D(A)}$ into X such that $Bx \in D(A^l)$ for all $x \in \overline{D(A)}$, then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying $T(\cdot)x = S(\cdot)x + D^\alpha S * BT(\cdot)x$ on $[0, T_0)$ for all $x \in X$ (Theorem 2.10); this has been obtained by Nicaise in [21, Corollary 4.2] using a Hille–Yosida space argument (see [4, 5]) when $\alpha \in \mathbb{N}$ and $T_0 = \infty$. Moreover, $T(\cdot)$ is exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is. We then show that $T(\cdot)$ is also locally Lipschitz continuous if $S(\cdot)$ is and $Bx \in D(A^{l-1})$ for all $x \in \overline{D(A)}$ (Theorem 2.12); this has been obtained by Kellermann and Hieber in [9] when $\alpha = 1$.

In Section 3, we first show that if B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and $A+B$ is a closed linear operator from $D(A)$ into X, then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying $T(\cdot)x = S(\cdot)x + D^{\alpha+1} S * B\widetilde{T}(\cdot)x$ on $[0, T_0)$ for all $x \in X$ (Theorem 3.1). Moreover, $T(\cdot)$

is exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is. We then show that $T(\cdot)$ is also locally Lipschitz continuous if $S(\cdot)$ is and $Bx \in D(A^l)$ for all $x \in D(A)$ (Theorem 3.2). Here $\widetilde{T}(\cdot) = j_0 * T(\cdot)$. We also show that the nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfies $T(\cdot)x = S(\cdot)x + D^\alpha S * (\lambda - A)B(\lambda - A)^{-1}T(\cdot)x$ on $[0, T_0)$ for all $x \in X$ if the assumption $AB = BA$ on $D(A^2)$ is added (Corollaries 3.5 and 3.6). Here $\lambda \in \rho(A)$ (the resolvent set of A) is fixed. An illustrative example concerning these theorems is also presented in the final part of this paper.

2. Bounded perturbation theorems. In this section, we first recall some basic properties of a nondegenerate local α-times integrated semigroup and known results about connections between the generator of such a semigroup and strong solutions of the abstract Cauchy problem

$$ACP(A, f, x) \begin{cases} u'(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\ u(0) = x, \end{cases}$$

where $x \in X$ and f is an X-valued function defined on $(0, T_0)$.

Proposition 2.1 (see [10, 14, 16, 18]). *Let A be the generator of a nondegenerate local α-times integrated semigroup $S(\cdot)$ on X. Then*

(2.1) $S(0) = 0$ *(the zero operator) on X;*
(2.2) *A is closed and $\rho(A)$ (the resolvent set of A) is nonempty;*
(2.3) *$S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$ for $x \in D(A)$ and $0 \le t < T_0$;*
(2.4) *$\int_0^t S(r)x\,dr \in D(A)$ and $A\int_0^t S(r)x\,dr = S(t)x - j_\alpha(t)x$ for $x \in X$ and $0 \le t < T_0$;*
(2.5) *$R(S(t)) \subset \overline{D(A)}$ for $0 \le t < T_0$;*
(2.6) *for each $\beta > \alpha$, $j_{\beta-\alpha-1} * S(\cdot)$ is a nondegenerate local β-times integrated semigroup on X with generator A.*

Definition 2.2. Let $A : D(A) \subset X \to X$ be a closed linear operator in a Banach space X with domain $D(A)$ and range $R(A)$. A function $u : [0, T_0) \to X$ is called a (strong) *solution* of $ACP(A, f, x)$ if $u \in C^1((0, T_0), X) \cap C([0, T_0), X) \cap C((0, T_0), [D(A)])$ and satisfies $ACP(A, f, x)$. Here $[D(A)]$ denotes the Banach space $D(A)$ with the norm $|\cdot|$ defined by $|x| = \|x\| + \|Ax\|$ for all $x \in D(A)$.

Remark 2.3. $u \in C([0, T_0), [D(A)])$ if $f \in C([0, T_0), X)$ and u is a (strong) solution of $ACP(A, f, x)$ in $C^1([0, T_0), X)$.

Theorem 2.4 (see [12]). *A generates a nondegenerate local α-times integrated semigroup $S(\cdot)$ on X if and only if for each $x \in X$, $ACP(A, j_\alpha(\cdot)x, 0)$ has a unique (strong) solution $u(\cdot, x)$ in $C^1([0, T_0), X)$. In this case, we have $u(\cdot, x) = j_0 * S(\cdot)x$ for all $x \in X$.*

We next recall some results concerning the αth derivative of a continuous function from a subinterval I of $[0, T_0)$ containing $\{0\}$ into X which have been given in [12].

