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Partially defined σ-derivations on semisimple Banach algebras

by

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Abstract. Let A be a semisimple Banach algebra with a linear automorphism σ and let $\delta\colon I \to A$ be a σ-derivation, where I is an ideal of A. Then $\Phi(\delta)(I \cap \sigma(I)) = 0$, where $\Phi(\delta)$ is the separating space of δ. As a consequence, if I is an essential ideal then the σ-derivation δ is closable. In a prime C^*-algebra, we show that every σ-derivation defined on a nonzero ideal is continuous. Finally, any linear map on a prime semisimple Banach algebra with nontrivial idempotents is continuous if it satisfies the σ-derivation expansion formula on zero products.

1. Results. Throughout the paper, A is always a unital Banach algebra over the complex field $\mathbb{C}$ and σ is a linear endomorphism of A. Let 1_A denote the identity automorphism of A. By a *σ-derivation* of A we mean a linear map $\delta\colon A \to A$ such that $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in A$. Clearly, the map $\sigma - 1_A$ is a σ-derivation and 1_A-derivations are just ordinary derivations. Thus the concept of σ-derivations can be regarded as a generalization of both derivations and endomorphisms. Let I be a nonzero ideal of A. A linear map $\delta\colon I \to A$ is called a *σ-derivation defined on I* if $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in I$. An ideal I of A is called *essential* if I has nontrivial intersection with any nonzero ideal of A. For a semisimple algebra A, this is equivalent to saying that $aI = 0$ where $a \in A$ implies $a = 0$. A σ-derivation $\delta\colon I \to A$ is called *essentially defined* on an ideal I if I is an essential ideal of A.

Kaplansky conjectured that every derivation on a C^*-algebra is continuous [16] and that every derivation on a semisimple Banach algebra is continuous [17]. Sakai confirmed Kaplansky's conjecture for C^*-algebras in [22]. The second conjecture was confirmed by Johnson and Sinclair in [15].

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Many related results have been obtained in the literature (see, for instance, [3, 4, 9, 15, 21, 24, 26, 27]). In [27] Villena proved that every derivation defined on an essential ideal of a semisimple Banach algebra is automatically closable. As an application, he showed that every derivation defined on a nonzero ideal of a prime C^*-algebra is continuous. Recently, several results concerning σ-derivations of Banach algebras have been studied (see [1, 5, 6, 9, 12, 13, 19, 21]). Brešar and Villena [9] proved that if A is a semisimple Banach algebra and σ is a linear automorphism of A, then every σ-derivation on A is automatically continuous. In this paper, instead of essential ideals, we investigate partially defined σ-derivations on any nonzero ideal.

To state our results precisely, we recall the definition of separating spaces. Let X and Y be normed spaces over the complex field $\mathbb{C}$ and let $T\colon X \to Y$ be a linear map. The *separating space* $\Phi(T)$ of T is defined as follows:

$$\Phi(T) = \{y \in Y \mid \text{there exists a sequence } (x_n) \text{ in } X \text{ with } \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} T(x_n) = y\}.$$

Clearly, $\Phi(T)$ is a subspace of Y. We say that T is *closable* if $\Phi(T) = \{0\}$. For Banach spaces X and Y, the closed graph theorem asserts that T is continuous if and only if it is closable. We are now ready to state the main theorem of the paper.

Theorem 1.1. *Let A be a semisimple Banach algebra with a linear epimorphism σ and let $\delta\colon I \to A$ be a σ-derivation, where I is an ideal of A. Then $\Phi(\delta)(I \cap \sigma(I)) = 0$. As a consequence, every essentially defined σ-derivation on A is closable if σ is a linear automorphism of A.*

As applications of Theorem 1.1, we have the following two results.

Corollary 1.2. *Let A be a semisimple Banach algebra with a linear epimorphism σ. Then every σ-derivation on A is continuous.*

Corollary 1.3. *Let A be a prime C^*-algebra with a linear automorphism σ and let I be a nonzero ideal of A. Then every σ-derivation $\delta\colon I \to A$ is continuous.*

Recently, there have been much work concerning maps preserving zero products in the literature (see [8, 10, 11, 14, 18, 28]). Applying Theorem 1.1 we obtain the continuity of linear maps which satisfy the σ-derivation expansion formula on zero products.

