STUDIA MATHEMATICA 160 (1) (2004)

The L^r Henstock–Kurzweil integral

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Abstract. We present a method of integration along the lines of the Henstock–Kurzweil integral. All L^r-derivatives are integrable in this method.

1. Introduction. During the early part of the twentieth century, A. Denjoy and O. Perron developed equivalent integrals which extend the Lebesgue integral and which integrate all derivatives.

Around 1960, R. Henstock and J. Kurzweil developed an integral which is equivalent to the integrals of Denjoy and Perron and which therefore integrates all derivatives. The construction of the Henstock–Kurzweil (HK) integral is quite similar to that of the Riemann integral and so is much easier than those of the Denjoy and Perron integrals. For a complete treatment of the Henstock–Kurzweil integral we refer the reader to [1], [4], or [5].

Instead of the classical derivative one may consider the approximate derivative, and with it integrals that will integrate such derivatives. Indeed, there exist both Perron-type and HK-type integrals that integrate approximate derivatives (see [4]).

Another notion of derivative, the L^r-derivative, useful in Harmonic Analysis, was developed by A. P. Calderón and A. Zygmund in [2]. L. Gordon, in [3], developed the P_r-integral (Perron r-integral) which integrates L^r-derivatives. In this paper we define a Henstock–Kurzweil type integral which integrates all functions that are integrated by the P_r-integral.

We are considering functions defined on a finite closed interval, $[a,b]$. Also the parameter r, throughout the paper, satisfies $1 \le r < \infty$. We use the phrase "nearly every" (abbreviated to n.e.) for "all but countably many." Finally, the symbol $\int$, without a modifier, stands for the Lebesgue integral.

2. The P_r-integral. To make our presentation reasonably self-contained we give an outline of the P_r-integral as developed by L. Gordon. For the full details of the proofs, see [3].

2000 *Mathematics Subject Classification*: Primary 26A39, 26A42; Secondary 26A24, 26A46.

Definition 1 ([3]). Let $f \in L^r(I)$ where $1 \le r < \infty$ and I is an open interval. For all $x \in I$ we define the r-Dini derivates. In all cases below, $h \to 0^+$.

The *upper-right L^r-derivate*:

$$D_r^+ f(x) = \inf\left\{a : \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - at]_+^r \, dt\right)^{1/r} = o(h)\right\}, \tag{2.1}$$

and similarly the *lower-right L^r-derivate*:

$$D_{+,r} f(x) = \sup\left\{a : \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - at]_-^r \, dt\right)^{1/r} = o(h)\right\}, \tag{2.2}$$

the *upper-left L^r-derivate*:

$$D_r^- f(x) = \inf\left\{a : \left(\frac{1}{h}\int_0^h [-f(x-t) + f(x) - at]_+^r \, dt\right)^{1/r} = o(h)\right\}, \tag{2.3}$$

and the *lower-left L^r-derivate*:

$$D_{-,r} f(x) = \sup\left\{a : \left(\frac{1}{h}\int_0^h [-f(x-t) + f(x) - at]_-^r \, dt\right)^{1/r} = o(h)\right\}. \tag{2.4}$$

Theorem 1 ([3]).

$$D_r^+ f(x) = \inf\left\{a : \int_0^h \left(\frac{f(x+t) - f(x)}{t} - a\right)_+^r dt = o(h)\right\}$$

with similar results for the other r-Dini derivates.

The next theorem simplifies the exposition.

Theorem 2. *Either $D_r^+ f(x) = \pm\infty$, or the extreme value in the definition is assumed. In other words*,

$$D_r^+ f(x) = \min\left\{a : \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - at]_+^r \, dt\right)^{1/r} = o(h)\right\}. \tag{2.5}$$

Also

$$D_r^+ f(x) = \min\left\{a : \int_0^h \left(\frac{f(x+t) - f(x)}{t} - a\right)_+^r dt = o(h)\right\} \tag{2.6}$$

with similar statements for $D_{+,r} f(x)$, $D_r^- f(x)$, and $D_{-,r} f(x)$.

Proof. Let us prove (2.5). Define $a_0 = D_r^+ f(x)$. For every $\delta > 0$ we have

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - (a_0 + \delta)t]_+^r \, dt\right)^{1/r} = o(h) \quad \text{as } h \to 0^+.$$

Let $\varepsilon > 0$ be given and let $\eta > 0$ be such that for all $0 < h < \eta$ we have

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - (a_0+\varepsilon)t]_+^r \, dt\right)^{1/r} \le \varepsilon h.$$

Thus

$$\begin{aligned}
&\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - a_0 t]_+^r \, dt\right)^{1/r} \\
&\quad = \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - (a_0+\varepsilon)t + \varepsilon t]_+^r \, dt\right)^{1/r} \\
&\quad \le \left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - (a_0+\varepsilon)t]_+^r \, dt\right)^{1/r} + \varepsilon\left(\frac{1}{h}\int_0^h t^r \, dt\right)^{1/r} < 2\varepsilon h
\end{aligned}$$

and so

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - a_0 t]_+^r \, dt\right)^{1/r} = o(h).$$

The proofs for the other r-Dini derivates and for (2.6) follow similarly. ■

Definition 2 ([3]). Let $1 \le r < \infty$. We define the L^r-*upper derivate* of f at x, $\overline{D}_r f(x)$, by

$$\overline{D}_r f(x) = \max\{D_r^- f(x), D_r^+ f(x)\}.$$

We define the L^r-*lower derivate* of f at x, $\underline{D}_r f(x)$, by

$$\underline{D}_r f(x) = \min\{D_{-,r} f(x), D_{+,r} f(x)\}.$$

If $\underline{D}_r f(x) = \overline{D}_r f(x)$ we say that f *has an* L^r-*derivative* and denote this common value by $f_r'(x)$.

Definition 3 ([3]). Let $1 \le r < \infty$. A function $f \in L^r[a,b]$ is said to be L^r-*continuous* at $x_0 \in [a,b]$ if

$$\int_{[a,b]\cap[x_0-h,x_0+h]} |f(x) - f(x_0)|^r dx = o(h).$$

Points of L^r-continuity are, of course, Lebesgue points of f.

Definition 4 ([3]). Let $f : [a,b] \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. A finite function $\Psi \in L^r$, $1 \le r < \infty$, is said to be an L^r-*major function* of f if

(1) $\Psi(a) = 0$,
(2) Ψ is L^r-continuous on $[a,b]$,
(3) for nearly every $x \in [a,b]$ we have

$$\underline{D}_r \Psi(x) > -\infty, \qquad \underline{D}_r \Psi(x) \ge f(x).$$

DEFINITION 5 ([3]). Let $f : [a,b] \to \mathbb{R}$. A finite function $\Phi \in L^r$, $1 \le r < \infty$, is said to be an *L^r-minor function* of f if $-\Phi$ is an L^r-major function of $-f$.

THEOREM 3 ([3]). *Let $1 \le r < \infty$. Suppose that Ψ is an L^r-major function and Φ an L^r-minor function of f on $[a,b]$. Then $\Psi - \Phi$ is non-decreasing in $[a,b]$.*

DEFINITION 6 ([3]). Let $1 \le r < \infty$. Let $f : [a,b] \to \overline{\mathbb{R}}$. If

$$\inf\{\Psi(b) : \Psi \text{ is an } L^r\text{-major function of } f\} = \sup\{\Phi(b) : \Phi \text{ is an } L^r\text{-minor function of } f\},$$

then the common value, denoted

$$(P_r)\int_a^b f,$$

is called the *P_r-integral* of f on $[a,b]$. If the P_r-integral exists we write $f \in P_r[a,b]$.

THEOREM 4 ([3]). *For $1 \le r < \infty$, the P_r-integral integrates all L^r-derivatives.*

REMARK 1 ([3]). There exist functions which are P_r-integrable but not Perron integrable.

3. The HK_r-integral. We recall some elementary notions from the theory of the HK-integral. Our notation in the main follows that of [4]. A *gauge function* is a strictly positive function on $[a,b]$. A *tagged interval* is a pair $(x,[c,d])$ where $x \in [c,d] \subseteq [a,b]$. We say that $(x,[c,d])$ is *subordinate* to δ, and write $(x,[c,d]) \prec \delta$, if $[c,d] \subset [a,b] \cap (x - \delta(x), x + \delta(x))$. If $\mathcal{P}$ is a finite collection of non-overlapping tagged intervals, each of which is subordinate to δ, then $\mathcal{P}$ is said to be *subordinate* to δ, and we write $\mathcal{P} \prec \delta$.

DEFINITION 7. For $1 \le r < \infty$, a real-valued function f is *L^r-Henstock–Kurzweil integrable* ($f \in HKI_r[a,b]$) if there exists a function $F \in L^r[a,b]$ so that for any $\varepsilon > 0$ there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i,[c_i,d_i])\}$ of non-overlapping tagged intervals in $[a,b]$ with

$$\mathcal{P} \prec \delta \tag{3.1}$$

we have

$$\sum_{i=1}^{n} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon. \tag{3.2}$$

If (3.1) implies (3.2) we say that δ is *HK_r-appropriate* for ε and f. We also say that F is *an HK_r-integral* of f.