Definition 2.5. Let $\alpha > 0$, $k = [\alpha] + 1$ and $v : I \to X$ for some subinterval I of $[0, T_0)$ containing $\{0\}$. We write $v \in C^\alpha(I, X)$ if $v = v(0) + j_{\alpha-k} * u$ on I for some $u \in C^{k-1}(I, X)$. In this case, we say that v is *α-times continuously differentiable* on I, and the $(k-1)$th derivative of u on I is called the αth *derivative* of v on I and denoted by $D^\alpha v$ (on I) or $D^\alpha v : I \to X$. Here $C^k(I, X)$ denotes the set of all k-times continuously differentiable functions from I into X, and $C^0(I, X) = C(I, X)$ the set of all continuous functions from I into X.

Remark 2.6 (see [10]). Let $k = [\alpha] + 1$ and $v \in C^\alpha(I, X)$ for some subinterval I of $[0, T_0)$ containing $\{0\}$. Assume that $v(0) = 0$. Then $j_{k-\alpha-1} * v \in C^k(I, X)$, $v \in C^{\alpha-i}(I, X)$ and $D^{\alpha-i}v = (j_{k-\alpha-1} * v)^{(k-i)}$ on I for all integers $0 \le i \le k-1$. In particular, $j_\alpha(\cdot) \in C^\alpha([0, T_0), \mathbb{C})$ and $D^{\alpha-i}j_\alpha(\cdot) = D^{k-i}j_k(\cdot) = j_i(\cdot)$ on $[0, T_0)$ for all integers $0 \le i \le k-1$.

Proposition 2.7 (see [10]). *Let A be the generator of a nondegenerate local α-times integrated semigroup $S(\cdot)$ on X, $x \in X$ and $f \in L^1_{\mathrm{loc}}([0, T_0), X) \cap C((0, T_0), X)$. Then $ACP(A, f, x)$ has a (strong) solution u in $C^1([0, T_0), X)$ if and only if $v(\cdot) = S(\cdot)x + S * f(\cdot) \in C^{\alpha+1}([0, T_0), X)$. In this case, $u = D^\alpha v$ on $[0, T_0)$.*

Lemma 2.8 (see [10]). *Let $V(\cdot)$ and $Z(\cdot)$ be strongly continuous families of bounded linear operators from X into some Banach space Y, and let $W(\cdot)$ be a strongly continuous family in $L(Y)$ such that $Z(\cdot)x = V(\cdot)x + W * Z(\cdot)x$ on $[0, T_0)$ for all $x \in X$. Then $Z(\cdot)$ is exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $V(\cdot)$ and $W(\cdot)$ both are.*

By slightly modifying the proof of [22, Lemma 2.11] we can obtain the next lemma.

Lemma 2.9. *Let $V(\cdot)$ be a locally Lipschitz continuous family of bounded linear operators from X into some Banach space Y, and let $W(\cdot)$ be a locally Lipschitz continuous family in $L(Y)$ with $W(0) = 0$ on Y. Then there exists a unique locally Lipschitz continuous family $Z(\cdot)$ of bounded linear operators from X into Y such that*

$$Z(t)x = V(t)x + \frac{d}{dt}W * Z(t)x$$

for all $x \in X$ and $t \in [0, T_0)$.

The next theorem is a bounded perturbation of local α-times integrated semigroups on X which has been established by Nicaise in [21, Corollary 4.2] using a Hille–Yosida space argument when $\alpha \in \mathbb{N}$ and $T_0 = \infty$.

THEOREM 2.10. *Let $S(\cdot)$ be a nondegenerate local α-times integrated semigroup on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into X such that $Bx \in D(A^l)$ for all $x \in \overline{D(A)}$. Then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying*

$$T(\cdot)x = S(\cdot)x + D^\alpha S * BT(\cdot)x \quad \text{on } [0, T_0) \tag{2.7}$$

for all $x \in X$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is.