Theorem 1.4. *Let A be a prime semisimple Banach algebra with nontrivial idempotents and let σ be a linear automorphism of A. Suppose that $\delta\colon A \to A$ is a linear map such that $\sigma(x)\delta(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$. Then δ is continuous.*

2. Preliminaries. We fix some notation and terminology. Let A be a semisimple Banach algebra. Recall that $\operatorname{soc}(A)$, the *socle* of A, is defined as the sum of all minimal left ideals of A. Therefore, each element in $\operatorname{soc}(A)$ lies in a sum of finitely many minimal left ideals of A. The socle $\operatorname{soc}(A)$ also coincides with the sum of all minimal right ideals of A. An element $a \in A$ is said to be of *rank one* if aA is a minimal right ideal of A. This is equivalent to saying that Aa is a minimal left ideal of A. Moreover, $a \in \operatorname{soc}(A)$ has rank one if and only if $aAa = \mathbb{C}a$. Thus, if $a \in \operatorname{soc}(A)$ has rank one, then Aa is a minimal left ideal and aA is a minimal right ideal of A. Let P be a primitive ideal of A and let π be a continuous irreducible representation of A with $\ker \pi = P$. Then $a + P \in \operatorname{soc}(A/P)$ if and only if $\pi(a)$ is a finite rank operator (see [7] for details).

We begin with several lemmas.

LEMMA 2.1. *Let A be a semisimple Banach algebra. If $a, b \in \operatorname{soc}(A)$ then aAb is finite-dimensional over $\mathbb{C}$.*

Proof. Obviously, we may assume $aAb \neq 0$. Suppose first that both a and b have rank one. Thus $aAa = \mathbb{C}a$ and Ab is a minimal left ideal of A. Choose $x \in A$ such that $axb \neq 0$. Then $Ab = Aaxb$ by minimality of Ab. Thus $aAb = aAaxb = (aAa)xb = \mathbb{C}axb$, implying $\dim_{\mathbb{C}} aAb = 1$, as desired.

Let $a, b \in \operatorname{soc}(A)$. There are finitely many elements $a_1, \ldots, a_m, b_1, \ldots, b_n$ in A of rank one such that $a = \sum_{i=1}^m a_i$ and $b = \sum_{j=1}^n b_j$. Note that $\dim_{\mathbb{C}} a_iAb_j \leq 1$ for all i, j. Then $aAb \subseteq \sum_{i=1}^m \sum_{j=1}^n a_iAb_j$, implying $\dim_{\mathbb{C}} aAb \leq mn$. This proves the lemma.

We also need the gliding hump argument due to Thomas [25, Proposition 1.3] and Johnson and Sinclair's lemma [25, Lemma 1.5], which are essential to our proofs.

LEMMA 2.2 (Gliding hump argument). *Let X, Y and $\{Y_i\}_{i=1}^\infty$ be Banach spaces. Let $\{T_i\}_{i=1}^\infty$ be a sequence of continuous linear operators from X into itself and let $\{U_i\}_{i=1}^\infty$ be a sequence of continuous linear operators, where each $U_i\colon Y \to Y_i$. If S is a linear operator from X to Y such that $U_nST_1T_2\cdots T_m$ is continuous for $m > n$, then $U_nST_1T_2\cdots T_n$ is continuous for sufficiently large n.*

LEMMA 2.3 (Johnson and Sinclair). *Let A be a Banach algebra and let π be a continuous irreducible representation of A on an infinite-dimensional normed complex linear space X. Let $\{x_i\}_{i=0}^\infty$ be a linearly independent subset of X. Then there exists a sequence $\{a_i\}_{i=1}^\infty$ in A such that $\pi(a_m \cdots a_1)x_n = 0$ for all $m > n \geq 0$ and $\{\pi(a_n \cdots a_1)x_l\}_{l=n}^\infty$ is a linearly independent subset of X for all $n \geq 1$.*

LEMMA 2.4. *Let A be a Banach algebra and let σ be a continuous epimorphism of A. Let π_i be a continuous irreducible representation of A on* *the Banach space X_i with* $\ker \pi_i = P_i$ *for* $i = 1, 2, \ldots$. *Suppose that* $\delta \colon I \to A$ *is a* σ*-derivation, where* I *is a nonzero ideal of* A. *Suppose that there exist a sequence* $\{c_i\}_{i=1}^{\infty}$ *in* I, *a sequence* $\{b_i\}_{i=0}^{\infty}$ *in* A *with* $b_0 \in I$, *and a sequence* $\{x_i\}_{i=1}^{\infty}$, *where* $x_i \in X_i$ *for* $i \geq 1$, *such that*

- $\sigma(c_n) \notin P_n$ *for all* $n \geq 1$,
- $\pi_n(b_n \cdots b_1 b_0) x_n \neq 0$,
- $\pi_n(b_m \cdots b_1 b_0) x_n = 0$ *for all* $m > n \geq 1$.

