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Note on distortion and Bourgain ℓ_1-index

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Abstract. Relations between different notions measuring proximity to ℓ_1 and distortability of a Banach space are studied. The main result states that a Banach space all of whose subspaces have Bourgain ℓ_1-index greater than ω^α, $\alpha < \omega_1$, contains either an arbitrarily distortable subspace or an ℓ_1^α-asymptotic subspace.

1. Preliminaries. The study of asymptotic properties and in particular complexity of the family of copies of ℓ_1^n in Banach spaces is closely related to investigating their distortability (cf. [17, 16, 19]). Investigation of arbitrarily distortion of Banach spaces is concentrated mainly on ℓ_1-asymptotic spaces. The first tool measuring the way ℓ_1 is represented in a Banach space is provided by the Bourgain ℓ_1-index. Another approach is given by higher order spreading models, studied extensively in mixed and modified mixed Tsirelson spaces. The ℓ_1-asymptoticity of higher order of a Banach space can be measured by the constants introduced in [19].

We present here an observation, in the spirit of the theorem of [17] recalled below, relating distortability of a Banach space to the "proximity" to ℓ_1 measured by the tools presented above.

THEOREM 1.1 ([17]). *Let X be a Banach space. Then X contains either an arbitrarily distortable subspace or an ℓ_p-asymptotic ($1 \leq p < \infty$) or c_0-asymptotic subspace.*

Our main result states that a Banach space with a basis whose block subspaces all have Bourgain ℓ_1-block index greater than ω^α, contains either an arbitrarily distortable subspace or an ℓ_1^α-asymptotic subspace. In particular, a space saturated with ℓ_1^α-spreading models generated by block sequences contains either an arbitrarily distortable subspace or an ℓ_1^α-asymptotic subspace. This implies Theorem 2.1 of [15]. Analogous results also hold in the c_0 case. As a corollary we show that the "stabilized" (with respect to block subspaces) Bourgain ℓ_1-block index of a space with bounded distortion not

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containing ℓ_1 is of the form ω^{ω^γ} for some non-limit $\gamma < \omega_1$. Let us also recall here the result of [21] stating that a Banach space with bounded distortion contains an unconditional basic sequence; the proof uses the notion of unconditional ordinal index.

We now recall the basic definitions and standard notation. Let X be a Banach space with a basis (e_i). The *support* of a vector $x = \sum_i x_i e_i$ is the set $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$. We shall also use the *interval support* of a vector $x \in X$, the smallest interval in $\mathbb{N}$ containing the support of x, and denote it by $\overline{\operatorname{supp}}\, x$.

Given any $x = \sum_i x_i e_i$ and finite $E \subset \mathbb{N}$ put $Ex = \sum_{i \in E} x_i e_i$. We write $x < y$ for vectors $x, y \in X$ if $\max(\operatorname{supp} x) < \min(\operatorname{supp} y)$. A *block sequence* is any sequence $(x_i) \subset X$ satisfying $x_1 < x_2 < \cdots$, and a *block subspace* of X is any closed subspace spanned by an infinite block sequence. If Y is a block subspace of X spanned by a block sequence (y_i) then Y_n, $n \in \mathbb{N}$, denotes the "tail" subspace spanned by $(y_i)_{i \geq n}$, and EY, $E \subset \mathbb{N}$, denotes the subspace spanned by $(y_i)_{i \in E}$.

A basic sequence $(x_1, \ldots, x_k)$ in a Banach space is *K-equivalent to the unit vector basis of k-dimensional ℓ_1* (resp. c_0) *space*, for some $K \geq 1$, if for any scalars $a_1, \ldots, a_k$ we have $K\|a_1 x_1 + \cdots + a_k x_k\| \geq |a_1| + \cdots + |a_k|$ (resp. $\|a_1 x_1 + \cdots + a_k x_k\| \leq K \max\{|a_1|, \ldots, |a_k|\}$).

Definition 1.2. A Banach space $(X, \|\cdot\|)$ is *λ-distortable*, for $\lambda > 1$, if there is an equivalent norm $|\cdot|$ on X such that for any infinite-dimensional subspace Y of X,

$$\sup\{|x|/|y| : x, y \in Y,\ \|x\| = \|y\| = 1\} \geq \lambda.$$

A Banach space X is *arbitrarily distortable* if it is λ-distortable for any $\lambda > 1$.

A Banach space X has *D-bounded distortion* if for any equivalent norm $|\cdot|$ and any infinite-dimensional subspace Y of X there is a further infinite-dimensional subspace Z of Y such that $|x|/\|x\| \leq D|y|/\|y\|$ for any non-zero $x, y \in Z$. A Banach space has *bounded distortion* if it has D-bounded distortion for some $D \geq 1$.

Notice that any Banach space X contains either an arbitrarily distortable subspace or a subspace with bounded distortion.

Given any $M \subset \mathbb{N}$, let $[M]^{<\infty}$ denote the family of finite subsets of M. A family $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ is *regular* if it is: *hereditary*, i.e. for any $G \subset F$, $F \in \mathcal{F}$ implies $G \in \mathcal{F}$; *spreading*, i.e. for any integers $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ with $n_i \leq m_i$, $i = 1, \ldots, k$, if $(n_1, \ldots, n_k) \in \mathcal{F}$ then also $(m_1, \ldots, m_k) \in \mathcal{F}$; and *compact* in the product topology of $2^{\mathbb{N}}$. If $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ is compact, let $\mathcal{F}'$ denote the set of limit points of $\mathcal{F}$. Define inductively $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})'$ for any ordinal α and $\mathcal{F}^{\alpha} = \bigcap_{\xi < \alpha} \mathcal{F}^{(\xi)}$ for any limit ordinal α. Set $\iota(\mathcal{F}) = \inf\{\alpha : \mathcal{F}^{(\alpha+1)} = \emptyset\}$.