Proof. Indeed, if we set $k = [\alpha] + 1$, we may define $\widetilde{S}(t) : X \to X$ for $0 \le t < T_0$ by $\widetilde{S}(t)x = j_{k-\alpha-1} * S(t)x$ for all $x \in X$. By (2.6), $\widetilde{S}(\cdot)$ is a nondegenerate local k-times integrated semigroup on X with generator A. It is also easy to see from (2.3) and (2.4) that

$$\widetilde{S}(t)y = j_{r-1} * \widetilde{S}(t)A^r y + \sum_{i=0}^{r-1} j_{k+i}(t)A^i y \tag{2.8}$$

for all $r \in \mathbb{N}$, $y \in D(A^r)$ and $0 \le t < T_0$. Combining (2.8) with Remark 2.6, we have

$$\begin{aligned} D^\alpha(S * Bf)(\cdot) &= D^k(\widetilde{S} * Bf)(\cdot) \\ &= \begin{cases} D^k(j_{k-2} * \widetilde{S} * A^{k-1}Bf + \sum_{i=0}^{k-2} j_{k+i} * A^i Bf)(\cdot) & \text{if } \alpha = k-1 \in \mathbb{N} \\ D^k(j_{k-1} * \widetilde{S} * A^k Bf + \sum_{i=0}^{k-1} j_{k+i} * A^i Bf)(\cdot) & \text{if } k-1 < \alpha < k \end{cases} \\ &= \begin{cases} S * A^{k-1}Bf(\cdot) + \sum_{i=0}^{k-2} j_i * A^i Bf(\cdot) & \text{if } \alpha = k-1 \in \mathbb{N} \\ \widetilde{S} * A^k Bf(\cdot) + \sum_{i=0}^{k-1} j_i * A^i Bf(\cdot) & \text{if } k-1 < \alpha < k \end{cases} \end{aligned} \tag{2.9}$$

on $[0, t_0]$ for all $0 < t_0 < T_0$ and $f \in C([0, t_0], \overline{D(A)})$. We shall show that for each $x \in X$ there exists a unique function w_x in $C([0, T_0), \overline{D(A)})$ such that $w_x(\cdot) = S(\cdot)x + D^\alpha S * Bw_x(\cdot)$ on $[0, T_0)$; this may be done by using Theorem 2.4. Indeed, fix $x \in X$ and $0 < t_0 < T_0$ and define $U : C([0, t_0], \overline{D(A)}) \to C([0, t_0], \overline{D(A)})$ by $U(f)(\cdot) = S(\cdot)x + D^\alpha(S * Bf)(\cdot)$ on $[0, t_0]$ for all $f \in C([0, t_0], \overline{D(A)})$. From (2.1), (2.5) and the assumption $Bx \in D(A^l)$ for all $x \in \overline{D(A)}$, we see that U is well-defined and $A^i B$ is a bounded linear operator from $\overline{D(A)}$ into X for all integers $0 \le i \le l$. We first claim that

$$\|D^{\alpha}S * Bf(t)\| \le M_{t_0}\int_0^t \|f(s)\|\,ds \tag{2.10}$$

for all $f \in C([0,t_0],\overline{D(A)})$ and $0 \le t \le t_0$, where

$$M_{t_0} = \begin{cases} \sup\limits_{0\le r\le t_0}\|S(r)\|\,\|A^{k-1}B\| + \sum\limits_{i=0}^{k-2} j_i(t_0)\|A^iB\| & \text{if } \alpha = k-1 \in \mathbb{N}, \\ \sup\limits_{0\le r\le t_0}\|\widetilde{S}(r)\|\,\|A^{k}B\| + \sum\limits_{i=0}^{k-1} j_i(t_0)\|A^iB\| & \text{if } k-1 < \alpha < k. \end{cases}$$

To see this, we consider only the case $\alpha = k-1 \in \mathbb{N}$, for the case $k-1 < \alpha < k$ can be treated similarly. Indeed, if $\alpha = k-1 \in \mathbb{N}$ and $f \in C([0,t_0],\overline{D(A)})$, then

$$\begin{aligned} \|S * A^{k-1}Bf(t)\| &\le \int_0^t \|S(t-s)A^{k-1}Bf(s)\|\,ds \\ &\le \int_0^t \sup_{0\le r\le t_0}\|S(r)\|\,\|A^{k-1}B\|\|f(s)\|\,ds \\ &= \sup_{0\le r\le t_0}\|S(r)\|\,\|A^{k-1}B\|\int_0^t \|f(s)\|\,ds \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \|j_i * A^iBf(t)\| &\le \int_0^t \|j_i(t-s)A^iBf(s)\|\,ds \\ &\le \int_0^t j_i(t_0)\|A^iB\|\,\|f(s)\|\,ds \\ &= j_i(t_0)\|A^iB\|\int_0^t \|f(s)\|\,ds \end{aligned} \tag{2.12}$$