Then $\Phi(\delta) \subseteq P_k$ *for some* $k \geq 1$.

Proof. Let $U_n \colon A \to X_n$, $T_n \colon A \to A$ and $R_{b_0} \colon A \to I$ be continuous linear operators given by

$$U_n(a) = \pi_n(\sigma(c_n)a)x_n, \quad T_n(a) = ab_n \quad \text{and} \quad R_{b_0}(a) = ab_0$$

for $a \in A$ and for $n \geq 1$. Notice that δR_{b_0} is a linear operator from A into itself. Then if $m > n$, we have

$$\begin{aligned}
&U_n(\delta R_{b_0})T_1 \cdots T_m(a) \\
&\quad = \pi_n(\sigma(c_n)\delta(ab_m \cdots b_1 b_0))x_n \\
&\quad = \pi_n(\delta(c_n a b_m \cdots b_1 b_0) - \delta(c_n) a b_m \cdots b_1 b_0) x_n \\
&\quad = \pi_n(\sigma(c_n a)\delta(b_m \cdots b_1 b_0) + \delta(c_n a) b_m \cdots b_1 b_0 - \delta(c_n) a b_m \cdots b_1 b_0) x_n \\
&\quad = \pi_n(\sigma(c_n)\sigma(a)\delta(b_m \cdots b_1 b_0))x_n.
\end{aligned}$$

Thus $U_n(\delta R_{b_0})T_1 \cdots T_m$ is continuous for all $m > n$. By Lemma 2.2, there exists an integer $n \geq 1$ such that $U_n(\delta R_{b_0})T_1 \cdots T_n$ is continuous. Let $b, c \in I$ and let $\{a_k\}_{k=1}^{\infty}$ be a sequence in I with $\lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} \delta(a_k) = a \in \Phi(\delta)$. Then $\lim_{k\to\infty} c a_k b = 0$. Since

$$\begin{aligned}
U_n(\delta R_{b_0})T_1 \cdots T_n(c a_k b)& \\
&= \pi_n(\sigma(c_n)\delta(c a_k b b_n \cdots b_1 b_0))x_n \\
&= \pi_n(\sigma(c_n))\pi_n(\sigma(c a_k)\delta(b b_n \cdots b_1 b_0) \\
&\qquad + \sigma(c)\delta(a_k) b b_n \cdots b_1 b_0 + \delta(c) a_k b b_n \cdots b_1 b_0) x_n \\
&= \pi_n(\sigma(c_n))\pi_n(\sigma(c))\pi_n(\sigma(a_k))\pi_n(\delta(b b_n \cdots b_1 b_0))x_n \\
&\qquad + \pi_n(\sigma(c_n))\pi_n(\sigma(c))\pi_n(\delta(a_k))\pi_n(b)\pi_n(b_n \cdots b_1 b_0)x_n \\
&\qquad + \pi_n(\sigma(c_n))\pi_n(\delta(c))\pi_n(a_k)\pi_n(b)\pi_n(b_n \cdots b_1 b_0)x_n,
\end{aligned}$$

it is easy to see that

$$\begin{aligned}
0 &= \lim_{k\to\infty} U_n \delta R_{b_0} T_1 \cdots T_n(c a_k b) \\
&= \pi_n(\sigma(c_n))\pi_n(\sigma(c))\pi_n(a)\pi_n(b)\pi_n(b_n \cdots b_1 b_0)x_n.
\end{aligned}$$

Thus

$$(1) \qquad \pi_n(\sigma(c_n))\pi_n(\sigma(I))\pi_n(a)\pi_n(I)\pi_n(b_n\cdots b_1b_0)x_n = 0.$$

Recall that $\sigma(c_n) \notin P_n$ and $b_0 \notin P_n$. So $\sigma(I) \not\subseteq P_n$ and $I \not\subseteq P_n$. In particular, we have $\pi_n(I)\pi_n(b_n\cdots b_1b_0)x_n = X_n$. It follows from (1) that $\sigma(c_n)\sigma(I)a \subseteq \ker\pi_n = P_n$. Recall that $\sigma(I)$ is an ideal of A and $\sigma(I) \not\subseteq P_n$. This implies $a \in P_n$, as desired.