We want to say that F in the definition above is *the* HK_r-integral of f. To do so we need to show that (3.2) determines F up to an additive constant.

If F_1 and F_2 are two L^r functions which satisfy (3.2) and $G = F_1 - F_2$ then

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|G(y)-G(x_i)|^r\,dy\right)^{1/r}\\ &\leq \sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F_1(y)-F_1(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}\\ &\quad+\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F_2(y)-F_2(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}\\ &< 2\varepsilon.\end{aligned}$$

We will show that any function, F, which satisfies (3.2) is L^r-continuous, and therefore approximately continuous, so that the following theorem proves the uniqueness, up to an additive constant, of F in (3.2).

THEOREM 5. *Suppose $F \in L^r[a,b]$, $1 \leq r < \infty$, is an approximately continuous function on $[a,b]$ and that $\alpha,\beta \in F([a,b])$ where $|\alpha-\beta| > 1$. Then for any gauge function, δ, there exists $\mathcal{P} = \{(x_m,[u_m,v_m])\} \prec \delta$ so that*

$$\sum_{m=1}^{N}\left(\frac{1}{v_m-u_m}\int_{u_m}^{v_m}|F(x)-F(x_m)|^r\,dx\right)^{1/r} > \frac{1}{12}. \tag{3.3}$$

The proof of the theorem will require some intermediate results.

We begin by recalling the definition of approximate continuity.

DEFINITION 8. A function, f, defined on $[a,b]$ is *approximately continuous* at x_0 if there exists $E \subseteq (a,b)$ so that x_0 is a point of density of E and $f|_E$ is continuous at x_0.

An application of Chebyshev's inequality and Theorem 14.5 of [4] prove:

THEOREM 6. *If f is L^r-continuous at x_0 then it is approximately continuous at x_0.*

We will make use of the well-known fact that approximately continuous functions on an interval have the Darboux property. See for example Theorem 14.9 in [4].

THEOREM 7. *Let $1 \leq r < \infty$. Assume that $f \in HKI_r[a,b]$ and let F be an HK_r-integral of f. Then F is L^r-continuous everywhere.*

Proof. Let $\varepsilon > 0$ and $x \in [a,b]$ be given. Let δ be a gauge function which is HK_r-appropriate for ε and f and so that $|f(x)|\delta(x) < \varepsilon$. Then

$$\left(\frac{1}{2\delta(x)}\int\limits_{x-\delta(x)}^{x+\delta(x)} |F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} \le \varepsilon.$$

Thus

$$\begin{aligned}\left(\int\limits_{x-\delta(x)}^{x+\delta(x)} |F(y)-F(x)|^r\,dy\right)^{1/r}& \\ &\le \left(\int\limits_{x-\delta(x)}^{x+\delta(x)} |F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} \\ &\quad + \left(\int\limits_{x-\delta(x)}^{x+\delta(x)} |f(x)(y-x)|^r\,dy\right)^{1/r} \\ &\le \varepsilon[2\delta(x)]^{1/r} + |f(x)|\delta(x)[2\delta(x)]^{1/r} < 2\varepsilon[2\delta(x)]^{1/r}. \ \blacksquare\end{aligned}$$

The essence of the assertion (3.3) is an estimate of the average deviation of a function by the measure of its range. We therefore want to relate $\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i} |F(x)-F(x_i)|^r\,dx\right)^{1/r}$ to $|F(v_i)-F(u_i)|$. Since

$$\begin{aligned}(3.4)\qquad |F(v_i)-F(u_i)| &= \left(\frac{1}{v_i-u_i}\int\limits_{u_i}^{v_i} |F(v_i)-F(u_i)|^r\,dx\right)^{1/r} \\ &\le \left(\frac{1}{v_i-u_i}\int\limits_{u_i}^{v_i} |F(x)-F(u_i)|^r\,dx\right)^{1/r} \\ &\quad + \left(\frac{1}{v_i-u_i}\int\limits_{u_i}^{v_i} |F(x)-F(v_i)|^r\,dx\right)^{1/r},\end{aligned}$$

at least one of

$$\left(\frac{1}{v_i-u_i}\int\limits_{u_i}^{v_i} |F(x)-F(u_i)|^r\,dx\right)^{1/r}, \qquad \left(\frac{1}{v_i-u_i}\int\limits_{u_i}^{v_i} |F(x)-F(v_i)|^r\,dx\right)^{1/r}$$

is at least as large as half of $|F(v_i)-F(u_i)|$. We want therefore to have the freedom to choose the tag for $[u_i,v_i]$ to be either u_i or v_i. This leads us to the concept of *doubly subordinate* tagged intervals:

Definition 9. Let δ be a gauge function on $[a,b]$ and let $[u,v] \subseteq [a,b]$. We will say that $[u,v]$ is *doubly subordinate* to δ, and write $[u,v] \prec\prec \delta$, if $(u,[u,v])$ and $(v,[u,v])$ are tagged intervals subordinate to δ. In other words,

$[u,v] \prec\prec \delta$ if both $[u,v] \subseteq (u-\delta(u), u+\delta(u))$ and $[u,v] \subseteq (v-\delta(v), v+\delta(v))$. We denote by $\mathcal{D} = \mathcal{D}(\delta)$ the collection of all doubly subordinate intervals.

There exist gauge functions for which we cannot find doubly subordinate partitions of the interval. For example take any gauge on $[0,1]$ so that for all $x \neq 1/2$ we have $\delta(x) < |x-1/2|$. Any partition which is subordinate to such a gauge can have only $x = 1/2$ as the tag for the interval which contains $1/2$ and so this interval cannot be doubly subordinate to δ. However, the following theorem proves that for any δ, the difficulty is confined to a countable set of points. For nearly every $x \in [a,b]$, in any open interval about x, there exist uncountably many $y > x$ so that $[x,y] \in \mathcal{D}$ and uncountably many $z < x$ so that $[z,x] \in \mathcal{D}$:

THEOREM 8. *Let $\delta > 0$ be a given gauge function on $[a,b]$. Let*

$$R_{x,n} = \{y : y - \delta(y) < x < y < x + \min(\delta(x), n^{-1})\},$$
$$L_{x,n} = \{y : x - \min(\delta(x), n^{-1}) < y < x < y + \delta(y)\}.$$

Then for nearly every $x \in [a,b]$, $R_{x,n}$ and $L_{x,n}$ are uncountable for all n.

Proof. Let

$$E = \{x \in [a,b] : \exists n \text{ so that } \#R_{x,n} \leq \aleph_0\}.$$

If $\#E > \aleph_0$ then there exists $\eta > 0$ so that $\#\{x \in E : \delta(x) > \eta\} > \aleph_0$. Define

$$E_1 = \{x \in E : \delta(x) > \eta\}.$$

There exists N so that $\#\{x \in E_1 : \#R_{x,N} \leq \aleph_0\} > \aleph_0$. We can assume $N^{-1} < \eta$. Define

$$E_2 = \{x \in E_1 : \#R_{x,N} \leq \aleph_0\}$$

There exists $I \subseteq [a,b]$ such that $\lambda(I) < N^{-1}$ and $\#(E_2 \cap I) > \aleph_0$. Let $x_0 \in E_2 \cap I$ be such that $\#(E_2 \cap I \cap (x_0, b)) > \aleph_0$. For all $y \in E_2 \cap I \cap (x_0, b)$ we have

$$0 < y - x_0 < N^{-1} < \eta < \delta(y).$$

Thus $y - \delta(y) < x_0$, contradicting the claim that $\#R_{x_0,N} \leq \aleph_0$. The same proof shows that for n.e. $x \in [a,b]$, $\#L_{x,n} > \aleph_0$ for all n. ∎

We can now prove Theorem 5.

Proof of Theorem 5. It suffices to prove the theorem for $r = 1$. We can assume that $F(a) = 0$ and $F(b) = 1$. Since F is approximately continuous, it has the Darboux property, so that $[0,1] \subseteq F[a,b]$.

For nearly every $x \in [a,b]$ there exist $z_n \searrow x$ so that for each n we have $[x, z_n] \in \mathcal{D}$. Let

$$Z = \{x \in [a,b] : \exists u \in F^{-1}(F(x)) \text{ so that } \#R_{u,n} \leq \aleph_0 \text{ for some } n > 0\}.$$

Since $F(Z)$ is countable, $\lambda(F(Z^{\mathrm{c}})) = \lambda(F[a,b]) \geq 1$.

Also define

$$E = \{x \in Z^{c} : F(x) \in [0,1] \;\&\; \lambda[F^{-1}(F(x))] = 0\}. \tag{3.5}$$

Let

$$N = \{y \in [0,1] : \lambda(F^{-1}(y)) > 0\}.$$

Then $\#N \leq \aleph_0$ and $F(E) \cup (N \cap F(Z^{c})) = F(Z^{c}) \cap [0,1]$ and therefore $\lambda(F(E)) = 1$.