A *tree on a set* S is a subset $\mathcal{T}$ of $\bigcup_{n=1}^{\infty} S^n$ such that $(x_1,\dots,x_k)\in\mathcal{T}$ whenever $(x_1,\dots,x_k,x_{k+1})\in\mathcal{T}$, $k\in\mathbb{N}$. A tree $\mathcal{T}$ is *well-founded* if there is no infinite sequence $(x_i)\subset S$ with $(x_1,\dots,x_k)\in\mathcal{T}$ for any $k\in\mathbb{N}$. Given a tree $\mathcal{T}$ on S put

$$D(\mathcal{T})=\{(x_1,\dots,x_k):(x_1,\dots,x_k,x)\in\mathcal{T}\text{ for some }x\in S\}.$$

Inductively define trees $D^\alpha(\mathcal{T})$: $D^0(\mathcal{T})=\mathcal{T}$, $D^{\alpha+1}=D(D^\alpha(\mathcal{T}))$ and $D^\alpha(\mathcal{T})=\bigcap_{\xi<\alpha}D^\xi(\mathcal{T})$ for α limit. The *order* of a well-founded tree $\mathcal{T}$ is given by the formula $o(\mathcal{T})=\inf\{\alpha:D^\alpha(\mathcal{T})=\emptyset\}$.

Let X be a Banach space with a basis. For $K\ge 1$, a tree $\mathcal{T}$ on X is an *ℓ_1-K-block tree on X* if any $(x_1,\dots,x_k)\in\mathcal{T}$ is a normalized block sequence K-equivalent to the unit vector basis of k-dimensional ℓ_1 space. An *ℓ_1-block tree* on X is an ℓ_1-K-block tree on X for some $K\ge 1$.

Let $I_{\rm b}(X,K)=\sup\{o(\mathcal{T}):\mathcal{T}\text{ is an }\ell_1\text{-}K\text{-block tree on }X\}$ for $K\ge 1$. The *Bourgain ℓ_1-block index of* X is defined by $I_{\rm b}(X)=\sup\{I_{\rm b}(X,K):K\ge 1\}$.

THEOREM 1.3 ([10]). *Let X be a Banach space with a basis not containing ℓ_1. Then $I_{\rm b}(X)=\omega^\alpha$ for some $\alpha<\omega_1$ and $I_{\rm b}(X)>I_{\rm b}(X,K)$ for any $K\ge 1$.*

REMARK 1.4. Recall the close relation ([10]) between $I_{\rm b}(X)$ and $I(X)$, the original Bourgain ℓ_1-index defined as the block index but by trees of not necessarily block sequences: for $I(X)\ge\omega^\omega$ we have $I_{\rm b}(X)=I(X)$, and if $I(X)=\omega^{n+1}$ for some $n\in\mathbb{N}$, then $I_{\rm b}(X)=\omega^{n+1}$ or ω^n.

The *generalized Schreier families* $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ of finite subsets of $\mathbb{N}$, introduced in [1], are defined by transfinite induction. First,

$$\mathcal{S}_0=\{\{n\}:n\in\mathbb{N}\}\cup\{\emptyset\}.$$

Suppose the families $\mathcal{S}_\xi$ are defined for all $\xi<\alpha$. If $\alpha=\beta+1$, put

$$\mathcal{S}_\alpha=\{F_1\cup\dots\cup F_m:m\in\mathbb{N},\ F_1,\dots,F_m\in\mathcal{S}_\beta,\ m\le F_1<\dots<F_m\}.$$

If α is a limit ordinal, choose $\alpha_n\nearrow\alpha$ and set

$$\mathcal{S}_\alpha=\{F:F\in\mathcal{S}_{\alpha_n}\text{ and }n\le F\text{ for some }n\in\mathbb{N}\}.$$

It is well known that any family $\mathcal{S}_\alpha$, $\alpha<\omega_1$, is regular with $\iota(\mathcal{S}_\alpha)=\omega^\alpha$, and considered as a tree on $\mathbb{N}$ it satisfies $o(\mathcal{S}_\alpha)=\omega^\alpha$ (cf. [1]).

Fix $\alpha<\omega_1$. A finite sequence (E_i) of subsets of $\mathbb{N}$ is *α-admissible* (resp. *α-allowable*) if $E_1<E_2<\cdots$ (resp. (E_i) are pairwise disjoint) and $(\min E_i)\in\mathcal{S}_\alpha$.

Let X be a Banach space with a basis (e_n), and fix $\alpha<\omega_1$. A finite sequence $(x_i)\subset X$ is *α-admissible* (resp. *α-allowable*) *with respect to the basis* (e_n) if $(\operatorname{supp}x_i)$ is α-admissible (resp. α-allowable).

DEFINITION 1.5. Fix $\alpha<\omega_1$. Let X be a Banach space with a basis (e_n). A normalized block sequence $(x_i)\subset X$ *generates an ℓ_1^α-spreading model* with constant $C \geq 1$ if for any $F \in \mathcal{S}_\alpha$ the sequence $(x_i)_{i\in F}$ is C-equivalent to the unit vector basis of $\sharp F$-dimensional ℓ_1 space.

The space X is ℓ_1^α-*asymptotic* (resp. ℓ_1^α-*strongly asymptotic*) with constant $C \geq 1$ if any sequence $(x_i)_{i=1}^k$ α-admissible (resp. α-allowable) with respect to (e_n) is C-equivalent to the unit vector basis of k-dimensional ℓ_1 space.

Obviously X is ℓ_1^α-asymptotic with constant C iff each normalized block sequence in X generates an ℓ_1^α-spreading model with constant C. By the properties of $\mathcal{S}_\alpha$'s, any block subspace of an ℓ_1^α-asymptotic (resp. strongly asymptotic) space with constant C is also ℓ_1^α-asymptotic (resp. strongly asymptotic) with the same constant. The relations between the Bourgain ℓ_1-block index and the notions introduced above are described by

Proposition 1.6. *Let X be a Banach space with a basis. Fix $\alpha < \omega_1$.*

- *If X admits an ℓ_1^α-spreading model, then $I_{\mathrm{b}}(X) > \omega^\alpha$.*
- *If X is an ℓ_1^α-asymptotic space, then $I_{\mathrm{b}}(X) \geq \omega^{\alpha\omega}$.*

Proof. The first part follows from Theorem 1.3 and the fact that $o(\mathcal{S}_\alpha) = \omega^\alpha$. The second part follows from the proof of Theorem 5.19 of [10]. We recall it briefly. For any $\mathcal{M}, \mathcal{N} \subset [\mathbb{N}]^{<\infty}$ put

$$\mathcal{M}[\mathcal{N}] = \{F_1 \cup \dots \cup F_k : F_1, \dots, F_k \in \mathcal{N},\ m_1 \leq F_1 < \dots < m_k \leq F_k \text{ for some } (m_1, \dots, m_k) \in \mathcal{M},\ k \in \mathbb{N}\}.$$