Lemma 2.5. *Let A be a Banach algebra and let σ be a continuous epimorphism of A. Suppose that $\delta\colon I \to A$ is a σ-derivation, where I is a nonzero ideal of A. If P is a primitive ideal of A satisfying $I \not\subseteq P$ and $\sigma(I) \not\subseteq P$, then either $\Phi(\delta) \subseteq P$ or $(I+P)/P = \mathrm{soc}(A/P)$.*

Proof. By assumption, there exists $c \in I$ such that $\sigma(c) \notin P$. Suppose that $(I+P)/P \neq \mathrm{soc}(A/P)$. Let π be a continuous irreducible representation of A on an infinite-dimensional Banach space X with $\ker\pi = P$. Then $\dim_{\mathbb{C}} \pi(b_0)X = \infty$ for some $b_0 \in I$. Hence $\{\pi(b_0)x_i\}_{i=0}^{\infty}$ is a linearly independent subset of X for some $x_i \in X$, $i \geq 0$. By Lemma 2.3, there exists a sequence $\{b_i\}_{i=1}^{\infty}$ in A such that $\pi(b_n\cdots b_1)\pi(b_0)x_n \neq 0$ and $\pi(b_m\cdots b_1)\pi(b_0)x_n = 0$ for all $m > n$. Now we let $c_i = c$, $\pi_i = \pi$ and $P_i = P$ for all $i \geq 1$. In view of Lemma 2.4, we obtain $\Phi(\delta) \subseteq P$, proving the lemma.

Lemma 2.6. *Let A be a Banach algebra, P a primitive ideal of A and σ a continuous epimorphism of A. Suppose that $\delta\colon I \to A$ is a σ-derivation defined on a nonzero ideal I of A. If there exist $c, b \in I$ such that $\sigma(c) \notin P$, $b \notin P$ and $\dim_{\mathbb{C}} cAb < \infty$, then $\Phi(\delta) \subseteq P$.*

Proof. Since $\dim_{\mathbb{C}} cAb < \infty$, the map $a \in A \mapsto \delta(cab)$ is continuous. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in I, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \delta(a_n) = a \in \Phi(\delta)$. Let $x, y \in I$. Since $\delta(cxa_nyb) = \sigma(cx)\sigma(a_n)\delta(yb) + \sigma(cx)\delta(a_n)yb + \delta(cx)a_nyb$, it is easy to see that $\lim_{n\to\infty} \delta(cxa_nyb) = \sigma(cx)ayb = 0$. This implies that $\sigma(c)\sigma(I)aIb = 0$. From $\sigma(c), b \notin P$ and $\sigma(I), I \not\subseteq P$, it follows that $a \in P$. Thus $\Phi(\delta) \subseteq P$, as desired.

3. Proofs

Proof of Theorem 1.1. By [23, Corollary 6.12], σ is continuous on A. Let Σ denote the set of all primitive ideals of A. Since A is semisimple, $\bigcap_{P\in\Sigma} P = 0$. Set

$$\Sigma_N = \{P \in \Sigma \mid I \not\subseteq P \text{ and } \sigma(I) \not\subseteq P\}.$$

For $P \in \Sigma$ and $P \notin \Sigma_N$, we have $I \cap \sigma(I) \subseteq P$. Thus $(I \cap \sigma(I)) \cap \bigcap_{P\in\Sigma_N} P = 0$. Next we set $\Sigma_I = \{P \in \Sigma_N \mid \Phi(\delta) \not\subseteq P\}$. So $\Phi(\delta) \subseteq P$ if $P \in \Sigma_N \setminus \Sigma_I$.

Suppose first that $\Sigma_I = \emptyset$. Then $\Phi(\delta) \subseteq \bigcap_{P\in\Sigma_N} P$. This implies that

$$\Phi(\delta)(I\cap\sigma(I)) \subseteq \Big(\bigcap_{P\in\Sigma_N} P\Big) \cap (I\cap\sigma(I)) = 0,$$

proving the theorem. Assume on the contrary that $\Sigma_I \neq \emptyset$. Let $K = \bigcap_{P\in\Sigma_N\setminus\Sigma_I} P$ and $J = (I\cap\sigma(I))\cap K$. Then J is an ideal of A contained in I and $J \subseteq \bigcap_{P\in\Sigma\setminus\Sigma_I} P$.