for all $0 \le t \le t_0$ and integers $0 \le i \le k-2$, and so

$$\begin{aligned} \|D^{\alpha}S * Bf(t)\| &\le \|S * A^{k-1}Bf(t)\| + \sum_{i=0}^{k-2}\|j_i * A^iBf(t)\| \\ &\le M_{t_0}\int_0^t \|f(s)\|\,ds \end{aligned}$$

for all $0 \le t \le t_0$. Hence (2.10) holds when $\alpha = k-1 \in \mathbb{N}$. By induction, we have

$$\begin{aligned}
(2.13)\qquad \|U^n f(t) - U^n g(t)\| &= \|U(U^{n-1}f)(t) - U(U^{n-1}g)(t)\| \\
&= \|D^\alpha S * B(U^{n-1}f - U^{n-1}g)(t)\| \\
&\le M_{t_0}^n \int_0^t j_{n-1}(t-s)\|f(s)-g(s)\|\,ds \\
&\le M_{t_0}^n j_n(t)\|f-g\| \le M_{t_0}^n j_n(t_0)\|f-g\|
\end{aligned}$$

for all $f, g \in C([0,t_0],\overline{D(A)})$, $0 \le t \le t_0$ and $n \in \mathbb{N}$, where $\|f-g\| = \max_{0\le s\le t_0}\|f(s)-g(s)\|$. It follows from the contraction mapping theorem that there exists a unique function w_{x,t_0} in $C([0,t_0],\overline{D(A)})$ such that $w_{x,t_0}(\cdot) = S(\cdot)x + D^\alpha S * Bw_{x,t_0}(\cdot)$ on $[0,t_0]$. In this case, we set $w_x(t) = w_{x,t_0}(t)$ for all $0 \le t \le t_0 < T_0$. Then $w_x(\cdot)$ is a unique function in $C([0,T_0),\overline{D(A)})$ such that $w_x(\cdot) = S(\cdot)x + D^\alpha S * Bw_x(\cdot)$ on $[0,T_0)$. Since $S*j_\alpha(\cdot)x + S*j_0*Bw_x \in C^{\alpha+1}([0,T_0),X)$ and $D^\alpha(S*j_\alpha(\cdot)x+S*j_0*Bw_x) = j_0*S(\cdot)x + j_0*D^\alpha S*Bw_x = j_0*w_x$ on $[0,T_0)$, we deduce from Proposition 2.7 that $u = j_0*w_x$ is the unique (strong) solution of $ACP(A, j_\alpha(\cdot)x + j_0*Bw_x, 0)$ in $C^1([0,T_0),X)$, and so $u = j_0*w_x$ is the unique function in $C^1([0,T_0),X)$ such that u' $(= Au + j_\alpha x + j_0 * Bw_x = Au + j_\alpha x + Bu) = (A+B)u + j_\alpha x$ on $[0,T_0)$. Hence $u = j_0 * w_x$ is the unique (strong) solution of $ACP(A+B, j_\alpha(\cdot)x, 0)$ in $C^1([0,T_0),X)$, which together with Theorem 2.4 implies that $A+B$ generates a nondegenerate α-times integrated semigroup $T(\cdot)$ on X satisfying (2.7). Combining Lemma 2.8 with (2.9), we find that $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is, by setting $Y = \overline{D(A)}$, $V(\cdot) = S(\cdot)$, $Z(\cdot) = T(\cdot)$ and

$$W(\cdot) = \begin{cases} S(\cdot)A^{k-1}B + \sum_{i=0}^{k-2} j_i(\cdot)A^iB & \text{if } \alpha = k-1 \in \mathbb{N}, \\ \widetilde{S}(\cdot)A^kB + \sum_{i=0}^{k-1} j_i(\cdot)A^iB & \text{if } k-1 < \alpha < k, \end{cases}$$

in Lemma 2.8.

Remark 2.11. Let $W(\cdot)$ be a locally Lipschitz continuous family in $L(Y)$ with $W(0) = 0$ for some Banach space Y and $g \in L^1_{\mathrm{loc}}([0,T_0),Y)$. Then $W * g \in C^1([0,T_0),Y)$ and for each $0 < t_0 < T_0$, we have $\|(W*g)'(t)\| \le K_{t_0}\int_0^t \|g(s)\|\,ds$ for all $0 \le t \le t_0$. Here K_{t_0} is given as in (1.3) with $S(\cdot)$ is replaced by $W(\cdot)$. Moreover, $(W*g)'(\cdot)$ is locally Lipschitz continuous if g is.

By slightly modifying the proof of Theorem 2.10, we can establish the next bounded perturbation theorem concerning locally Lipschitz continuous local α-times integrated semigroups on X, which has been obtained by Kellermann and Hieber in [9] when $\alpha = 1$.