Put $[\mathcal{S}_\alpha]^n = \mathcal{S}_\alpha[\dots[\mathcal{S}_\alpha]]$ (n times). If X is ℓ_1^α-asymptotic with constant C, then any normalized block sequence $(x_1, \dots, x_k)$ with $(\min(\operatorname{supp} x_i)) \in [\mathcal{S}_\alpha]^n$ is C^n-equivalent to the unit vector basis of k-dimensional ℓ_1 space. Since $o([\mathcal{S}_\alpha]^n) = \omega^{\alpha n}$ ([1]), we have $I_{\mathrm{b}}(X) > \omega^{\alpha n}$ for any $n \in \mathbb{N}$, which ends the proof. ∎

Remark 1.7. Definition 1.5 extends the well-known notions of ℓ_1-asymptotic space (introduced in [17]) and spreading model generated by a basic sequence. The higher order ℓ_1-spreading models were introduced in [11] and investigated in [4, 14, 15]. The constants describing ℓ_1-asymptoticity of higher order were introduced and studied in [19]. The term ℓ_1-asymptoticity of higher order was explicitly introduced in [9], where also a criterion for arbitrary distortion in terms of ℓ_1-spreading models was given. The ℓ_p-strongly asymptotic spaces were introduced and studied in [6]. The Bourgain ℓ_1-index and ℓ_1-block index of various spaces in relation to existence of higher order spreading models, distortability and quasiminimality were investigated in [10, 12, 13, 14].

We shall need additional norms given by the ℓ_1-asymptoticity of the space:

Definition 1.8. Let U be a Banach space with a basis. Fix $\alpha < \omega_1$. If U is ℓ_1^α-asymptotic with constant C, define an associated norm $|\cdot|_\alpha$ on U by

$$|x|_\alpha = \sup\Big\{\sum_{i=1}^k \|E_i x\| : E_1 < \cdots < E_k \ \alpha\text{-admissible}, \ k \in \mathbb{N}\Big\}, \ x \in U.$$

Clearly $\|\cdot\| \le |\cdot|_\alpha \le C\|\cdot\|$. If U is ℓ_1^α-strongly asymptotic, we define analogously the norm $|\cdot|_\alpha^{\mathrm{s}}$ using allowable sequences instead of admissible ones.

Simpler versions of these norms were used to show arbitrary distortion of the famous Schlumprecht space, the first Banach space known to be arbitrarily distortable; these norms also distort some mixed and modified mixed Tsirelson spaces [3, 4, 14].

2. Main result. Now we present the main result, which shows that we can reverse the implication in Proposition 1.6 in spaces with bounded distortion.

Theorem 2.1. *Let X be a Banach space with a basis. Fix $\alpha < \omega_1$. Assume that $I_{\mathrm{b}}(Y) > \omega^\alpha$ for any block subspace Y of X. Then X contains either an arbitrarily distortable subspace or an ℓ_1^α-asymptotic subspace.*

If, additionally, X is ℓ_1^1-strongly asymptotic, then X contains either an arbitrarily distortable subspace or an ℓ_1^α-strongly asymptotic subspace.

By Proposition 1.6 a Banach space admitting, in any block subspace, ℓ_1^α-spreading models generated by normalized block sequences satisfies the assumption of Theorem 2.1. We also have the following corollary, implying Theorem 2.1 of [15]:

Corollary 2.2. *Fix $1 < \alpha < \omega_1$. Suppose that X is a Banach space with a basis that admits for any $\xi < \alpha$, in every block subspace, an ℓ_1^ξ-spreading model generated by a normalized block sequence with a universal constant $C \ge 1$. Then X contains either an arbitrarily distortable subspace or an ℓ_1^α-asymptotic subspace.*

If, additionally, X is ℓ_1^1-strongly asymptotic, then X contains either an arbitrarily distortable subspace or an ℓ_1^α-strongly asymptotic subspace.

Proof. Assume X has no arbitrarily distortable subspaces. If $\alpha = \beta + 1$ for some $\beta < \omega_1$, then by Theorem 2.1, there is an ℓ_1^β-asymptotic subspace W with some constant $C \ge 1$. By Proposition 3.2 of [19] there is $n_0 \in \mathbb{N}$ such that $F \in \mathcal{S}_\beta$ for any $n_0 \le F \in \mathcal{S}_1$. Thus W_{n_0} is also ℓ_1^1-asymptotic with constant C, and therefore also ℓ_1^α-asymptotic (with constant C^2).

If α is a limit ordinal, then by assumption $I_{\mathrm{b}}(Y) > I_{\mathrm{b}}(Y, C) \ge \omega^\alpha$ for any block subspace Y of X, and Theorem 2.1 ends the proof. The case of ℓ_1-strong asymptoticity follows analogously. ∎

Remark 2.3. By Lemma 6.5 (and Remark 6.6(iii)) of [10] the universal constant C (arbitrarily close to 1) in the assumption of Corollary 2.2 is automatic for $\alpha = \omega^\gamma$, with γ a limit ordinal.

Remark 2.4. We collect some known examples:

(i) $I_{\mathrm{b}}(X) > \omega$ iff 1 belongs to the Krivine set of X, i.e. ℓ_1 is finitely (almost isometrically) represented on block sequences in X.

(ii) For any $\alpha < \omega_1$ by Theorem 5.19 of [10], and Proposition 1.6, any block subspace Y of the Tsirelson type space $T(\mathcal{S}_\alpha, 1/2)$ (which is clearly ℓ_1^α-asymptotic) satisfies $I_{\mathrm{b}}(Y) = \omega^{\alpha\omega}$.

(iii) By Theorem 4.2 of [2], the mixed Tsirelson space $X = T[(\mathcal{S}_n, \theta_n)_n]$ with $\theta_n \searrow 0$ contains no ℓ_1^ω-asymptotic subspace. On the other hand, it was shown in [14] that $I_{\mathrm{b}}(Y) > \omega^\omega$ for any block subspace Y of X iff any block subspace Y admits an ℓ_1^ω-spreading model. In such a case X is arbitrarily distortable. This holds in particular if $\lim \sqrt[n]{\theta_n} = 1$.