If $J \subseteq P$ for some $P \in \Sigma_I$, then $I\sigma(I)K \subseteq J \subseteq P$. Since $I \nsubseteq P$ and $\sigma(I) \nsubseteq P$, we have $K \subseteq P$. Thus $\Phi(\delta) \subseteq K \subseteq P$, a contradiction. Hence we may choose $P_0 \in \Sigma_I$ such that $J \nsubseteq P_0$. By Lemma 2.5, $0 \neq (J+P_0)/P_0 \subseteq (I+P_0)/P_0 = \mathrm{soc}(A/P_0)$. Let π_0 be a continuous irreducible representation of A on a Banach space X_0 with $\ker\pi_0 = P_0$. So there exist $c_0 \in I$, $b_0 \in J$, $0 \neq x_0 \in X_0$ such that $\sigma(c_0) \notin P_0$, $0 \neq b_0 + P_0 \in \mathrm{soc}(A/P_0)$ and $\pi_0(b_0)x_0 = x_0$. By Lemma 2.1, $\dim_{\mathbb{C}} \bar{c}_0(A/P_0)\bar{b}_0 = n_0 < \infty$, where $\bar{x} = x + P_0$ for $x \in A$. Then there exist maps $\lambda_{0i}\colon A \to \mathbb{C}$, $i = 1,\dots,n_0$, and $a_{01},\dots,a_{0n_0} \in A$ such that $c_0ab_0 - \sum_{i=1}^{n_0}\lambda_{0i}(a)c_0a_{0i}b_0 \in P_0$ for all $a \in A$.

Let $J_0 = J\cap P_0$. We claim that there exists $P_1 \in \Sigma_I\setminus\{P_0\}$ such that $J_0 \nsubseteq P_1$ and $b_0 \notin P_1$. Otherwise, $c_0ab_0 - \sum_{i=1}^{n_0}\lambda_{0i}(a)c_0a_{0i}b_0 \in (J\cap P_0)\cap \bigcap_{P\in\Sigma_I\setminus\{P_0\}} P = 0$ for all $a \in A$. This implies $\dim_{\mathbb{C}} c_0Ab_0 = n_0 < \infty$. By Lemma 2.6, $\Phi(\delta) \subseteq P_0$, a contradiction. This proves the claim. By Lemma 2.5,

$$0 \neq (J_0+P_1)/P_1 \subseteq (I+P_1)/P_1 = \mathrm{soc}(A/P_1).$$

Let π_1 be a continuous irreducible representation of A on a Banach space X_1 with $\ker\pi_1 = P_1$. So there exist $c_1 \in I$, $b_1 \in J_0$, $0 \neq x_1 \in X_1$ such that

$$\sigma(c_1) \notin P_1, \qquad 0 \neq b_1 + P_1 \in \mathrm{soc}(A/P_1), \qquad \pi_1(b_1)\pi_1(b_0)x_1 = x_1.$$

By Lemma 2.1, $\dim_{\mathbb{C}} \bar{c}_1(A/P_1)\bar{b}_1\bar{b}_0 = n_1 < \infty$. Notice that $\pi_0(b_1b_0)x_0 = 0$ since $b_1 \in P_0$.

Suppose now that we have primitive ideals $P_0, P_1, \dots, P_k \in \Sigma_I$ and elements $b_0, b_1, \dots, b_k \in I$ and $c_1, \dots, c_k \in I$ such that

- $b_i \in J_{i-1} = J\cap P_0\cap P_1\cap\cdots\cap P_{i-1}$ for all $1 \le i \le k$,
- $\dim_{\mathbb{C}} \bar{c}_i(A/P_i)\bar{b}_i\cdots\bar{b}_0 = n_i < \infty$ for all $1 \le i \le k$,
- $\sigma(c_i) \notin P_i$ for all $1 \le i \le k$.

Further, for each $i \ge 1$, there exist a continuous irreducible representation π_i of A on a Banach space X_i with $\ker\pi_i = P_i$ and $x_i \in X_i$ satisfying

$$\pi_j(b_j\cdots b_1b_0)x_j \neq 0 \quad\text{and}\quad \pi_i(b_j\cdots b_1b_0)x_i = 0 \quad \text{for all } 0 \le i < j \le k.$$

Since $\dim_{\mathbb{C}} \bar{c}_k(A/P_k)\bar{b}_k\cdots\bar{b}_1\bar{b}_0 = n_k$, there exist maps $\lambda_{ki}\colon A \to \mathbb{C}$, $i = 1,\dots,n_k$ and $a_{k1},\dots,a_{kn_k} \in A$ such that $c_kab_k\cdots b_1b_0 - \sum_{i=1}^{n_k}\lambda_{ki}(a)c_ka_{ki}b_k \cdots b_1b_0 \in P_k$ for all $a \in A$. Let $J_k = J_{k-1}\cap P_k = J\cap P_0\cap P_1\cap\cdots\cap P_k$. We claim that there exists $P_{k+1} \in \Sigma_I \setminus \{P_0, P_1, \dots, P_k\}$ such that $J_k \nsubseteq P_{k+1}$ and $b_k \cdots b_1 b_0 \notin P_{k+1}$. Otherwise,

$$c_k a b_k \cdots b_1 b_0 - \sum_{i=1}^{n_k} \lambda_{ki}(a) c_k a_{ki} b_k \cdots b_1 b_0 \in J_k \cap \Big(\bigcap_{P \in \Sigma_I \setminus \{P_0, P_1, \dots, P_k\}} P \Big) = 0$$