We are going to divide E into two sets. In one we will have all points, x, so that $F(y) \neq F(x)$ for all y in an open interval to the right of x. The second set will consist of all other points in E. Let

$$A = \{x \in E : \exists z \in (x,b) \text{ so that } F^{-1}(F(x)) \cap (x,z) = \emptyset\},$$
$$B = E \setminus A.$$

For $x \in A$ let $z_n(x) \searrow x$ be such that $[x, z_n(x)] \in \mathcal{D}$ and $F(z_n(x)) \neq F(x)$. Since $[x, z_n(x)] \in \mathcal{D}$ we are free to choose for the interval $[x, z_n(x)]$ a tag, x_n, which is either x or $z_n(x)$, so that (3.4) applies and

$$\frac{1}{z_n(x) - x} \int_x^{z_n(x)} |F(w) - F(x_n)|\,dw \geq \frac{1}{2}\,|F(z_n(x)) - F(x)|. \tag{3.6}$$

Now consider $x \in B$. In this case we may fail to find $z_n(x) \searrow x$ such that both $[x, z_n(x)] \in \mathcal{D}$ and $F(z_n(x)) \neq F(x)$. We therefore need a different argument.

Since $x \in B$, there exists a sequence $\{t_n(x)\}$ with $t_1(x) < x + \delta(x)$ and $t_n(x) \searrow x$ so that $F(x) = F(t_n(x))$. We claim that for each n we can find $\xi_n(x)$ so that $x < \xi_n(x) < t_n(x)$ and

$$\frac{1}{\xi_n(x) - x} \int_x^{\xi_n(x)} F \neq F(x). \tag{3.7}$$

If for some $h > 0$ and for all $\xi \in (x, x+h)$, we had $\int_x^{\xi} F = F(x)(\xi - x)$, then differentiating with respect to ξ, we would have $F(\xi) = F(x)$ at almost every $\xi \in (x, x+h)$, and by the approximate continuity of F, $F(\xi) = F(x)$ for all $\xi \in (x, x+h)$. But $x \in E$, so that $F^{-1}(F(x))$ does not contain an interval, a contradiction. Thus there exists a sequence $\{\xi_n(x)\}$ so that $\xi_n(x) \searrow x$ and (3.7) holds. Since $\xi_n(x) \searrow x$, we can also assume $x < \xi_n(x) < t_n(x)$.

Let us see that there exists $\eta_n(x) = \eta_n(x, \xi_n(x))$ so that $x < \eta_n(x) < \xi_n(x)$ and

$$F(\eta_n(x)) = \frac{1}{\xi_n(x) - x} \int_x^{\xi_n(x)} F \neq F(x). \tag{3.8}$$

Since F cannot be a constant in $(x, \xi_n(x))$, the interval

$$(\inf_{x \le w \le \xi_n(x)} F(w), \sup_{x \le w \le \xi_n(x)} F(w)) \tag{3.9}$$

is not empty. The average

$$\frac{1}{\xi_n(x) - x} \int_x^{\xi_n(x)} F$$

is in

$$[\inf_{x \le w \le \xi_n(x)} F(w), \sup_{x \le w \le \xi_n(x)} F(w)] \tag{3.10}$$

but if it is an endpoint of (3.10) then F is a constant a.e. in $(x, \xi_n(x))$ and by the approximate continuity of F, everywhere in $(x, \xi_n(x))$, which is ruled out. Thus the average is in (3.9) and since F has the Darboux property, it assumes every value in (3.9) and there exists $\eta_n(x)$ with $x < \eta_n(x) < \xi_n(x)$ so that (3.8) holds. We therefore have

$$0 < |F(\eta_n(x)) - F(x)| \le \frac{1}{\xi_n(x) - x} \int_x^{\xi_n(x)} |F(w) - F(x)| \, dw.$$

We have at this point associated with each $x \in B$ a sequence of intervals, $\{[x, \xi_n(x)]\}$, converging to x, and a sequence of intervals on the vertical axis, $\{[F(\eta_n(x)) \wedge F(x), F(\eta_n(x)) \vee F(x)]\}$, so that the average deviations of the function on the horizontal intervals are bounded from below by the lengths of the corresponding intervals on the vertical axis.

Similarly, by (3.6), we have associated with each $x \in A$ a sequence of intervals, $\{[x, z_n(x)]\}$, converging to x, and a sequence of intervals on the vertical axis, $\{[F(z_n(x)) \wedge F(x), F(z_n(x)) \vee F(x)]\}$, so that the average deviations of the function on the horizontal intervals are bounded from below by half the lengths of the corresponding intervals on the vertical axis.

For $a \le x < z \le b$, define

$$Q(x, z) = Q(x, z; F) = [F(x) \wedge F(z), F(x) \vee F(z)],$$
$$Q_n(x) = \begin{cases} Q(x, z_n(x)) & \text{if } x \in A \text{ and } n \in \mathbb{Z}_+, \\ Q(x, \eta_n(x)) & \text{if } x \in B \text{ and } n \in \mathbb{Z}_+. \end{cases}$$

Using a selection process related to the Vitali Covering Lemma, we make an effective selection from the vertical intervals, which in turn makes certain that the average deviation of the function on the horizontal intervals is large.

Define

$$\gamma_n(r) = \begin{cases} z_n(r) & \text{if } r \in A, \\ t_n(r) & \text{if } r \in B, \end{cases} \qquad H_1 = \{[r, \gamma_n(r)] : r \in E,\ n \in \mathbb{Z}_+\}.$$

Since every $y \in F(E)$ is an endpoint of an interval in

$$V_1 := \{Q_n(r) : r \in E,\, n \in \mathbb{Z}_+\},$$

V_1 covers $F(E)$. Let

$$t_1^* = \sup\{\lambda(Q) : Q \in V_1\}.$$

If $t_1^* > 1/6$, choose $Q_{n_1}(u_1) \in V_1$ so that $\lambda(Q_{n_1}(u_1)) > 1/6$ and stop collecting intervals. Otherwise, choose $Q_{n_1}(u_1) \in V_1$ so that $\lambda(Q_{n_1}(u_1)) > t_1^*/2$.

Let H_2 be the collection of all intervals $[r, \gamma_n(r)] \in H_1$ so that either $[r, \gamma_n(r)] \times Q_n(r)$ is to the right and above $[u_1, \gamma_{n_1}(u_1)] \times Q_{n_1}(u_1)$, or $[r, \gamma_n(r)] \times Q_n(r)$ is to the left and below $[u_1, \gamma_{n_1}(u_1)] \times Q_{n_1}(u_1)$.

In other words, $[r, \gamma_n(r)] \in H_2$ iff $[r, \gamma_n(r)] \in H_1$ and either

$$r > \gamma_{n_1}(u_1) \quad \text{and} \quad \min Q_n(r) > \max Q_{n_1}(u_1),$$

or

$$\gamma_n(r) < u_1 \quad \text{and} \quad \max Q_n(r) < \min Q_{n_1}(u_1).$$

Observe that the definition of H_2 implies that if $[r, \gamma_n(r)] \in H_2$ then both $[r, \gamma_n(r)] \cap [u_1, y_{n_1}(u_1)] = \emptyset$ and $Q_n(r) \cap Q_{n_1}(u_1) = \emptyset$.

Define also

$$V_2 = \{Q_n(u) : u \in E \text{ and } [u, \gamma_n(u)] \in H_2\}.$$

Let us show that $F(E) \setminus 5Q_{n_1}(u_1)$ is covered by intervals in V_2.

Observe that from the definition of E (see (3.5)),

$$F^{-1}(F(E)) = E. \tag{3.11}$$

If there exists

$$c \in \big[[0, \min Q_{n_1}(u_1)) \cap F(E)\big] \setminus 5Q_{n_1}(u_1)$$

then applying the argument which we used for the interval $[a, b]$ to the interval $[a, u_1]$ (observe that this includes the application of the Darboux property of F on $[a, u_1]$), there exists $u_{1,1} \in [a, u_1)$ so that $F(u_{1,1}) = c$. From (3.11) it follows that $u_{1,1} \in E$.

There exists $n_{1,1}$ so that $\gamma_{n_{1,1}}(u_{1,1}) < u_1$ and $\lambda(Q_{n_{1,1}}(u_{1,1})) > 0$. Since $c \notin 5Q_{n_1}(u_1)$ and since

$$\lambda(Q_{n_{1,1}}(u_{1,1})) \le t_1^* < 2\lambda(Q_{n_1}(u_1)),$$

we have $Q_{n_{1,1}}(u_{1,1}) \cap Q_{n_1}(u_1) = \emptyset$, which shows

$$\min Q_{n_1}(u_1) - \max Q_{n_{1,1}}(u_{1,1}) > 0$$

and so $[u_{1,1}, \gamma_{n_{1,1}}(u_{1,1})] \in H_2$. To summarize, $c = F(u_{1,1})$ and $c \in Q_{n_{1,1}}(u_{1,1}) \in V_2$.

Consider

$$c \in \big[(\max Q_{n_1}(u_1), 1] \cap F(E)\big] \setminus 5Q_{n_1}(u_1).$$

There are two cases: $u_1 \in A$ and $u_1 \in B$.