(iv) In [13] the Bourgain ℓ_1-block index of mixed Tsirelson spaces is computed, and as a consequence it is proved that for any α not of the form ω^γ, γ a limit ordinal, there is a Banach space X_α with $I_{\mathrm{b}}(X_\alpha) = \omega^\alpha$. In particular, it is proved that (with a special choice of sequences in the definition of Schreier families) $I_{\mathrm{b}}(T(\mathcal{S}_{\beta_n}, \theta_n)_n)$ is either $\omega^{\omega^\xi 2}$ or ω^{ω^ξ}, where $\beta_n \nearrow \omega^\xi$ with $\xi < \omega_1$ successor.

Proof of Theorem 2.1. We can assume that X has a bimonotone basis. Assume X contains no arbitrarily distortable subspaces, and pick a block subspace Y of X with D-bounded distortion, for some $D \geq 1$. We restrict our considerations to Y and use transfinite induction.

The idea of the proof of the initial step and the limit case of the inductive step (the successor case is trivial) could be described as follows: we consider equivalent norms, whose uniform equivalence to the original norm would give asymptoticity of the desired order. We "glue" the norms on some special vectors provided by the high ℓ_1-index of the space (Lemmas 2.5 and 2.6), using methods now standard in the study of Tsirelson type spaces, and by bounded distortion of the space we obtain uniform equivalence of these norms to the original one on some subspace. In the proof we will also show the relation between the constants involved, which will be useful for the next corollary.

Initial step. The result for $\alpha = 1$ follows from Theorem 1.1, but we present here a shorter proof, whose idea was used in the proof of Theorem 1.1 given in [16], and whose scheme will also be applied in the inductive step. Define new equivalent norms on Y as follows:

$$\|y\|_n = \sup\Big\{\sum_{j=1}^n \|E_j y\| : E_1 < \cdots < E_n \text{ intervals}\Big\}, \qquad y \in Y,\ n \in \mathbb{N}.$$

We recall a standard observation providing vectors "gluing" the original norm and the new norms:

LEMMA 2.5. *Let U be a Banach space with a bimonotone basis. Fix $n \in \mathbb{N}$ and assume $I_{\mathrm{b}}(U) > \omega$. Then for any $\varepsilon > 0$ there is a vector $x \in U$ with $1/(1+\varepsilon) \le \|x\| \le \|x\|_n \le 1+\varepsilon$.*

Proof of Lemma 2.5. Pick $k \in \mathbb{N}$ such that $n^{2-k} < \varepsilon$ and by Remark 2.4(i) take a normalized block sequence $(x_i)_{i=1}^{n^k} \subset U$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of n^k-dimensional ℓ_1 space, and put $x = n_k^{-1} \sum_{i=1}^{n^k} x_i$. Obviously $\|x\| \ge 1/(1+\varepsilon)$. Take any $E_1 < \cdots < E_n$ and put

$$I = \{i : \min E_j \in \overline{\operatorname{supp}}\, x_i \text{ for some } j\}.$$

Since $\sharp I \le n$ we have

$$\sum_{j=1}^{n} \Big\| E_j \frac{1}{n^k} \sum_{i \in I} x_i \Big\| \le n \Big\| \frac{1}{n^k} \sum_{i \in I} x_i \Big\| \le n^{2-k} < \varepsilon.$$

Notice that $\overline{\operatorname{supp}}\, x_i$ intersects at most one E_j if $i \notin I$, hence

$$\sum_{j=1}^{n} \Big\| E_j \frac{1}{n^k} \sum_{i \notin I} x_i \Big\| \le \frac{1}{n^k} \sum_{i \notin I} \|x_i\| = 1,$$

which ends the proof of the lemma.

Now we finish the proof of the initial step. Fix $\varepsilon > 0$. Applying bounded distortability of Y and standard diagonalization pick a block subspace Z of Y such that $\|y\|_n/\|y\| \le (1+\varepsilon)D\|z\|_n/\|z\|$ for any non-zero $y, z \in Z_n$ and $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and take $x \in Z_n$ as in Lemma 2.5. By the choice of Z we have $\|y\|_n \le (1+\varepsilon)^3 D\|y\|$ for any $y \in Z_n$. By definition of $\|\cdot\|_n$,

$$\|E_1 y\| + \cdots + \|E_n y\| \le (1+\varepsilon)^3 D\|y\|$$

for any $y \in Z$ and any intervals E_i with $n \le E_1 < \cdots < E_n$, which shows that Z is ℓ_1^1-asymptotic with constant $(1+\varepsilon)^3 D$.

INDUCTIVE STEP. Take $1 < \alpha < \omega_1$ and suppose that the assertion holds true for all $\xi < \alpha$. If $\alpha = \beta + 1$ for some $\beta < \omega_1$ then by inductive hypothesis there is a block subspace W of Y which is ℓ_1^{β}-asymptotic, and thus ℓ_1^{α}-asymptotic.

If α is a limit ordinal pick $(\alpha_n)_n$ with $\alpha_n \nearrow \alpha$ as in the definition of $\mathcal{S}_\alpha$. By the inductive hypothesis we can pick a block subspace W of Y such that W is $\ell_1^{\alpha_n}$-asymptotic for any $n \in \mathbb{N}$.

Let $|\cdot|_n$, $n \in \mathbb{N}$, denote the norm on W given by $\ell_1^{\alpha_n}$-asymptoticity of W (Definition 1.8, norms defined with respect to the block basis of W). As before we will use some special vectors in order to "glue" the original norm and the new norms. Those vectors—so-called special convex combinations, introduced in [3]—are the crucial tool in studying properties of mixed and modified mixed Tsirelson spaces. In order to construct the vectors on ℓ_1-K-block trees we will slightly generalize the reasoning from Lemma 4 of [12] (cf. also [4, Lemma 4.9]).

LEMMA 2.6. *Let U be a Banach space with a bimonotone basis. Fix $1 \leq \eta < \xi < \omega_1$ and assume that U is ℓ_1^η-asymptotic and $I_{\mathrm{b}}(U, K) > \omega^\xi$ for some $K \geq 1$. Then for any $\varepsilon > 0$ there is $x \in U$ with $1/K \leq \|x\| \leq |x|_\eta \leq 1 + \varepsilon$.*

If, additionally, U is ℓ_1^1-strongly asymptotic with a constant C_1, then for any $\varepsilon > 0$ there is $x \in U$ satisfying $1/K \leq \|x\| \leq |x|_\eta^{\mathrm{s}} \leq C_1 + \varepsilon$.