for all $a \in A$. Then we conclude that $\dim_{\mathbb{C}} c_k A b_k \cdots b_1 b_0 \leq n_k < \infty$. By Lemma 2.6, $\Phi(\delta) \subseteq P_k$, a contradiction. This proves the claim. By Lemma 2.5, $0 \neq (J_k + P_{k+1})/P_{k+1} \subseteq (I + P_{k+1})/P_{k+1} = \mathrm{soc}(A/P_{k+1})$. Let π_{k+1} be a continuous irreducible representation of A on a Banach space X_{k+1} with $\ker \pi_{k+1} = P_{k+1}$. So there exist $c_{k+1} \in I$, $b_{k+1} \in J_k$, $0 \neq x_{k+1} \in X_{k+1}$ such that $\sigma(c_{k+1}) \notin P_{k+1}$, $0 \neq b_{k+1} + P_{k+1} \in \mathrm{soc}(A/P_{k+1})$, $\pi_{k+1}(b_{k+1})\pi_{k+1}(b_k \cdots b_1 b_0) x_{k+1} = x_{k+1}$. By Lemma 2.1, $\dim_{\mathbb{C}} \overline{c}_{k+1}(A/P_1) \overline{b}_{k+1} \overline{b}_k \cdots \overline{b}_1 \overline{b}_0 = n_{k+1} < \infty$. Moreover, $\pi_i(b_{k+1} \cdots b_1 b_0) x_i = 0$ for all $1 \leq i \leq k$ since $b_{k+1} \in P_0 \cap \dots \cap P_k$.

Proceeding in the same way as above, we may obtain a sequence $\{b_i\}_{i=0}^{\infty}$ in I, a sequence $\{c_i\}_{i=1}^{\infty}$ in I and a sequence $\{P_i\}_{i=1}^{\infty}$ of primitive ideals in Σ_I such that $\sigma(c_n) \notin P_n$, $\pi_n(b_n \cdots b_1 b_0) x_n \neq 0$ and $\pi_n(b_m \cdots b_1 b_0) x_n = 0$ for all $m > n \geq 1$, where π_n is a continuous irreducible representation of A on the Banach space X_n with $\ker \pi_n = P_n$. In view of Lemma 2.4, $\Phi(\delta) \subseteq P_i$ for some $i \geq 1$, a contradiction. This forces $\Sigma_I = \emptyset$, as desired. Finally, if I is an essential ideal of A and σ is a linear automorphism of A, then $\sigma(I)$ is an essential ideal of A. In particular, $I \cap \sigma(I)$ is also an essential ideal of A. Then from $\Phi(\delta)(I \cap \sigma(I)) = 0$, it follows that $\Phi(\delta) = 0$. The proof is now complete.

Clearly, Corollary 1.2 follows directly from Theorem 1.1. Also, we have

Corollary 3.1. *Let A be a semisimple Banach algebra with a linear automorphism σ and let I be a closed essential ideal of A. Suppose that $\delta\colon I \to A$ is a σ-derivation defined on I. Then δ is continuous.*

Recall that an automorphism σ of a unital algebra A is called *inner* if there exists an invertible element $u \in A$ such that $\sigma(a) = uau^{-1}$ for all $a \in A$. Given any derivation d of A and an invertible element $u \in A$, the map defined by $a \in A \mapsto ud(a)$ is a σ_u-derivation, where $\sigma_u\colon a \in A \mapsto uau^{-1}$ is an inner automorphism. Obviously, every inner automorphism is continuous. Moreover, if P is a primitive ideal of A and σ is inner, then $I \subseteq P$ if and only if $\sigma(I) \subseteq P$. The next result can be regarded as an extension of the corresponding theorem for derivations and is an immediate consequence of Theorem 1.1.

Corollary 3.2. *Let A be a semisimple Banach algebra, σ an inner automorphism of A and $\delta\colon I \to A$ a σ-derivation, where I is a nonzero ideal of A. Then $\Phi(\delta)I = 0$.*

A Banach algebra A is called *ultraprime* if there exists $K > 0$ such that $K\|a\|\,\|b\| \le \|M_{a,b}\|$ for all $a, b \in A$, where $M_{a,b}$ denotes the two-sided multiplication operator on A defined by $M_{a,b}(x) = axb$ for $x \in A$. Obviously, every ultraprime Banach algebra is a prime algebra. By [20, Proposition 2.3], every prime C^*-algebra is ultraprime and semisimple.