Theorem 2.12. *Let A be the generator of a locally Lipschitz continuous nondegenerate local α-times integrated semigroup $S(\cdot)$ on X for some*

$\alpha \geq 1$. Assume that B is a bounded linear operator from $\overline{D(A)}$ into X. Then $A+B$ generates a locally Lipschitz continuous nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying (2.7), *if either $\alpha = 1$ or $\alpha > 1$ with $Bx \in D(A^{l-1})$ for all $x \in \overline{D(A)}$.*

Proof. Just as in the proof of Theorem 2.10, we shall first show that $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying (2.7), and need only show that

$$\|D^\alpha S * Bf(t)\| \leq N_{t_0} \int_0^t \|f(s)\|\, ds \tag{2.14}$$

for all $0 < t_0 < T_0$, $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq t \leq t_0$. Here

$$N_{t_0} = \begin{cases} K_{t_0}\|B\| & \text{if } \alpha = 1, \\ K_{t_0} j_{k-\alpha}(t_0)\|A^{k-1}B\| + \sum_{i=0}^{k-2} j_i(t_0)\|A^iB\| & \text{if } 1 \leq k-1 < \alpha < k, \\ K_{t_0}\|A^{k-2}B\| + \sum_{i=0}^{k-3} j_i(t_0)\|A^iB\| & \text{if } \alpha = k-1 \geq 2, \end{cases}$$

and K_{t_0} is given as in (1.3). Indeed, the local Lipschitz continuity of $S(\cdot)$ implies that $\widetilde{S}(\cdot)$ is also locally Lipschitz continuous with a Lipschitz constant $K_{t_0} j_{k-\alpha}(t_0)$ on $[0,t_0]$ for all $0 < t_0 < T_0$. Combining Remarks 2.6 and 2.11, (2.8) with the assumption $Bx \in D(A^{l-1})$ for all $x \in \overline{D(A)}$, we have $S * Bf \in C^\alpha([0,t_0], \overline{D(A)})$ and

$$\begin{aligned} &D^\alpha S * Bf(\cdot) \\ &= \begin{cases} (S*Bf)'(\cdot) & \text{if } \alpha = 1, \\ D^k\Big(j_{k-2} * \widetilde{S} * A^{k-1}Bf + \sum_{i=0}^{k-2} j_{k+i} * A^iBf\Big)(\cdot) & \\ \quad = (\widetilde{S} * A^{k-1}Bf)'(\cdot) + \sum_{i=0}^{k-2} j_i * A^iBf(\cdot) & \text{if } 1 \leq k-1 < \alpha < k, \\ D^k\Big(j_{k-3} * \widetilde{S} * A^{k-2}Bf + \sum_{i=0}^{k-3} j_{k+i} * A^iBf\Big)(\cdot) & \\ \quad = (S * A^{k-2}Bf)'(\cdot) + \sum_{i=0}^{k-3} j_i * A^iBf(\cdot) & \text{if } \alpha = k-1 \geq 2, \end{cases} \end{aligned} \tag{2.15}$$

on $[0,t_0]$ for all $0 < t_0 < T_0$ and $f \in C([0,t_0], \overline{D(A)})$. Now if $0 < t_0 < T_0$ is fixed, then for each $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq t \leq t_0$, from Remark 2.11 and the continuity of A^iB on $\overline{D(A)}$ for integers $1 \leq i \leq k-1$ we obtain

$$\|(S * A^{k-2}Bf)'(t)\| \le K_{t_0} \int_0^t \|A^{k-2}Bf(s)\|\,ds \le K_{t_0}\|A^{k-2}B\| \int_0^t \|f(s)\|\,ds$$

if $\alpha = k-1 \ge 1$, and

$$\begin{aligned}\|(\widetilde{S} * A^{k-1}Bf)'(t)\| &\le K_{t_0} j_{k-\alpha}(t_0) \int_0^t \|A^{k-1}Bf(s)\|\,ds \\ &\le K_{t_0} j_{k-\alpha}(t_0)\|A^{k-1}B\| \int_0^t \|f(s)\|\,ds\end{aligned}$$

if $k-1 < \alpha < k$. Consequently, (2.14) holds, showing that $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying (2.7). We deduce from (2.15) and Lemma 2.9 that $T(\cdot)$ is also locally Lipschitz continuous: it suffices to set $Y = \overline{D(A)}$, $V(\cdot) = S(\cdot)$, $Z(\cdot) = T(\cdot)$ and

$$W(\cdot) = \begin{cases} S(\cdot)B & \text{if } \alpha = 1, \\ \widetilde{S}(\cdot)A^{k-1}B + \sum_{i=0}^{k-2} j_{i+1}(\cdot)A^iB & \text{if } 1 \le k-1 < \alpha < k, \\ S(\cdot)A^{k-2}B + \sum_{i=0}^{k-3} j_{i+1}(\cdot)A^iB & \text{if } \alpha = k-1 \ge 2, \end{cases}$$

in Lemma 2.9.