The important part of the lemma is that the estimates of the norms in the assertion do not depend on the ℓ_1^η-asymptoticity constant.

Proof of Lemma 2.6. Let U be ℓ_1^η-asymptotic with constant C. Let $\mathcal{T}$ be an ℓ_1-K-block tree on U with $o(\mathcal{T}) > \omega^\xi$. We can assume that for any $(x_i) \in \mathcal{T}$ also any subsequence (x_{i_m}) is in $\mathcal{T}$. Put

$$\mathcal{F} = \{(m_1, \ldots, m_l) \subset \mathbb{N} : m_i \geq \max(\operatorname{supp} x_i),\ 1 \leq i \leq l, \\ \text{for some } (x_1, \ldots, x_l) \in \mathcal{T}\}.$$

The family $\mathcal{F}$ is hereditary and either non-compact or, by Proposition 13 of [13], compact with $\iota(\mathcal{F}) \geq o(\mathcal{T}) > \omega^\xi = \iota(\mathcal{S}_\xi)$. Hence by Theorem 1.1 of [7], there is an infinite $M \subset \mathbb{N}$ with

$$\mathcal{S}_\xi \cap [M]^{<\infty} \subset \mathcal{F}.$$

Using Proposition 3.6 of [19], we get $F \in \mathcal{S}_\xi \cap [M]^{<\infty}$ and positive scalars $(a_m)_{m\in F}$ such that $\sum_{m\in F} a_m = 1$ and $\sum_{m\in G} a_m < \varepsilon/C$ for any $G \in \mathcal{S}_\eta$ with $G \subset F$. By definition of $\mathcal{F}$ there is $(x_i) \in \mathcal{T}$ such that $F = (m_1, \ldots, m_l)$ with $m_i \geq \max(\operatorname{supp} x_i)$ for $1 \leq i \leq l$. Let $x = \sum_{m_i\in F} a_{m_i} x_i$. Since $(x_i) \in \mathcal{T}$, we have $\|x\| \geq 1/K$.

Take now any η-admissible sequence $E_1 < \cdots < E_k$. Put

$$J = \left\{j \in \{1, \ldots, k\} : \min E_j \in \overline{\operatorname{supp}}\, x_{i_j} \text{ for some } i_j\right\}.$$

Let $I = \{i_j : j \in J\}$ and split the sum of the norms as follows:

$$\sum_{j=1}^k \|E_j x\| \leq \sum_{j=1}^k \Big\|E_j \sum_{i\in I} a_{m_i} x_i\Big\| + \sum_{j=1}^k \Big\|E_j \sum_{i\notin I} a_{m_i} x_i\Big\|.$$

In order to estimate the first part of the sum notice that $G = \{m_{i_j} : j \in J\}$ belongs to $\mathcal{S}_\eta$ since $(\min E_j)_{j=1}^k \in \mathcal{S}_\eta$ and $m_{i_j} \geq \min E_j$ for any $j \in J$. Hence by ℓ_1^η-asymptoticity of U and the choice of the scalars (a_m) we have

$$\sum_{j=1}^k \Big\|E_j \sum_{i\in I} a_{m_i} x_i\Big\| \leq C \Big\|\sum_{i\in I} a_{m_i} x_i\Big\| \leq C \sum_{j\in J} a_{m_{i_j}} \|x_{i_j}\| = C \sum_{m\in G} a_m \leq \varepsilon.$$

On the other hand, notice that $\overline{\operatorname{supp}}\, x_i$ intersects at most one E_j, provided $i \notin I$. Therefore

$$\sum_{j=1}^{k} \Big\| E_j \sum_{i\notin I} a_{m_i} x_i \Big\| \le \sum_{i\notin I} \|a_{m_i} x_i\| \le \sum_{m\in F} a_m = 1.$$

Putting together those two estimates we obtain $\sum_{j=1}^{k} \|E_j x\| \le 1+\varepsilon$.

Now we consider the strongly asymptotic case. We repeat the whole proof up to the estimation of $\sum_{j=1}^{k} \|E_j \sum_{i\notin I} a_{m_i} x_i\|$, which must be dealt with in a different way. To handle this notice that for any $i \notin I$ and $j = 1,\dots,k$ we have $\min E_j < \min(\operatorname{supp} x_i)$ whenever $\overline{\operatorname{supp}}\, x_i \cap E_j \neq \emptyset$. Therefore the sets $J_i = \{j : E_j \cap \overline{\operatorname{supp}}\, x_i \neq \emptyset\}$, $i \notin I$, satisfy $\sharp J_i < \min(\operatorname{supp} x_i)$. Hence for any $i \notin I$ the sequence $(E_j \cap \overline{\operatorname{supp}}\, x_i)_{j\in J_i}$ is 1-admissible and thus

$$\sum_{j=1}^{k} \Big\| E_j \sum_{i\notin I} a_{m_i} x_i \Big\| \le \sum_{i\notin I} a_{m_i} \sum_{j\in J_i} \|E_j x_i\| \le \sum_{i\notin I} a_{m_i} C_1 \|x_i\| \le C_1.$$

Putting those estimates together we obtain $\sum_{j=1}^{k} \|E_j x\| \le C_1+\varepsilon$, which ends the proof of the lemma.

Now we return to the proof of the inductive step. Fix $\varepsilon > 0$ and take a block subspace Z of W such that $|y|_n/\|y\| \le (1+\varepsilon)D|z|_n/\|z\|$ for any non-zero y, z in Z_n and $n \in \mathbb{N}$.