THEOREM 3.3. *Let A be an ultraprime Banach algebra with a linear automorphism σ and let I be a nonzero ideal of A. If $\delta\colon I \to A$ is a nonzero closable σ-derivation, then both δ and σ are continuous.*

It is clear that every nonzero ideal in a prime algebra is essential. Applying Theorem 1.1 and 3.3, we have

COROLLARY 3.4. *Let A be an ultraprime semisimple Banach algebra with a linear automorphism σ and let I be an ideal of A. Then every σ-derivation defined on I is continuous.*

Since every prime C^*-algebra is ultraprime [20, Proposition 2.3], Corollary 1.3 follows directly from Corollary 3.4. We now turn to the

Proof of Theorem 3.3. For $b \in I$, let $L_b\colon A \to I$ and $R_b\colon A \to I$ be the linear operators given by $L_b(x) = bx$ and $R_b(x) = xb$ for $x \in A$. We claim that the operator $\delta R_b\colon A \to A$ is continuous. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in A with

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \delta R_b(x_n) = \lim_{n\to\infty} \delta(x_n b) = x.$$

Since δ is closable and $\lim_{n\to\infty} x_n b = 0$, $x_n b \in I$, we have $x = 0$. That is, $\Phi(\delta R_b) = 0$. By the closed graph theorem, δR_b is continuous. Similarly, δL_b is also continuous.

We claim that σ is continuous. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in A with $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \sigma(x_n) = x$. For $b, c \in I$, since δR_{bc} and δR_b are continuous, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \delta R_{bc}(x_n) = \lim_{n\to\infty} \delta(x_n bc) = \lim_{n\to\infty} (\sigma(x_n b)\delta(c) + \delta(x_n b)c) \\ &= \lim_{n\to\infty} (\sigma(x_n)\sigma(b)\delta(c) + \delta R_b(x_n)c) = x\sigma(b)\delta(c). \end{aligned}$$

This implies that $x\sigma(b)\delta(c) = 0$ for all $b, c \in I$. Hence $x\sigma(I)\delta(I) = 0$. By primeness of A and $\delta \ne 0$, we see that $\delta(I) \ne 0$ and so $x = 0$, implying the continuity of σ.

For $a, b \in I$ and $x \in A$, we have $\delta(axb) = \sigma(a)\delta(xb) + \delta(a)xb$. That is,

$$M_{\delta(a),b}(x) = \delta R_b L_a(x) - L_{\sigma(a)}\delta R_b(x).$$

Note that $\delta R_b L_a$ and $L_{\sigma(a)}\delta R_b$ are continuous. Thus

$$\|M_{\delta(a),b}\| \le \|\delta R_b L_a\| + \|L_{\sigma(a)}\delta R_b\| \le \|a\|(1 + \|\sigma\|)\|\delta R_b\| \quad \text{for all } a, b \in I.$$

By assumption, there exists $K > 0$ such that $K\|\delta(a)\|\,\|b\| \le \|M_{\delta(a),b}\|$ for all $a, b \in I$. So $\|\delta(a)\| \le K'\|a\|$ for some $K' > 0$. This proves the theorem.

Before proving our last result, we refer the reader to [2, Chapter 2] for the notion of the symmetric algebra of quotients of a semisimple algebra. Theorem 1.4 is an immediate consequence of Theorem 3.5 below.

THEOREM 3.5. *Let A be a prime semisimple Banach algebra and let Q be the symmetric algebra of quotients of A. Suppose that Q contains a nontrivial idempotent and $\delta\colon A \to A$ is a linear map. If $\sigma(x)\delta(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$, where σ is a linear automorphism of A, then there exists a nonzero ideal J of A such that $\delta\colon J \to A$ is closable. In addition, if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$, then δ is continuous.*

Proof. In view of [18, Theorem 1.1], there exist $a, b \in Q$, a nonzero ideal J of A and a σ-derivation $d\colon A \to Q$ such that $\delta(x) = d(x) + \sigma(x)b = d(x) + ax$ for all $x \in J$. Moreover, $J = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$. Choose a nonzero ideal K of A such that $K \subseteq J$ and $bK \cup Kb \subseteq A$. Set $I = K \cap \sigma^{-1}(K)$. Then K is a nonzero ideal of A such that $I \subseteq J$ and $\sigma(I)b \cup bI \subseteq A$. Since $d(x) = \delta(x) - \sigma(x)b$ for $x \in I$, we see that $d(I) \subseteq A$.