Consider $u_1 \in A$. The existence of c implies $\max Q_{n_1}(u_1) < 1 = F(b)$. If $u_1 \in A$ then $\gamma_{n_1}(u_1) = z_{n_1}(u_1)$. Since $F(z_{n_1}(u_1)) \le \max Q_{n_1}(u_1) < c$, by the Darboux property of F, there exists $u_{1,2} \in (\gamma_{n_1}(u_1), b] \cap E$ so that $F(u_{1,2}) = c$. If $u_1 \in B$ then $\gamma_{n_1}(u_1) = t_{n_1}(u_1)$ so that $F(\gamma_{n_1}(u_1)) = F(u_1) \le \max Q_{n_1}(u_1) < c$ and so, by the Darboux property of F, there exists $u_{1,2} \in (\gamma_{n_1}(u_1), b] \cap E$ so that $F(u_{1,2}) = c$. As in the previous argument we have $[u_{1,2}, \gamma_{n_{1,2}}(u_{1,2})] \in H_2$.

Let

$$t_2^* = \sup\{\lambda(Q) : Q \in V_2\}$$

and choose $[u_2, \gamma_{n_2}(u_2)] \in H_2$ so that

$$\lambda(Q_{n_2}(u_2)) > \frac{1}{2}\, t_2^*.$$

We proceed inductively, and as in the proof of the Vitali Lemma we get a sequence of disjoint intervals, $\{Q_{n_m}(u_m)\}$, so that $[u_m, \gamma_{n_m}(u_m)]$ are disjoint and

$$\sum_{m=1}^{\infty} \lambda(Q_{n_m}(u_m)) \ge \frac{1}{5}.$$

There exists N so that

$$\sum_{m=1}^{N} \lambda(Q_{n_m}(u_m)) > \frac{1}{6}.$$

For all $x \in E$ we set

$$y_n(x) = \begin{cases} z_n(x) & \text{if } x \in A, \\ \xi_n(x) & \text{if } x \in B. \end{cases}$$

We now assign a tag, x_m, to each $[u_m, y_{n_m}(u_m)]$. If $u_m \in B$ we have seen that

$$\begin{aligned} \lambda(Q_{n_m}(u_m)) &= |F(\eta_{n_m}(u_m)) - F(u_m)| \\ &\le \frac{1}{\xi_{n_m}(u_m) - u_m} \int_{u_m}^{\xi_{n_m}(u_m)} |F(w) - F(u_m)|\, dw \end{aligned}$$

so that we take $x_m = u_m$. Observe that $\xi_{n_m}(u_m) < u_m + \delta(u_m)$ so that $(u_m, [u_m, \xi_{n_m}(u_m)]) \prec \delta$.

If $u_m \in A$ then $[u_m, z_{n_m}(u_m)] \prec\prec \delta$ so that we are free to choose $x_m = u_m$ or $x_m = z_{n_m}(u_m)$. By (3.4) we can choose the tag so that

$$\begin{aligned} \frac{1}{2}\, \lambda(Q_{n_m}(u_m)) &= \frac{1}{2}\, |F(u_m) - F(z_{n_m}(u_m))| \\ &\le \frac{1}{z_{n_m}(u_m) - u_m} \int_{u_m}^{z_{n_m}(u_m)} |F(w) - F(x_m)|\, dw. \end{aligned}$$

Therefore, $\mathcal{P} := \{(x_m, [u_m, y_{n_m}(u_m)])\} \prec \delta$ and

$$\sum_{m=1}^{N} \frac{1}{y_{n_m}(u_m) - u_m} \int\limits_{u_m}^{y_{n_m}(u_m)} |F(w) - F(x_m)|\, dw > \frac{1}{12}. \ \blacksquare$$

Having shown that for $1 \le r < \infty$ the indefinite HK_r-integral of f is defined up to an additive constant, we can define

$$(HK_r)\int\limits_a^b f = F(b) - F(a)$$

with F as in (3.2).

It is easy to see that if $f, g \in HKI_r[a,b]$ then $\alpha f + \beta g \in HKI_r[a,b]$ and

$$(HK_r)\int\limits_a^b (\alpha f + \beta g) = \Big(\alpha \cdot (HK_r)\int\limits_a^b f\Big) + \Big(\beta \cdot (HK_r)\int\limits_a^b g\Big)$$

Let us see that the HK_r-integral generalizes the HK-integral.

THEOREM 9. *Let $1 \le r < \infty$, $f \in HKI[a,b]$, and $F(x) := (HK)\int_a^x f$. Then $f \in HKI_r[a,b]$ and $F(x) = (HK_r)\int_a^x f$. Moreover if δ is such that for all $\mathcal{P} = \{(x_j, [c_j, d_j])\} \prec \delta$ we have*

$$\sum_j |F(d_j) - F(c_j) - f(x_j)(d_j - c_j)| < \varepsilon$$

then for all such $\mathcal{P}$ we also have

$$\sum_j \bigg(\frac{1}{d_j - c_j}\int\limits_{c_j}^{d_j} |F(y) - F(x_j) - f(x_j)(y - x_j)|^r\, dy\bigg)^{1/r} < \varepsilon.$$

Proof. Consider

$$\sum_j \bigg(\frac{1}{d_j - c_j}\int\limits_{c_j}^{d_j} |F(y) - F(x_j) - f(x_j)(y - x_j)|^r\, dy\bigg)^{1/r}.$$

For each j the integrand is a continuous function of y so that in each interval $[c_j, d_j]$ there exists $y_j \neq x_j$ so that for all $y \in [c_j, d_j]$ we have

$$|F(y) - F(x_j) - f(x_j)(y - x_j)| \le |F(y_j) - F(x_j) - f(x_j)(y_j - x_j)|.$$

We define $\mathcal{P}' := \{x_j, [x_j \wedge y_j, x_j \vee y_j]\}$. Since $\mathcal{P}' \prec \delta$,

$$\sum_j \bigg(\frac{1}{d_j - c_j}\int\limits_{c_j}^{d_j} |F(y) - F(x_j) - f(x_j)(y - x_j)|^r\, dy\bigg)^{1/r} \le \sum_j |F(y_j) - F(x_j) - f(x_j)(y_j - x_j)| < \varepsilon. \ \blacksquare$$

Assume that f and g are two real-valued functions, $f = g$ almost everywhere, and $f \in HKI_r[a,b]$. Let us write $g = f + h$ so that $h = 0$ almost everywhere. This implies that h is Lebesgue integrable and $\int h = 0$. Therefore $h \in HKI[a,b]$ and by the previous theorem $h \in HKI_r[a,b]$ and

$$(HK_r)\int_a^b h = 0.$$

By the additivity of the HK_r-integral we have $g \in HKI_r[a,b]$, and the HK_r-integrals of f and g are equal.

We therefore define the HK_r-integral for functions that are defined only almost everywhere on $[a,b]$: given f which is defined on $[a,b] \setminus Z$ where $\lambda(Z) = 0$ we define

$$g(x) := \begin{cases} f(x) & \text{if } x \notin Z, \\ 0 & \text{if } x \in Z, \end{cases}$$

and say that $f \in HKI_r[a,b]$ iff $g \in HKI_r[a,b]$. We also define

$$(HK_r)\int_a^b f = (HK_r)\int_a^b g.$$

It is also easy to see that if $f \in HKI_r[a,b]$ and if $[c,d]$ is a subinterval of $[a,b]$, then $f \in HKI_r[c,d]$.

We next consider convergence theorems for the HK_r-integral.

THEOREM 10. *Let $1 \le r < \infty$. Assume that $\{f_n\} \in HKI_r[a,b]$ and that $\{f_n\}$ converges uniformly to a function f. Let*

$$F_n(x) = (HK_r)\int_a^x f_n$$

and assume that $\{F_n\}$ converges uniformly to a function F. Then $f \in HKI_r[a,b]$ and

$$F(x) = (HK_r)\int_a^x f.$$

Proof. Given $\varepsilon > 0$ we choose n so that both

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon \quad \text{and} \quad \sup_{x\in[a,b]} |F_n(x) - F(x)| < \varepsilon.$$

Let δ be an HK_r-appropriate gauge for ε and f_n and let $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$. Then

$$\sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r\, dy\right)^{1/r}$$

$$\le \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F(y) - F_n(y) - (F(x_i) - F_n(x_i))|^r\,dy\right)^{1/r}$$
$$+ \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F_n(y) - F_n(x_i) - f_n(x_i)(y - x_i)|^r\,dy\right)^{1/r}$$
$$+ \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |(f_n(x_i) - f(x_i))(y - x_i)|^r\,dy\right)^{1/r}$$
$$\le 2\varepsilon + \varepsilon + \varepsilon(b-a). \ \blacksquare$$

Theorem 5 enables us to extend to the HK_r-integral a convergence theorem for the HK-integral.

Definition 10. Let $1 \le r < \infty$. Assume that $\{f_n\}$ is a sequence of $HKI_r[a,b]$ functions. We say that $\{f_n\}$ is *uniformly in* $HKI_r[a,b]$, and write $\{f_n\} \in UHKI_r[a,b]$, if for each $\varepsilon > 0$ there exists a gauge function which is HK_r-appropriate for ε and f_n, for all $n \ge 1$.