Since $I_{\mathrm{b}}(Z) > \omega^\alpha$, Lemma 5.8 of [10] implies that $I_{\mathrm{b}}(Z_n, K) \ge \omega^\alpha$ for some $K \ge 1$ and any $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. We use Lemma 2.6 for $U = Z_n$, $\eta = \alpha_n$, $\xi = \alpha_{n+1}$ to get a vector $x \in Z_n$ with $1/K \le \|x\|$ and $|x|_n \le 1+\varepsilon$. Therefore, by the choice of Z, $|y|_n \le (1+\varepsilon)^2 KD\|y\|$ for any $y \in Z_n$. Hence, by the definition of the norms $|\cdot|_n$,

$$\|E_1 y\| + \dots + \|E_k y\| \le (1+\varepsilon)^2 KD\|y\|, \qquad y \in Z,$$

for any α_n-admissible sequence $n \le E_1 < \dots < E_k$ and any $n \in \mathbb{N}$, i.e. for any α-admissible sequence $E_1 < \dots < E_k$, which shows that Z is ℓ_1^α-asymptotic with constant $(1+\varepsilon)^2 KD$.

The second part of the theorem can be proved in the same way, replacing α-admissible sequences by α-allowable sequences. The initial step is provided by the assumptions on the space, and the inductive step follows analogously by use of Lemma 2.6, the only difference showing up in a worse ℓ_1^α-asymptoticity constant, depending also on the ℓ_1^1-strong asymptoticity constant C_1, namely $(C_1+\varepsilon)(1+\varepsilon)KD$.

The following corollary characterizes "stabilized" Bourgain ℓ_1-block index and provides a quantitative version of Theorem 2.1 for spaces with bounded distortion.

Corollary 2.7. *Let X be a Banach space of D-bounded distortion, $D > 1$, with a bimonotone basis, not containing ℓ_1. If $I_{\rm b}(Y) = I_{\rm b}(X)$ for any block subspace Y of X, then $I_{\rm b}(X) = \omega^{\omega^\gamma}$ for some non-limit $\gamma < \omega_1$.*

If, additionally, ℓ_1 is finitely block represented in X, i.e. $\gamma = \beta + 1$ for some $\beta < \omega_1$, then X is block saturated with $\ell_1^{\omega^\beta}$-asymptotic subspaces with constant $D + \varepsilon$, for any $\varepsilon > 0$.

Proof. Let $I_{\rm b}(X) = \omega^\alpha$. For any $\beta < \alpha$, by Theorem 2.1, X has an ℓ_1^β-asymptotic subspace, thus $I_{\rm b}(X) > \omega^{\beta 2}$ by Proposition 1.6. Hence for any $\beta < \alpha$ also $\beta 2 < \alpha$, thus $\alpha = \omega^\gamma$ for some $\gamma < \omega_1$. By Remark 5.15(iii) of [10], γ is not a limit ordinal.

Assume now that $\gamma = \beta + 1$, $\beta < \omega_1$, fix $\varepsilon > 0$ and pick any block subspace Y of X. It is enough to repeat the reasoning from the proof of Theorem 2.1 for $\alpha = \omega^\beta$.

If $\beta = 0$ we repeat the initial step obtaining some block subspace Z which is ℓ_1^1-asymptotic with constant $(1 + \varepsilon)^3 D$.

If $\beta > 0$ we use the limit inductive step: we pick a subspace W which is $\ell_1^{\omega^\beta}$-asymptotic (by Theorem 2.1), and, using D-bounded distortability, we find a subspace Z stabilizing the norms $|\cdot|_n$ given by a sequence (α_n) with $\alpha_n \nearrow \omega^\beta$.

We shall need the following result from [10]: if $I_{\rm b}(U) > \omega^{\omega^\beta}$ for a Banach space U with a basis, then $I_{\rm b}(U, 1 + \varepsilon) \geq \omega^{\omega^\beta}$ for any $\varepsilon > 0$ ([10, proof of Theorem 1.1, Remark 4.3]). Now we repeat the reasoning for Z as in the proof of Theorem 2.1 with $K = 1 + \varepsilon$, showing that Z is $\ell_1^{\omega^\beta}$-asymptotic with constant $(1 + \varepsilon)^3 D$, which ends the proof. ■

Remark 2.8.

(i) Observe that any Banach space X has a block subspace Y with $I_{\rm b}(Z) = I_{\rm b}(Y)$ for any block subspace Z of Y. Indeed, either X contains ℓ_1, or $I_{\rm b}(X) < \omega_1$ ([5]) and we can use standard diagonalization.

(ii) It follows that in spaces with bounded distortion, with the notation as in Corollary 2.7, the spectral index $I_\Delta(X)$ defined in [19, Def. 4.22] is equal to ω^β (use [19, Theorem 4.23]).

(iii) The norms $(\|\cdot\|_n)_n$ appearing in the first step of the proof of Theorem 2.1 were used in the proof of arbitrary distortion of the Schlumprecht space [20]. These norms give $(2-\varepsilon)$-distortion of the Tsirelson space $T = T[\mathcal{S}_1, 1/2]$ for any $\varepsilon > 0$; moreover, the norms $(|\cdot|_n)_n$ given by ℓ_1^n-asymptoticity of T do not arbitrarily distort T ([18, Thm. 2.1, Prop. 1.1]). In the case of mixed and modified mixed Tsirelson spaces $T[(\mathcal{S}_{\alpha_n}, \theta_n)_n]$ and $T_M[(\mathcal{S}_{\alpha_n}, \theta_n)_n]$ studied in [3, 4, 14], the norms $(|\cdot|_{\alpha_n})_n$ distort the whole space under certain conditions on

$(\alpha_n, \theta_n)_n$. In [3, 4] the special convex combinations, which we used in our proof, are applied to produce an asymptotic biorthogonal system.

3. The c_0 case. In an obvious way, we can formulate analogous definitions of the c_0-block index, denoted here by $J_{\mathrm{b}}(X)$, c_0^α-spreading models and c_0^α-asymptotic spaces, obtaining various measures of "proximity" of a Banach space to c_0. The notion of c_0-block index was investigated in particular in [10]; in [8] higher order c_0-spreading models were used to construct a strictly singular operator on a reflexive ℓ_1-asymptotic HI space.

We will sketch here briefly the variant of the reasoning presented in the previous section, proving Theorem 2.1 in the c_0 case.

THEOREM 3.1. *Let X be a Banach space with a basis. Fix $\alpha < \omega_1$. Assume that $J_{\mathrm{b}}(Y) > \omega^\alpha$ for any block subspace Y of X. Then X contains either an arbitrarily distortable subspace or a c_0^α-asymptotic subspace.*

If, additionally, X is c_0^1-strongly asymptotic, then X contains either an arbitrarily distortable subspace or a c_0^α-strongly asymptotic subspace.