Let $x \in J$ and $y \in I$. Then $\delta(x)y = d(x)y + \sigma(x)by = d(xy) - \sigma(x)d(y) + \sigma(x)(by)$. That is, $R_y\delta(x) = dR_y(x) - R_{d(y)}\sigma(x) + R_{by}\sigma(x)$. By Theorem 1.1, $d\colon I \to A$ is closable. By the same proof given in Theorem 3.3, $dR_y\colon A \to A$ is continuous. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in J, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \delta(x_n) = x \in \Phi(\delta)$. Since σ is continuous [23, Corollary 6.12], it is easy to see that $0 = \lim_{n\to\infty} R_y\delta(x_n) = xy$. Hence $xI = 0$ and then $xAI = 0$. By primeness of A, $x = 0$. Thus $\Phi(\delta) = 0$. This proves the theorem.

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References

[1] Z. Abdelali, *On Φ-derivation in Banach algebras*, Comm. Algebra 34 (2006), 2437–2452.

[2] P. Ara and M. Mathieu, *Local Multipliers of C^*-algebras*, Springer, London, 2003.

[3] M. I. Berenguer and A. R. Villena, *Continuity of Lie mappings of the skew elements of Banach algebras with involution*, Proc. Amer. Math. Soc. 126 (1998), 2717–2720.

[4] —, —, *Continuity of Lie derivations on Banach algebras*, Proc. Edinburgh Math. Soc. 41 (1998), 625–630.

[5] M. Brešar, *On automorphisms of Banach algebras*, Arch. Math. (Basel) 78 (2002), 297–302.

[6] M. Brešar, A. Fošner and M. Fošner, *A Kleinecke–Shirokov type condition with Jordan automorphisms*, Studia Math. 147 (2001), 237–242.

[7] M. Brešar and P. Šemrl, *Finite rank elements in semisimple Banach algebras*, ibid. 128 (1998), 287–298.

[8] M. Brešar and P. Šemrl, *Commutativity preserving linear maps on central simple algebras*, J. Algebra 284 (2005), 102–110.
[9] M. Brešar and A. R. Villena, *The noncommutative Singer–Wermer conjecture and ϕ-derivations*, J. London Math. Soc. 66 (2002), 710–720.
[10] M. A. Chebotar, W.-F. Ke and P.-H. Lee, *Maps characterized by actions on zero products*, Pacific J. Math. 216 (2004), 217–228.
[11] M. A. Chebotar, W.-F. Ke, P.-H. Lee and N.-C. Wong, *Mappings preserving zero products*, Studia Math. 155 (2003), 77–94.
[12] C.-L. Chuang and C.-K. Liu, *Extended Jacobson density theorem for rings with skew derivations*, Comm. Algebra 35 (2007), 1391–1413.
[13] —, —, *Skew derivations mapping into the socle*, manuscript.
[14] W. Jing, S. Lu and P. Li, *Characterizations of derivations on some operator algebras*, Bull. Austral. Math. Soc. 66 (2002), 227–232.
[15] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. 90 (1968), 1067–1073.
[16] I. Kaplansky, *Functional Analysis. Some Aspects of Analysis and Probability*, Surveys Appl. Math. 4, Wiley, New York, and Chapman & Hall, London, 1958.
[17] —, *Derivations of Banach algebras*, in: Seminars on Analytic Functions, Vol. II, Institute for Advanced Study, Princeton, 1958, 254–258.
[18] T.-K. Lee, *Generalized skew derivations characterized by acting on zero products*, Pacific J. Math. 216 (2004), 293–301.
[19] T.-K. Lee and C.-K. Liu, *Spectrally bounded ϕ-derivations on Banach algebras*, Proc. Amer. Math. Soc. 133 (2005), 1427–1435.
[20] M. Mathieu, *Elementary operators on prime C^*-algebras I*, Math. Ann. 284 (1989), 223–244.
[21] M. Mirzavaziri and M. S. Moslehian, *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Amer. Math. Soc. 134 (2006), 3319–3327.
[22] S. Sakai, *On a conjecture of Kaplansky*, Tôhoku Math. J. 12 (1960), 31–33.
[23] A. M. Sinclair, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge Univ. Press, Cambridge, 1976.
[24] —, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. 24 (1970), 209–214.
[25] M. P. Thomas, *Primitive ideals and derivations on noncommutative Banach algebras*, Pacific J. Math. 159 (1993), 139–152.
[26] A. R. Villena, *Derivations on Jordan–Banach algebras*, Studia Math. 118 (1996), 205–229.
[27] —, *Essentially defined derivations on semisimple Banach algebras*, Proc. Edinburgh Math. Soc. 40 (1997), 175–179.
[28] W. J. Wong, *Maps on simple algebras preserving zero products. I. The associative case*, Pacific J. Math. 89 (1980), 229–247.

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