Theorem 11. *Let $1 \le r < \infty$. Assume that $\{f_n\} \in UHKI_r[a,b]$ and that $\{f_n\}$ converges uniformly to f. Let*

$$F_n(x) = (HK_r)\int_a^x f_n.$$

Then $\{F_n\}$ converges uniformly to a function F, and $F = (HK_r)\int f$.

Proof. Let $\varepsilon > 0$ be given and let N be such that if $m, n > N$ then for all $x \in [a,b]$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Let δ be an HK_r-appropriate gauge function for ε and for all f_n, and let $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$. Then for all $m, n > N$,

$$\sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |(F_n - F_m)(y) - (F_n - F_m)(x_i)|^r\,dy\right)^{1/r}$$
$$\le \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F_n(y) - F_n(x_i) - f_n(x_i)(y - x_i)|^r\,dy\right)^{1/r}$$
$$+ \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F_m(y) - F_m(x_i) - f_m(x_i)(y - x_i)|^r\,dy\right)^{1/r}$$
$$+ \sum_i \left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |(f_m - f_n)(x_i)(y - x_i)|^r\,dy\right)^{1/r}$$
$$< 2\varepsilon + \varepsilon(b-a).$$

This, by Theorem 5, implies

$$\sup_{x\in[a,b]} |F_n(x) - F_m(x)| \le 12[2\varepsilon + \varepsilon(b-a)].$$

The claim $F(x) = (HK_r)\int_a^x f$ follows from Theorem 10. ∎

THEOREM 12. *Let $1 \le r < \infty$. If $f \in HKI_r[a,b]$ and F is the HK_r-integral of f then the L^r-derivative of F, F_r', is equal to f a.e. on $[a,b]$.*

Proof. If either $F_r'(x)$ does not exist or else it exists but is not equal to $f(x)$ then

$$\overline{\lim_{h\to 0^+}} \frac{1}{h^{r+1}} \int_{x-h}^{x+h} |F(y) - F(x) - f(x)(y-x)|^r\,dy > 0. \tag{3.12}$$

Suppose that the set of points where (3.12) holds has positive exterior measure. Then there exists $\eta > 0$ so that

$$A_\eta := \left\{x : \overline{\lim_{h\to 0^+}} \frac{1}{2h^{r+1}} \int_{x-h}^{x+h} |F(y) - F(x) - f(x)(y-x)|^r\,dy > \eta^r\right\}$$

satisfies $\lambda_{\mathrm{e}}(A_\eta) > 0$. Thus for each $x \in A_\eta$ there exist arbitrarily small h so that

$$\left(\frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x) - f(x)(y-x)|^r\,dy\right)^{1/r} > \eta h. \tag{3.13}$$

Let $0 < \alpha < \lambda_{\mathrm{e}}(A_\eta)$. Since F is the HK_r-integral of f, there exists a gauge function δ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$ then

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r\,dy\right)^{1/r} < \eta\alpha.$$

But

$$\{[x-h, x+h] : x \in A_\eta,\ 0 < h < \delta(x),\ (3.13)\text{ holds}\}$$

is a Vitali cover of A_η. Thus we can find a disjoint subcollection

$$\{[x_i-h_i, x_i+h_i] : x_i \in A_\eta \text{ and } 0 < h_i < \delta(x_i),\ (3.13)\text{ holds}\}$$

with $\sum_{i=1}^n h_i > \alpha$. Thus

$$\sum_{i=1}^{n} \left(\frac{1}{2h_i} \int_{x_i-h_i}^{x_i+h_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r\,dy\right)^{1/r} > \sum_{i=1}^{n} \eta h_i > \eta\alpha,$$

a contradiction. ∎

Let us see that the HK_r-integral integrates all L^r-derivatives. We do that by showing that it integrates all P_r-integrable functions.

THEOREM 13. *For $1 \le r < \infty$, $P_r[a,b] \subseteq HKI_r[a,b]$.*

Proof. Let $f \in P_r[a,b]$. We can assume that f is finite-valued. Given $\varepsilon > 0$ there exist an L^r-major function, Ψ, and an L^r-minor function, Φ, of f so that $0 \le \Psi(b) - \Phi(b) < \varepsilon$. We know that $\underline{D}_r\Psi(x) \ge f(x)$ n.e. and similarly $\overline{D}_r\Phi(x) \le f(x)$ n.e. Define

$$F(x) = (P_r)\int_a^x f, \qquad H(x) = \Psi(x) - F(x), \qquad J(x) = F(x) - \Phi(x).$$

We have already mentioned that $\Psi - \Phi$ is a non-decreasing function. It follows that H and J are non-decreasing functions as well. Indeed, let $a \le x_1 < x_2 \le b$. Thus $\Psi(x) - \Psi(x_1)$ is an L^r-major function for f on $[x_1, x_2]$. Let

$$G(x) = (P_r)\int_{x_1}^x f = F(x) - F(x_1).$$

Thus $\Psi(x_2) - \Psi(x_1) \ge G(x_2) = F(x_2) - F(x_1)$, which implies $\Psi(x_2) - F(x_2) \ge \Psi(x_1) - F(x_1)$ and $H \nearrow$. The proof that $J \nearrow$ is the same.

We define the gauge function. For x outside a countable set, $Z = \{z_k\}$, there exists $\delta(x) > 0$ so that if $0 < h < \delta(x)$ we have the following four inequalities:

$$\int_x^{x+h} [\Psi(y) - \Psi(x) - f(x)(y-x)]_-^r \, dy < \varepsilon^r h^{r+1}, \tag{3.14}$$

$$\int_{x-h}^{x} [\Psi(y) - \Psi(x) - f(x)(y-x)]_+^r \, dy < \varepsilon^r h^{r+1}, \tag{3.15}$$

$$\int_x^{x+h} [\Phi(y) - \Phi(x) - f(x)(y-x)]_+^r \, dy < \varepsilon^r h^{r+1}, \tag{3.16}$$

$$\int_{x-h}^{x} [\Phi(y) - \Phi(x) - f(x)(y-x)]_-^r \, dy < \varepsilon^r h^{r+1}. \tag{3.17}$$

On Z we define $\delta(z_k) > 0$ to be such that both

$$\delta(z_k) < \frac{\varepsilon 2^{-k}}{1 + |f(z_k)|}$$

and for all $0 < h_1, h_2 < \delta(z_k)$,

$$\left(\frac{1}{h_1 + h_2}\int_{z_k - h_1}^{z_k + h_2} |\Psi(y) - \Psi(z_k)|^r \, dy\right)^{1/r} + \left(\frac{1}{h_1 + h_2}\int_{z_k - h_1}^{z_k + h_2} |\Phi(y) - \Phi(z_k)|^r \, dy\right)^{1/r} < \varepsilon 2^{-k}.$$

The last condition is possible since Ψ and Φ are L^r-continuous everywhere. Let $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$. We then have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}$$
$$= \sum_{x_i \in Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}$$
$$+ \sum_{x_i \notin Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}.$$

Consider $(x_i, [c_i, d_i])$ with $x_i \in Z$:

$$\left(\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}$$
$$\leq \left(\int_{c_i}^{d_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} + |f(x_i)| \left(\int_{c_i}^{d_i} |y - x_i|^r \, dy \right)^{1/r}$$
$$\leq \left(\int_{c_i}^{d_i} |\Phi(y) - \Phi(x_i)|^r \, dy \right)^{1/r} + \left(\int_{c_i}^{d_i} |J(y) - J(x_i)|^r \, dy \right)^{1/r}$$
$$+ |f(x_i)| \left(\int_{c_i}^{d_i} |y - x_i|^r \, dy \right)^{1/r}.$$

Consider each one of the last three terms:

$$\sum_{x_i \in Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |\Phi(y) - \Phi(x_i)|^r \, dy \right)^{1/r} \leq \varepsilon \sum_{i=1}^{\infty} 2^{-i} < \varepsilon.$$

The second term:

$$\sum_{x_i \in Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |J(y) - J(x_i)|^r \, dy \right)^{1/r}$$
$$\leq \sum_{x_i \in Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{x_i} (J(x_i) - J(y))^r \, dy \right)^{1/r}$$
$$+ \sum_{x_i \in Z} \left(\frac{1}{d_i - c_i} \int_{x_i}^{d_i} (J(y) - J(x_i))^r \, dy \right)^{1/r}$$

$$\leq \sum_{x_i\in Z}\left(\left(\frac{x_i-c_i}{d_i-c_i}\right)^{1/r}(J(x_i)-J(c_i))+\left(\frac{d_i-x_i}{d_i-c_i}\right)^{1/r}(J(d_i)-J(x_i))\right)$$
$$\leq \sum_{x_i\in Z}(J(d_i)-J(c_i))\leq J(b)-J(a)<\varepsilon.$$