Proof. We shall need suitable norms reflecting c_0-asymptoticity of a space.

DEFINITION 3.2. Let U be a Banach space with a basis. Fix $\alpha < \omega_1$ and assume U is c_0^α-asymptotic with constant C. The associated norm $|\cdot|_\alpha$ is given by $|x|_\alpha = \sup\{|\phi(x)| : \phi \in U^*,\ |\phi|_\alpha^* \le 1\}$ for $x \in U$, where

$$|\phi|_\alpha^* = \sup\Big\{\sum_{i=1}^k \|\phi|_{E_j U}\|^* : E_1 < \cdots < E_k \ \alpha\text{-admissible}\Big\}, \qquad \phi \in U^*.$$

As before, if U is c_0^α-strongly asymptotic, in the definition of the corresponding norm $|\cdot|_\alpha^{\mathrm{s}}$ we use allowable sequences instead of admissible ones.

REMARK 3.3. Clearly $\|\cdot\|^* \le |\cdot|_\alpha^* \le C\|\cdot\|^*$ and $|x|_\alpha \le \max\{\|E_j x\| : 1 \le j \le k\}$ for any $x \in U$ and α-admissible $E_1 < \cdots < E_k$, and the same relations hold in the strongly asymptotic case.

As before, we will restrict the considerations to the case where X has a block subspace Y with D-bounded distortion.

INITIAL STEP. Define on Y^* new equivalent norms as follows:

$$\|\phi\|_n^* = \sup\Big\{\sum_{j=1}^n \|\phi|_{E_j Y}\|^* : E_1 < \cdots < E_n \text{ intervals}\Big\}, \qquad \phi \in Y^*,\ n \in \mathbb{N}.$$

Let $\|y\|_n = \sup\{|\phi(y)| : \phi \in Y^*,\ \|\phi\|_n^* \le 1\}$ for $y \in Y$ and $n \in \mathbb{N}$.

LEMMA 3.4. *Let U be a Banach space with a bimonotone basis. Fix $n \in \mathbb{N}$ and assume $J_{\mathrm{b}}(U) > \omega$. Then for any $\varepsilon > 0$ there is a vector $x \in U$ with $1/(1+\varepsilon) \le \|x\|_n \le \|x\| \le 1+\varepsilon$.*

Proof of Lemma 3.4. Pick $k \in \mathbb{N}$ such that $n^{2-k} < \varepsilon$, take a normalized block sequence $(x_i)_{i=1}^{n^k} \subset U$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of a c_0 space of dimension n^k, and put $x = \sum_{i=1}^{n^k} x_i$. Obviously $\|x\| \leq 1+\varepsilon$. Since the basis is bimonotone we can take normalized functionals $(\phi_i)_{i=1}^{n^k} \subset U^*$ with $\phi_i(x_i) = 1$ and $\phi_i(y) = 0$ for any $y \in U$ with $\operatorname{supp} y \cap \overline{\operatorname{supp}}\, x_i = \emptyset$ for $1 \leq i \leq n^k$. Put $\phi = n^{-k} \sum_{i=1}^{n^k} \phi_i$. Since $\phi(x) = 1$ it is enough to show that $\|\phi\|_n^* \leq 1+\varepsilon$. Take any $E_1 < \cdots < E_n$, define the set I as in the proof of Lemma 2.5 and proceed by computing the norms of $\phi_i|_{E_j U}$ instead of $E_j x_i$. The second estimate follows from the fact that by the choice of (ϕ_i), for any $i \notin I$ there is at most one $1 \leq j \leq n$ with $\phi_i|_{E_j U} \not\equiv 0$.

Returning to the proof of the initial step, fix $\varepsilon > 0$ and take a block subspace Z of Y such that $\|y\|_n/\|y\| \leq (1+\varepsilon)D\|z\|_n/\|z\|$ for any non-zero $y, z \in Z_n$ and any $n \in \mathbb{N}$. For a fixed $n \in \mathbb{N}$ take $x \in Z_n$ as in Lemma 3.4 to get $\|y\| \leq (1+\varepsilon)^3 D\|y\|_n$ for any $y \in Z_n$. It follows that

$$\|y\| \leq (1+\varepsilon)^3 D \max_{j=1,\ldots,n} \|E_j y\|$$

for any $y \in Y$, $n \in \mathbb{N}$, and $n \leq E_1 < \cdots < E_n$, which shows that Z is c_0^1-asymptotic with constant $(1+\varepsilon)^3 D$.

Inductive step. Take $1 < \alpha < \omega_1$ and assume that the conclusion holds true for all $\xi < \alpha$. If $\alpha = \beta + 1$ for some $\beta < \omega_1$ then by the inductive hypothesis there is a block subspace W of Y which is c_0^β-asymptotic, and hence also c_0^α-asymptotic.

If α is a limit ordinal take $(\alpha_n)_n$ with $\alpha_n \nearrow \alpha$ as in the definition of $\mathcal{S}_\alpha$. By the inductive hypothesis we can pick a subspace W of Y such that W is $c_0^{\alpha_n}$-asymptotic for any $n \in \mathbb{N}$.

Let $|\cdot|_n$ and $|\cdot|_n^*$ denote the norms on W and W^* respectively given by the $c_0^{\alpha_n}$-asymptoticity of W. We need the following analogue of Lemma 2.6:

Lemma 3.5. *Let U be a Banach space with a bimonotone basis. Fix ordinals $\eta < \xi < \omega_1$ and assume U is c_0^η-asymptotic and $\mathrm{J_b}(U,K) > \omega^\xi$ for some $K \geq 1$. Then for any $\varepsilon > 0$ there is $x \in U$ with $1/(1+\varepsilon) \leq |x|_\eta \leq \|x\| \leq K$.*

If, additionally, U is c_0^1-strongly asymptotic with constant C_1, then for any $\varepsilon > 0$ there is $x \in U$ satisfying $1/(C_1+\varepsilon) \leq |x|_\eta^{\mathrm{s}} \leq \|x\| \leq K$.

Proof of Lemma 3.5. Let U be c_0^η-asymptotic with constant C. Proceed as in the proof of Lemma 2.6, obtaining a normalized block sequence $(x_i)_{i=1}^l$ in a c_0-K-block tree, a set $F = (m_1, \ldots, m_l) \in \mathcal{S}_\xi$ with $m_i \geq \max(\operatorname{supp} x_i)$, $1 \leq i \leq l$ and suitable positive scalars $(a_m)_{m\in F}$.