REMARK 2.13. An example in [5, Example 19.11] shows that there exists a nondegenerate α-times integrated semigroup on X with a generator A such that $A+B$ does not generate a nondegenerate α-times integrated semigroup on X for some bounded linear operator B from X into $D(A^{l-1})$.

3. Unbounded perturbation theorems. By slightly modifying the proof of Theorem 2.10, we can establish the next unbounded perturbation theorem concerning local α-times integrated semigroups on X which has been obtained by Wang et al. in [25] when $\alpha \in \mathbb{N}$ except for the growth propertis of $T(\cdot)$.

THEOREM 3.1. *Let $S(\cdot)$ be a nondegenerate local α-times integrated semigroup on X with generator A. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and $A+B$ is a closed linear operator from $D(A)$ into X. Then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying*

$$T(\cdot)x = S(\cdot)x + D^{\alpha+1}S * B\widetilde{T}(\cdot)x \quad \text{on } [0, T_0) \tag{3.1}$$

for all $x \in X$. Here $\widetilde{T}(\cdot) = j_0 * T(\cdot)$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is.

Proof. We consider only the case $\alpha = k-1 \in \mathbb{N}$, for the case $k-1 < \alpha < k$ can be treated similarly. Just as in the proof of Theorem 2.10, for each $0 < t_0 < T_0$, we can apply (2.9) and the fact that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ to establish the following inequalities analogous to (2.10)–(2.13):

$$|S * A^{k-1}Bf(t)| \le \sup_{0\le r\le t_0} \|S(r)\|\,|A^{k-1}B| \int_0^t |f(s)|\,ds, \tag{3.2}$$

$$|j_i * A^iBf(t)| \le j_i(t_0)|A^iB| \int_0^t |f(s)|\,ds \tag{3.3}$$

for all $0 \le t \le t_0$ and integers $0 \le i \le k-2$,

$$|D^\alpha S * Bf(t)| \le M_{t_0} \int_0^t |f(s)|\,ds \tag{3.4}$$

for all $0 \le t \le t_0$, and

$$|U^n f(t) - U^n g(t)| \le M_{t_0}^n j_n(t_0)|f-g| \tag{3.5}$$

for all $f, g \in C([0,t_0],[D(A)])$, $0 \le t \le t_0$ and $n \in \mathbb{N}$. Here $|A^iB|$ denotes the norm of A^iB in $L([D(A)])$ for all integers $0 \le i \le k-1$, $|f-g| = \max_{0\le s\le t_0} |f(s) - g(s)|$ and $U : C([0,t_0],[D(A)]) \to C([0,t_0],[D(A)])$ is defined by $U(f)(\cdot) = j_0 * S(\cdot)x + D^\alpha(S * Bf)(\cdot)$ on $[0,t_0]$, and

$$M_{t_0} = \begin{cases} \sup\limits_{0\le r\le t_0} \|S(r)\|\,|A^{k-1}B| + \sum\limits_{i=0}^{k-2} j_i(t_0)|A^iB| & \text{if } \alpha = k-1 \in \mathbb{N}, \\ \sup\limits_{0\le r\le t_0} \|\widetilde{S}(r)\|\,|A^kB| + \sum\limits_{i=0}^{k-1} j_i(t_0)|A^iB| & \text{if } k-1 < \alpha < k. \end{cases}$$

Combining (3.2)–(3.5), we conclude that for each $x \in X$ there exists a unique function w_x in $C([0,T_0),[D(A)])$ such that $w_x(\cdot) = j_0 * S(\cdot)x + D^\alpha S * Bw_x(\cdot)$ on $[0,T_0)$ as in the proof of Theorem 2.10, and then show that $u = j_0 * w_x$ is the unique (strong) solution of $ACP(A, j_{\alpha+1}(\cdot)x + j_0 * Bw_x, 0)$ in $C^1([0,T_0),X)$, and so $u = j_0 * w_x$ is the unique (strong) solution of $ACP(A+B, j_{\alpha+1}(\cdot)x, 0)$ in $C^1([0,T_0),X)$. Hence $A+B$ generates a nondegenerate local $(\alpha+1)$-times integrated semigroup $\widetilde{T}(\cdot)$ on X satisfying