The last term:

$$\sum_{x_i\in Z}|f(x_i)|\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|y-x_i|^r\,dy\right)^{1/r}\leq \sum_{x_i\in Z}|f(x_i)|(d_i-c_i)\leq \sum_i \varepsilon 2^{-i}<\varepsilon.$$

Therefore

$$\sum_{x_i\in Z}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}<3\varepsilon.$$

Consider $(x_i,[c_i,d_i])$ with $x_i\notin Z$:

$$\sum_{x_i\notin Z}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}$$
$$\leq \sum_{x_i\notin Z}\left(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}$$
$$+\sum_{x_i\notin Z}\left(\frac{1}{d_i-c_i}\int_{x_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}.$$

Consider first

$$\sum_{x_i\notin Z}\left(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}.$$

Using the L^r-major and L^r-minor functions, we obtain

$$\left(\int_{c_i}^{x_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r}$$
$$\leq\left(\int_{c_i}^{x_i}[F(y)-F(x_i)-f(x_i)(y-x_i)]_+^r\,dy\right)^{1/r}$$
$$+\left(\int_{c_i}^{x_i}[F(y)-F(x_i)-f(x_i)(y-x_i)]_-^r\,dy\right)^{1/r}$$

$$\le \Big(\int_{c_i}^{x_i}[\Psi(y)-\Psi(x_i)-f(x_i)(y-x_i)]_+^r\,dy\Big)^{1/r}+\Big(\int_{c_i}^{x_i}[H(y)-H(x_i)]_-^r\,dy\Big)^{1/r}$$
$$+\Big(\int_{c_i}^{x_i}[\Phi(y)-\Phi(x_i)-f(x_i)(y-x_i)]_-^r\,dy\Big)^{1/r}+\Big(\int_{c_i}^{x_i}[J(y)-J(x_i)]_-^r\,dy\Big)^{1/r}.$$

We consider each one of the last four terms. By (3.15),

$$\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[\Psi(y)-\Psi(x_i)-f(x_i)(y-x_i)]_+^r\,dy\bigg)^{1/r}$$
$$\le\sum_{x_i\notin Z}\bigg(\frac{\varepsilon^r}{d_i-c_i}(x_i-c_i)^{r+1}\bigg)^{1/r}\le\varepsilon\sum_{x_i\notin Z}(x_i-c_i)\le\varepsilon(b-a).$$

Since $[H(y)-H(x_i)]_-=H(x_i)-H(y)$ for $y\in[c_i,x_i]$, we have

$$\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[H(y)-H(x_i)]_-^r\,dy\bigg)^{1/r}$$
$$=\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[H(x_i)-H(y)]^r\,dy\bigg)^{1/r}$$
$$\le\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[H(x_i)-H(c_i)]^r\,dy\bigg)^{1/r}$$
$$\le\sum_{x_i\notin Z}[H(x_i)-H(c_i)]\le H(b)-H(a)<\varepsilon.$$

By (3.17),

$$\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[\Phi(y)-\Phi(x_i)-f(x_i)(y-x_i)]_-^r\,dy\bigg)^{1/r}$$
$$\le\varepsilon\sum_{x_i\notin Z}(x_i-c_i)\le\varepsilon(b-a).$$

Since $[J(y)-J(x_i)]_-=J(x_i)-J(y)$ for $y\in[c_i,x_i]$, we have

$$\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[J(y)-J(x_i)]_-^r\,dy\bigg)^{1/r}$$
$$=\sum_{x_i\notin Z}\bigg(\frac{1}{d_i-c_i}\int_{c_i}^{x_i}[J(x_i)-J(y)]^r\,dy\bigg)^{1/r}$$

$$\le \sum_{x_i \notin Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{x_i} [J(x_i) - J(c_i)]^r \, dy \right)^{1/r}$$

$$\le \sum_{x_i \notin Z} [J(x_i) - J(c_i)] \le J(b) - J(a) < \varepsilon.$$

Therefore

$$\sum_{x_i \notin Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{x_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \le C\varepsilon.$$

The estimate of

$$\sum_{x_i \notin Z} \left(\frac{1}{d_i - c_i} \int_{x_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}$$

follows in the same manner and we have

$$\sum_{x_i \notin Z} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \le C\varepsilon. \blacksquare$$

4. HK_r-absolute continuity. We present a concept of absolute continuity which characterizes indefinite HK_r-integrals. We write

$$\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec_E \delta$$

if $\mathcal{P} \prec \delta$ and $x_i \in E$ for all i.

Definition 11. Let $1 \le r < \infty$. We say that $F \in AC_r(E)$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge function $\delta(x)$ defined on E so that for all $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec_E \delta$ such that $\sum_{i=1}^{q}(d_i - c_i) < \eta$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

Definition 12. Let $1 \le r < \infty$. We say that $F \in ACG_r(E)$ if E can be written as $E = \bigcup_{n=1}^{\infty} E_n$ and $F \in AC_r(E_n)$ for all n.

Theorem 14. *Let $1 \le r < \infty$. Then $f \in HKI_r[a,b]$ iff there exists $F \in ACG_r[a,b]$ so that $F'_r = f$ a.e.*

Proof. If $f \in HKI_r[a,b]$ then there exists an F so that $F'_r = f$ a.e. and for any $\varepsilon > 0$ there exists a gauge function, δ, HK_r-appropriate for ε and f. Define

$$E_n = \{x \in [a,b] : n - 1 \le |f(x)| < n\}.$$

Clearly $[a,b] = \bigcup_{n=1}^{\infty} E_n$. Fix n. Let $\mathcal{P}_n = \{(x_{n,i}, [c_{n,i}, d_{n,i}])\} \prec_{E_n} \delta$ satisfy

$$\sum_{i=1}^{q_n}(d_{n,i}-c_{n,i}) < \varepsilon/n.$$

Then

$$\begin{aligned}\sum_{i=1}^{q_n}&\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|F(y)-F(x_{n,i})|^r\,dy\right)^{1/r}\\ &\le \sum_{i=1}^{q_n}\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|F(y)-F(x_{n,i})-f(x_{n,i})(d_{n,i}-c_{n,i})|^r\,dy\right)^{1/r}\\ &\quad+\sum_{i=1}^{q_n}\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|f(x_{n,i})(d_{n,i}-c_{n,i})|^r\,dy\right)^{1/r}.\end{aligned}$$

But, since $f\in HKI_r[a,b]$, δ is HK_r-appropriate for ε and f, and $\mathcal{P}_n\prec\delta$, we have

$$\sum_{i=1}^{q_n}\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|F(y)-F(x_{n,i})-f(x_{n,i})(d_{n,i}-c_{n,i})|^r\,dy\right)^{1/r}<\varepsilon$$

so that

$$\begin{aligned}\sum_{i=1}^{q_n}&\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|F(y)-F(x_{n,i})|^r\,dy\right)^{1/r}\\ &\le\varepsilon+\sum_{i=1}^{q_n}|f(x_{n,i})|(d_{n,i}-c_{n,i})\le\varepsilon+n\,\frac{\varepsilon}{n}=2\varepsilon.\end{aligned}$$

Thus $F\in AC_r(E_n)$ for all n.

Conversely, suppose that there exists $F\in ACG_r[a,b]$ and $F'_r=f$ a.e. and show that F is the HK_r-integral of f. Let

$$E=\{x\in[a,b]:F'_r(x)=f(x)\}.$$

Let $\varepsilon>0$ be given. By Lemma 9.15 of [4] there exists δ_0 so that if $\mathcal{P}=\{(x_i,[c_i,d_i])\}\prec_{E^{\mathrm{c}}}\delta_0$ then $|f|(\mathcal{P})<\varepsilon$. Since $F\in ACG_r[a,b]$ there exist E_n, disjoint, so that $\bigcup_{n=1}^{\infty}E_n=E^{\mathrm{c}}$ and $F\in AC_r(E_n)$ for all n. For each n define δ_n on E_n and η_n so that

$$\sum_{i=1}^{q_n}\left(\frac{1}{d_{n,i}-c_{n,i}}\int_{c_{n,i}}^{d_{n,i}}|F(y)-F(x_{n,i})|^r\,dy\right)^{1/r}<\varepsilon 2^{-n}$$

whenever $\mathcal{P}_n=\{(x_{n,i},[c_{n,i},d_{n,i}])\}\prec_{E_n}\delta_n$ and

$$\sum_{i=1}^{q_n}(d_{n,i}-c_{n,i})<\eta_n.$$

Each E_n is a set of measure 0. We therefore choose open sets O_n so that $E_n \subseteq O_n$ and $\lambda(O_n) < \eta_n$. For each $x \in E^{\mathrm{c}}$, there exists a unique n so that $x \in E_n$. For $x \in E^{\mathrm{c}}$, we define

$$\delta(x) = \min\{\delta_0(x), \delta_n(x), \operatorname{dist}(x, O_n^{\mathrm{c}})\}.$$

For all $x \in E$ there exists $\delta(x) > 0$ so that for all $0 < h < \delta(x)$,

$$\frac{1}{h_1 + h_2} \int_{x-h_1}^{x+h_2} |F(y) - F(x) - f(x)(y - x)|^r \, dy < \varepsilon^r (h_1 + h_2)^r.$$

Let $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$. Then

$$\begin{aligned}
&\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \\
&\quad = \sum_{\{i \,:\, x_i \in E\}} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \\
&\qquad + \sum_{\{i \,:\, x_i \in E^{\mathrm{c}}\}} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \\
&\quad \le \sum_{\{i \,:\, x_i \in E\}} (d_i - c_i)\varepsilon + \sum_{\{i \,:\, x_i \in E^{\mathrm{c}}\}} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} \\
&\qquad + \sum_{\{i \,:\, x_i \in E^{\mathrm{c}}\}} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \\
&\quad \le (b - a)\varepsilon + \sum_{n=1}^{\infty} \sum_{\{i \,:\, x_i \in E_n\}} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} \\
&\qquad + \sum_{\{i \,:\, x_i \in E^{\mathrm{c}}\}} |f(x_i)|(d_i - c_i) \le \varepsilon(b - a + 2). \ \blacksquare
\end{aligned}$$

We denote the space of absolutely continuous functions on $[a, b]$ by $AC[a, b]$.