Pick normalized functionals $(\phi_i)_{i=1}^l \subset U^*$ with $\phi_i(x_i) = 1$ and $\phi_i(y) = 0$ for any $y \in U$ with $\operatorname{supp} y \cap \overline{\operatorname{supp}}\, x_i = \emptyset$ for $1 \leq i \leq l$. Put $x = \sum_{i=1}^l x_i$ and $\phi = \sum_{i=1}^l a_{m_i}\phi_i$. Then $\|x\| \leq K$. Since $\phi(x) = 1$ it is enough to show that $|\phi|_\eta^* \leq 1+\varepsilon$ in the asymptotic case and $\|\phi\|_\eta \leq C_1+\varepsilon$ in the strongly asymptotic case.

Take any η-admissible sequence $E_1 < \cdots < E_k$, define the sets J, I and J_i for $i \notin I$ as in the proof of Lemma 2.6 and proceed to compute the norms of $\phi_i|_{E_jU}$ instead of $E_j x_i$. To estimate $\sum_{j=1}^k \|(\sum_{i\in I} a_{m_i}\phi_i)|_{E_jU})\|^*$ use Remark 3.3. To estimate $\sum_{j=1}^k \|(\sum_{i\notin I} a_{m_i}\phi_i)|_{E_jU})\|^*$ use Remark 3.3 and the fact that $\phi_i|_{E_jU} \equiv 0$ whenever $j \notin J_i$.

Now we return to the proof of the inductive step. Fix $\varepsilon > 0$ and take a block subspace Z of W such that $|y|_n/\|y\| \leq (1+\varepsilon)D|z|_n/\|z\|$ for all $n \in \mathbb{N}$ and any non-zero $y, z \in Z_n$.

Since $J_{\mathrm{b}}(Z) > \omega^\alpha$, we have $J_{\mathrm{b}}(Z_n, K) \geq \omega^\alpha$ for some $K \geq 1$ and any $n \in \mathbb{N}$ (it is easy to check that Lemma 5.8 of [10] is also valid in the c_0 case). Fix $n \in \mathbb{N}$. Apply Lemma 3.5 for Z_n, α_n, α_{n+1} to get $x \in Z_n$ with $1/(1+\varepsilon) \leq |x|_n \leq \|x\| \leq K$. It follows that $\|y\| \leq (1+\varepsilon)^2 KD|y|_n$ for any $y \in Z_n$ and thus

$$\|y\| \leq (1+\varepsilon)^2 KD \max_{j=1,\ldots,k} \|E_j y\|$$

for any $y \in Y$ and α-admissible $E_1 < \cdots < E_k$, which shows that Z is c_0^α-asymptotic with constant $(1+\varepsilon)^2 KD$.

The part for c_0^α-strongly asymptotic spaces follows easily as in the ℓ_1 case. ∎

REMARK 3.6. Corollary 2.7 also remains true in the c_0 case, since the result of [10] also holds for c_0: if $J_{\mathrm{b}}(U) > \omega^{\omega^\beta}$ for a Banach space U, then $J_{\mathrm{b}}(U, 1+\varepsilon) \geq \omega^{\omega^\beta}$ for any $\varepsilon > 0$ ([10, Remark 4.3]).

One can consider Theorem 2.1 also in the ℓ_p case, $1 < p < \infty$, using the corresponding notions partly analyzed in [10]. Recall here that by [16] any ℓ_p-asymptotic Banach space, $1 < p < \infty$, not containing finite copies of ℓ_1 uniformly is arbitrarily distortable. Theorem 2.1 remains true in the ℓ_p-asymptotic variant.

Notice that ℓ_p^α-asymptoticity of a space U with a basis is characterized by two norms equivalent to the original one, namely the norm

$$|x|_\alpha = \sup\{(\|E_1 x\|^p + \cdots + \|E_k x\|^p)^{1/p} : E_1 < \cdots < E_k \ \alpha\text{-admissible}\}$$

and the predual norm $|||\cdot|||_\alpha$ to

$$|||\phi|||_\alpha^* = \sup\{((\|\phi|_{E_1U}\|^*)^q + \cdots + (\|\phi|_{E_kU}\|^*)^q)^{1/q} : E_1 < \cdots < E_k \ \alpha\text{-admissible}\}$$

where $1/p+1/q = 1$, thus in order to show Theorem 2.1 in the ℓ_p-asymptotic case one should combine the proofs for the ℓ_1 and c_0 cases.

The initial step of the proof follows from Theorem 1.1 of [17]. In the inductive step for a limit ordinal α, and $\alpha_n \nearrow \alpha$, working in a subspace Y

with bounded distortion we pick a block subspace Z stabilizing the norms $(|\cdot|_{\alpha_n})_n$ and $(|||\cdot|||_{\alpha_n})_n$.

We need the following version of [19, Prop. 3.6] (proved by induction analogously to the original version): for any $\eta < \xi < \omega_1$, $1 < r < \infty$, infinite $M \subset \mathbb{N}$, and $\varepsilon > 0$ there are $F \in \mathcal{S}_\xi \cap [M]^{<\infty}$ and positive scalars $(a_m)_{m\in F}$ such that $\sum_{m\in F} a_m^r = 1$ and $\sum_{m\in G} a_m < \varepsilon$ for any $G \in \mathcal{S}_\eta$ with $G \subset F$.

Having this we prove an analogue of Lemma 2.6 choosing a vector $x = \sum_{m_i\in F} a_{m_i} x_i$ with $(a_m)_{m\in F}$ given by the observation above for $p = r$. An analogue of Lemma 3.5 also holds true: to prove it one uses a vector $x = \sum_{m_i\in F} a_{m_i}^{q/p} x_i$ and a functional $\phi = \sum_{m_i\in F} a_{m_i}\phi_i$, with $(a_m)_{m\in F}$ given by the observation above for $r = q$. Once we have suitable versions of Lemmas 2.6, 3.5 we are able to "glue" the norms $(|\cdot|_{\alpha_n})_n$ and $(|||\cdot|||_{\alpha_n})_n$ on Z and obtain ℓ_p^α-asymptoticity.

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