$$\widetilde{T}(\cdot)x = j_0 * S(\cdot)x + D^\alpha S * B\widetilde{T}(\cdot)x \quad \text{on } [0,T_0) \tag{3.6}$$

for all $x \in X$. From the assumption $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and (2.9) we see that $\widetilde{T}(\cdot)x$ is continuously differentiable on $[0,T_0)$ for all $x \in X$, and so $T(\cdot)$ defined by $T(t)x = \frac{d}{dt}\widetilde{T}(t)x$ for all $x \in X$ and $0 \le t < T_0$ is a nondegenerate local α-times integrated semigroup on X with generator $A+B$ satisfying $T(\cdot)x = S(\cdot)x + D^{\alpha+1}S * B\widetilde{T}(\cdot)x$ on $[0,T_0)$ for all $x \in X$. Clearly, $j_0 * S(\cdot)$

is exponentially Lipschitz continuous if $S(\cdot)$ is exponentially bounded. Applying Lemma 2.8, (2.9) and (3.6), we find that $\widetilde{T}(\cdot)$ is exponentially Lipschitz continuous if $S(\cdot)$ is exponentially bounded: just set $Y = [D(A)]$, $V(\cdot) = j_0 * S(\cdot)$, $Z(\cdot) = \widetilde{T}(\cdot)$ and $W(\cdot) = S(\cdot)A^{k-1}B + \sum_{i=0}^{k-2} j_i(\cdot)A^i B$ in Lemma 2.8. This implies that $T(\cdot)$ is also exponentially bounded if $S(\cdot)$ is. Next if $S(\cdot)$ is norm continuous (resp., exponentially Lipschitz continuous), then applying Lemma 2.8 again, we infer that $\widetilde{T}(\cdot)$ is also norm continuous (resp., exponentially Lipschitz continuous), and so $A^i B\widetilde{T}(\cdot)$ for $0 \le i \le k$ are norm continuous (resp., exponentially Lipschitz continuous). Combining this with (2.9), we see that $D^{\alpha+1}S * B\widetilde{T}(\cdot)$ is norm continuous (resp., exponentially Lipschitz continuous), which together with (3.1) implies that $T(\cdot)$ is also norm continuous (resp., exponentially Lipschitz continuous).

By slightly modifying the proof of Theorem 2.12, the next new unbounded perturbation theorem concerning locally Lipschitz continuous local α-times integrated semigroups on X is also obtained.

THEOREM 3.2. *Let $S(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated semigroup on X with generator A for some $\alpha \ge 1$. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^l)$ for all $x \in D(A)$ and $A + B$ is a closed linear operator from $D(A)$ into X. Then $A + B$ generates a nondegenerate locally Lipschitz continuous local α-times integrated semigroup $T(\cdot)$ on X satisfying* (3.1).

Proof. Just as in the proof of Theorem 3.1, we consider only the case $\alpha = k-1 \in \mathbb{N}$, and so for each $0 < t_0 < T_0$ and $f \in C([0,t_0],[D(A)])$, we deduce from Remark 2.11 and the fact $(S*A^{k-1}Bf)'(\cdot) = A(S*A^{k-1}Bf)(\cdot) + j_{k-2} * A^{k-1}Bf$ that (3.4) holds, which implies that $A + B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying (3.1). Clearly, $\widetilde{T}(\cdot)$ is locally Lipschitz continuous and $\widetilde{T}(0) = 0$ on X. It follows that $A^i B\widetilde{T}(\cdot)$ is also locally Lipschitz continuous and $A^i B\widetilde{T}(0) = 0$ on X for all integers $0 \le i \le k-1$. Combining this with the local Lipschitz continuity of $S(\cdot)$, we conclude from Remark 2.11 that $(S * A^{k-1}B\widetilde{T})'(\cdot)$ is locally Lipschitz continuous, which together with (2.9) in which f is replaced by $\widetilde{T}(\cdot)$, and (3.1), implies that $T(\cdot)$ is also locally Lipschitz continuous.

COROLLARY 3.3. *Let $S(\cdot)$ be a nondegenerate local α-times integrated semigroup on X with generator A. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and $\rho(A+B)$ is nonempty. Then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying* (3.1) *for all $x \in X$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is.*

Corollary 3.4. *Let $S(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated semigroup on X with generator A for some $\alpha \geq 1$. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^l)$ for all $x \in D(A)$ and $\rho(A+B)$ is nonempty. Then $A+B$ generates a nondegenerate locally Lipschitz continuous local α-times integrated semigroup $T(\cdot)$ on X satisfying* (3.1).