Theorem 15. *For any* $1 \le r < \infty$, $AC_r[a, b] = AC[a, b]$.

Proof. Let us first show that $AC[a, b] \subseteq AC_r[a, b]$. Let $F \in AC[a, b]$, and let $\varepsilon > 0$ be given. There exists $\eta > 0$ such that if $\{[a_j, b_j]\}$ is a finite collection of non-overlapping subintervals of $[a, b]$ so that $\sum_{j=1}^{q} (b_j - a_j) < \eta$ then $\sum_{j=1}^{q} |F(b_j) - F(a_j)| < \varepsilon$. This implies a seemingly stronger statement that if $\sum_{j=1}^{q}(b_j - a_j) < \eta$ then

$$\sum_{j=1}^{q}(\max_{x\in[a_j,b_j]} F(x) - \min_{x\in[a_j,b_j]} F(x)) < \varepsilon.$$

Thus for any choice of $x_j \in [a_j, b_j]$,

$$\sum_{j=1}^{q}\left(\frac{1}{b_j - a_j}\int_{a_j}^{b_j} |F(y) - F(x_j)|^r \, dy\right)^{1/r} \leq \sum_{j=1}^{q}(\max_{x\in[a_j,b_j]} F(x) - \min_{x\in[a_j,b_j]} F(x)) < \varepsilon.$$

Observe that this holds for any gauge function δ.

For the converse it suffices to show that $AC_1[a,b] \subseteq AC[a,b]$. Assume that $F \in AC_1[a,b]$ and let $\varepsilon > 0$. There exist $\eta > 0$ and a gauge function δ defined on $[a,b]$ so that if $\mathcal{P} = \{(x_n, [c_n, d_n])\} \prec \delta$ and $\sum_{n=1}^{q}(d_n - c_n) < \eta$ then

$$\sum_{n=1}^{q}\frac{1}{d_n - c_n}\int_{c_n}^{d_n} |F(y) - F(x_n)| \, dy < \varepsilon.$$

Let $\{[c_n, d_n]\}$ be any finite collection of non-overlapping intervals so that $\sum_{n=1}^{q}(d_n - c_n) < \eta$. The function F is L^1-continuous and so certainly approximately continuous. By Theorem 5 there exist $\mathcal{P}_n := \{(x_{n,i}, [c_{n,i}, d_{n,i}])\} \prec \delta$, where $[c_{n,i}, d_{n,i}] \subseteq [c_n, d_n]$ for all n and all i, so that

$$\sum_{i=1}^{q_n}\frac{1}{d_{n,i} - c_{n,i}}\int_{c_{n,i}}^{d_{n,i}} |F(y) - F(x_{n,i})| \, dy \geq \frac{1}{12}\,|F(d_n) - F(c_n)|.$$

Since $\mathcal{P} := \bigcup_{n=1}^{q}\mathcal{P}_n$ is subordinate to δ, we have

$$\sum_{n=1}^{q}|F(d_n) - F(c_n)| \leq 12\sum_{n=1}^{q}\sum_{i=1}^{q_n}\frac{1}{d_{n,i} - c_{n,i}}\int_{c_{n,i}}^{d_{n,i}} |F(y) - F(x_{n,i})| \, dy < 12\varepsilon. \blacksquare$$

Definition 13 ([6]). We say that F is *absolutely continuous* on E, and write $F \in AC(E)$, if for all $\varepsilon > 0$ there exists $\eta > 0$ so that for all finite collections of non-overlapping intervals $[a_j, b_j]$ such that $a_j, b_j \in E$ and $\sum_j (b_j - a_j) < \eta$ we have

$$\sum_j |F(b_j) - F(a_j)| < \varepsilon.$$

Note that if E is a finite set then all functions are absolutely continuous on E.

THEOREM 16. *If $F \in AC_1(E)$ then we can find E_n so that $E = \bigcup_{n=1}^{\infty} E_n$ and $F \in AC(E_n)$ for all n.*

Proof. Since $F \in AC_1(E)$, for each $\varepsilon > 0$ there exists $\eta > 0$ and a gauge function δ on E so that

$$\sum_i \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|\, dy < \varepsilon$$

whenever $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec_E \delta$ and $\sum_i (d_i - c_i) < \eta$. Let

$$S_n = \{x \in E : \delta(x) > 1/n\}.$$

Then $E = \bigcup_{n=1}^{\infty} S_n$. Let $c_n = \inf S_n$ and $d_n = \sup S_n$. Fix n. Let q_n be a sufficiently large integer so that

$$\frac{d_n - c_n}{q_n} < \min\{1/n, \eta\}.$$

Let

$$y_{n,j} = c_n + j\,\frac{d_n - c_n}{q_n} \qquad \text{for } j = 0, \ldots, q_n - 1.$$

Let $\{[s_{n,j,i}, t_{n,j,i}]\}_i$ be a collection of non-overlapping intervals with endpoints in $S_n \cap [y_{n,j}, y_{n,j+1}]$. Then

$$\begin{aligned}
\sum_i |F(t_{n,j,i}) - F(s_{n,j,i})| &= \sum_i \frac{1}{t_{n,j,i} - s_{n,j,i}} \int_{s_{n,j,i}}^{t_{n,j,i}} |F(t_{n,j,i}) - F(s_{n,j,i})|\, dy \\
&\le \sum_i \frac{1}{t_{n,j,i} - s_{n,j,i}} \int_{s_{n,j,i}}^{t_{n,j,i}} |F(y) - F(s_{n,j,i})|\, dy \\
&\quad + \sum_i \frac{1}{t_{n,j,i} - s_{n,j,i}} \int_{s_{n,j,i}}^{t_{n,j,i}} |F(y) - F(t_{n,j,i})|\, dy.
\end{aligned}$$

But since $t_{n,j,i} - s_{n,j,i} < \min\{\delta(s_{n,j,i}), \delta(t_{n,j,i})\}$ we can consider the intervals $[s_{n,j,i}, t_{n,j,i}]$ as tagged intervals with tags at $s_{n,j,i}$ or at $t_{n,j,i}$. Thus

$$\sum_i \left(\frac{1}{t_{n,j,i} - s_{n,j,i}} \int_{s_{n,j,i}}^{t_{n,j,i}} |F(y) - F(s_{n,j,i})|^r\, dy \right)^{1/r} < \varepsilon$$

and similarly

$$\sum_i \left(\frac{1}{t_{n,j,i} - s_{n,j,i}} \int_{s_{n,j,i}}^{t_{n,j,i}} |F(y) - F(t_{n,j,i})|^r\, dy \right)^{1/r} < \varepsilon.$$

Thus $F \in AC(S_n \cap [y_{n,j}, y_{n,j+1}])$ and

$$E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{q_n} (S_n \cap [y_{n,j}, y_{n,j+1}]). \ \blacksquare$$

COROLLARY 1. *Let $1 \le r < \infty$. If $F \in ACG_r[a,b]$ then we can find E_n so that $[a,b] = \bigcup_{n=1}^{\infty} E_n$ and $F \in AC(E_n)$ for all n* (1).

THEOREM 17 ([4, p. 97]). *If $F \in AC(E)$ then F satisfies Lusin's condition N, in other words, if $Z \subseteq E$ and $\lambda(Z) = 0$ then $\lambda(F(E)) = 0$.*

THEOREM 18. *Let $1 \le r < \infty$. If F is an indefinite HK_r-integral then F satisfies Lusin's condition N.*

Proof. We have shown that if $F(x) = (HK_r)\int_a^x f$ then $F \in ACG_r[a,b]$ and that this implies that we can find E_n so that $[a,b] = \bigcup_{n=1}^{\infty} E_n$ and $F \in AC(E_n)$ and so satisfies Lusin's condition N on each E_n. If $Z \subseteq [a,b]$ and $\lambda(Z) = 0$ then

$$\lambda(F(Z)) = \lambda\Big(F\Big(\bigcup_{n=1}^{\infty} Z \cap E_n\Big)\Big) = \lambda\Big(\bigcup_{n=1}^{\infty} F(Z \cap E_n)\Big)$$
$$\le \sum_{n=1}^{\infty} \lambda(F(Z \cap E_n)) = 0. \ \blacksquare$$

The HK_r-integral of a non-negative function is non-decreasing.