When the assumption that $A+B$ is a closed linear operator from $D(A)$ into X is replaced by assuming that $AB = BA$ on $D(A^2)$, we can obtain the next unbounded perturbation result which has been obtained by Wang et al. in [25] when $\alpha \in \mathbb{N}$ except for the growth properties of $T(\cdot)$.

Corollary 3.5. *Let $S(\cdot)$ be a nondegenerate local α-times integrated semigroup on X with generator A. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^{l+1})$ for all $x \in D(A)$ and $AB = BA$ on $D(A^2)$. Then $A+B$ generates a nondegenerate local α-times integrated semigroup $T(\cdot)$ on X satisfying*

$$T(\cdot)x = S(\cdot)x + D^\alpha S * (\lambda - A)B(\lambda - A)^{-1}T(\cdot)x \quad \text{on } [0, T_0) \tag{3.7}$$

for all $x \in X$. Here $\lambda \in \rho(A)$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is.

Proof. Just as in the proof of [25, Theorem 3.1], we can show that $A+B$ is a closed linear operator from $D(A)$ into X, or equivalently, $\lambda - (A+B)$ is. Here $\lambda \in \rho(A)$ is fixed. By slightly modifying the proof of Theorem 2.10, we also deduce that for each $x \in X$ there exists a unique function w_x in $C([0,T_0), X)$ such that $w_x = S(\cdot)x + D^\alpha S * (\lambda - A)B(\lambda - A)^{-1}w_x$, and so $j_0 * w_x$ is the unique solution of

$$\begin{aligned} &ACP(A, j_\alpha x + j_0 * (\lambda - A)B(\lambda - A)^{-1}w_x, 0) \\ &\qquad = ACP(A, j_\alpha x + (\lambda - A)B(\lambda - A)^{-1}j_0 * w_x, 0) \\ &\qquad = ACP(A, j_\alpha x + (\lambda - A)B(\lambda - A)^{-1}j_0 * w_x, 0) \\ &\qquad = ACP(A, j_\alpha x + Bj_0 * w_x, 0) \end{aligned}$$

in $C^1([0,T_0), X)$. Hence $u = j_0 * w_x$ is the unique function in $C^1([0,T_0), X)$ such that $u' = Au + j_\alpha x + Bu = (A+B)u + j_\alpha x$ on $[0,T_0)$ and $u(0) = 0$. Applying Theorem 2.4 again, we find that $A+B$ generates a nondegenerate local α-times integrated semigroup on X satisfying (3.7) which is defined by $T(\cdot)x = w_x(\cdot)$ for all $x \in X$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous or exponentially Lipschitz continuous) if $S(\cdot)$ is.

By slightly modifying the proof of Theorem 2.12, the next unbounded perturbation result concerning locally Lipschitz continuous local α-times integrated semigroups on X is also obtained.

Corollary 3.6. *Let $S(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated semigroup on X with generator A for some $\alpha \geq 1$. Assume that B is a bounded linear operator from $[D(A)]$ into X such that $Bx \in D(A^l)$ for all $x \in D(A)$ and $AB = BA$ on $D(A^2)$. Then $A + B$ generates a nondegenerate locally Lipschitz continuous local α-times integrated semigroup $T(\cdot)$ on X satisfying* (3.7).

We end this paper with a simple illustrative example. Let $X = L^\infty(\mathbb{R})$, and $A : D(A) \subset X \to X$ be defined by $D(A) = W^{1,\infty}(\mathbb{R})$ and $Af = -f'$ for all $f \in D(A)$. Then A generates a locally Lipschitz continuous local 1-times integrated semigroup $S(\cdot)$ ($= \{S(t) \mid 0 \leq t < T_0\}$) on X and $\overline{D(A)} = C_0(\mathbb{R})$ (see [1, Example 3.3.10]). Here $0 < T_0 \leq \infty$ is fixed. Applying Theorem 2.12, we find that $A + B$ generates a locally Lipschitz continuous local 1-times integrated semigroup $T(\cdot)$ on $L^\infty(\mathbb{R})$ satisfying (2.7) when B is a bounded linear operator from $C_0(\mathbb{R})$ into $L^\infty(\mathbb{R})$ defined by $B(f)(t) = \int_{-\infty}^{\infty} f(t-s)\, d\mu(s)$ for all $f \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$. Here μ is a fixed finite regular Borel measure on $\mathbb{R}$.

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