DEFINITION 14. We denote by $D(A)$ the set of points of density of A.

DEFINITION 15 ([4, p. 249]). The *lower approximate derivate* of F at a point, x_0, is defined by

$$\underline{D}_{\mathrm{app}} F(x_0) = \sup\Big\{\beta : x_0 \in D\Big(\Big\{x : \frac{F(x) - F(x_0)}{x - x_0} > \beta\Big\}\Big)\Big\}.$$

THEOREM 19 ([3]). *Assume that $E' \subseteq [a,b]$ is such that $F(E')$ contains no intervals. Assume also that for all $x \in [a,b] \setminus E'$,*

$$\underline{D}_{\mathrm{app}} F(x) \ge 0$$

and that F is approximately continuous for all $x \in E'$. Then $F \nearrow$ on $[a,b]$.

Proof. The theorem follows from Lemma 4 and Remark 1 of [3]. ■

DEFINITION 16. A function, F, is *approximately differentiable* at x_0 if there exists a number, A, so that the function

$$G(x) := \begin{cases} \dfrac{F(x) - F(x_0)}{x - x_0} & \text{if } x \ne x_0, \\ A & \text{if } x = x_0, \end{cases}$$

(1) This is not quite $F \in ACG[a,b]$ since for $F \in ACG[a,b]$ we also postulate that F is continuous. See [4, p. 90].

is approximately continuous at x_0. We then say that A is the *approximate derivative* of F at x_0 and denote it by $D_{\mathrm{app}}F(x_0)$.

It is easy to see that if F is approximately differentiable at x_0 then $\underline{D}_{\mathrm{app}}F(x_0) = D_{\mathrm{app}}F(x_0)$.

Clearly A is the L^r-derivative of F at x_0 iff the function G is L^r-continuous at x_0. By Theorem 6, if F is L^r-differentiable at x_0, then G is approximately continuous at x_0, which proves that F is approximately differentiable at x_0 and the approximate and L^r-derivatives are equal.

THEOREM 20. *Let $1 \le r < \infty$. Suppose that $f \in HKI_r[a,b]$ and*

$$F(x) = (HK_r)\int_a^x f.$$

If $f \ge 0$ a.e. then $F \nearrow$ in $[a,b]$.

Proof. We have shown that $F_r' = f$ a.e. and so $D_{\mathrm{app}}F = f \ge 0$ a.e. The complement, E', of $\{D_{\mathrm{app}}F \ge 0\}$ is a set of measure 0 and since we have shown that F satisfies condition N we see that $F(E')$ is a set of measure 0 and so contains no intervals. Finally, since F is approximately continuous, $F \nearrow$ in $[a,b]$ by Theorem 19. ∎

THEOREM 21. *Suppose that $f \in HKI_r[a,b]$, $1 \le r < \infty$, and $f \ge 0$ a.e. Then $f \in L^1[a,b]$.*

Proof. Let

$$F(x) = (HK_r)\int_a^x f.$$

We have shown that $F \nearrow$. By Lebesgue's theorem F is differentiable a.e. and $F' \in L^1[a,b]$. By Theorem 12, f is the L^r-derivative of F a.e. We saw that f is therefore also the approximate derivative of F. The approximate and the usual derivatives are clearly consistent so that $f = F' \in L^1[a,b]$. ∎

The next two theorems show that if $f \in HKI_r[a,b]$ then we can choose a measurable gauge function which is HK_r-appropriate for ε and f.

THEOREM 22. *Let $1 \le r < \infty$. Assume that E is a measurable subset of $[a,b]$ and that f is a measurable function on $[a,b]$. Suppose that there exists F defined on $[a,b]$ so that $F_r'(x) = f(x)$ for all $x \in E$. Let $\varepsilon > 0$ be given. For each $x \in E$, let $\delta(x)$ be the supremum of all η so that for closed intervals $I \subseteq [a,b]$ with $\lambda(I) < \eta$ and $x \in I$ we have*

$$\left(\frac{1}{\lambda^{r+1}(I)}\int_I |F(y) - F(x) - f(x)(y-x)|^r\,dy\right)^{1/r} \le \varepsilon.$$

Then $\delta : E \to (0,\infty)$ is a measurable function.

Proof. We can assume that $E \subseteq (a,b)$. Let $r > 0$ and define $A = \{x \in E : \delta(x) \geq r\}$. Let $\{I_k\}$ be the sequence of all closed intervals in $[a,b]$ with rational endpoints which satisfy $\lambda(I_k) < r$. For each k, define

$$H_k = \left\{ x \in I_k : \left(\frac{1}{\lambda^{r+1}(I_k)} \int_{I_k} |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} \leq \varepsilon \right\},$$
$$G_k = (([a,b] \setminus I_k) \cup H_k) \cap E.$$

Since f and F are measurable functions, G_k are measurable sets.

Let us show that $A = \bigcap_{k=1}^{\infty} G_k$. Let $x \in A$, that is to say, $\delta(x) \geq r$. For each k if $x \notin I_k$ then $x \in G_k$; if $x \in I_k$, then since $\delta(x) \geq r$ and $\lambda(I_k) < r$, we have

$$\left(\frac{1}{\lambda^{r+1}(I_k)} \int_{I_k} |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} \leq \varepsilon$$

and so $x \in G_k$. Thus $A \subseteq \bigcap_{k=1}^{\infty} G_k$.

Conversely, if $x \in \bigcap_{k=1}^{\infty} G_k$, then if I is any closed subinterval of $[a,b]$ so that $x \in I$ and $\lambda(I) < r$ choose a subsequence $\{I_{k_j}\}$ so that $I_{k_j} \to I$ and $x \in I_{k_j}$ for all j. We have

$$\begin{aligned}
&\left(\frac{1}{\lambda^{r+1}(I)} \int_{I} |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} \\
&\qquad = \lim_{j \to \infty} \left(\frac{1}{\lambda^{r+1}(I_{k_j})} \int_{I_{k_j}} |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} \leq \varepsilon
\end{aligned}$$

so that $\delta(x) \geq r$. Thus A is a measurable set and hence δ is a measurable function. ∎

THEOREM 23. *Let $1 \leq r < \infty$. Suppose $f \in HKI_r[a,b]$ and*

$$F(x) = (HK_r) \int_a^x f.$$

Then for any $\varepsilon > 0$ there exists a measurable gauge function, δ, so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\} \prec \delta$, then

$$\sum_{i=1}^{n} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

Proof. Let $E = \{x \in [a,b] : F_r'(x) = f(x)\}$ so that $\lambda(E^{\mathrm{c}}) = 0$. Let δ_1 be the measurable gauge function defined on E as in the previous theorem. Let

$\delta_2 = \frac{1}{2}\delta_1$, so that whenever $(x,[c,d]) \prec_E \delta_2$ we have

$$\left(\frac{1}{d-c}\int\limits_c^d |F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} \le \varepsilon(d-c).$$

Let δ_3 be a gauge function defined on E^{c} (see Theorem 14) so that if $\mathcal{P} = \{(x_i,[c_i,d_i])\} \prec_{E^{\mathrm{c}}} \delta_3$, then

$$\sum_{i=1}^n \left(\frac{1}{d_i-c_i}\int\limits_{c_i}^{d_i} |F(y)-F(x_i)|^r\,dy\right)^{1/r} < \varepsilon, \qquad \sum_{i=1}^n |f(x_i)|(d_i-c_i) < \varepsilon.$$

Define

$$\delta(x) = \begin{cases} \delta_2(x) & \text{if } x \in E, \\ \delta_3(x) & \text{if } x \in E^{\mathrm{c}}. \end{cases}$$

Clearly δ is measurable. Let $\mathcal{P} = \{(x_i,[c_i,d_i])\}$ be a partition of $[a,b]$ which is subordinate to δ. Then

$$\begin{aligned} &\sum_i \left(\frac{1}{d_i-c_i}\int\limits_{c_i}^{d_i} |F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} \\ &\quad\le \sum_{\{i\,:\,x_i\in E\}} \left(\frac{1}{d_i-c_i}\int\limits_{c_i}^{d_i} |F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} \\ &\qquad + \sum_{\{i\,:\,x_i\in E^{\mathrm{c}}\}} \left(\frac{1}{d_i-c_i}\int\limits_{c_i}^{d_i} |F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} \\ &\quad\le \varepsilon(b-a) + \sum_{\{i\,:\,x_i\in E^{\mathrm{c}}\}} \left(\frac{1}{d_i-c_i}\int\limits_{c_i}^{d_i} |F(y)-F(x_i)|^r\,dy\right)^{1/r} \\ &\qquad + \sum_{\{i\,:\,x_i\in E^{\mathrm{c}}\}} |f(x_i)|(d_i-c_i) \\ &\quad< \varepsilon(b-a+2). \ \blacksquare \end{aligned}$$

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Received May 6, 2002
Revised version November 18, 2002 (4939)