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Canonical Banach function spaces generated by Urysohn universal spaces. Measures as Lipschitz maps

by

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Abstract. It is proved (independently of the result of Holmes [Fund. Math. 140 (1992)]) that the dual space of the uniform closure $\mathrm{CFL}(\mathbb{U}_r)$ of the linear span of the maps $x \mapsto d(x,a) - d(x,b)$, where d is the metric of the Urysohn space $\mathbb{U}_r$ of diameter r, is (isometrically if $r = +\infty$) isomorphic to the space $\mathrm{LIP}(\mathbb{U}_r)$ of equivalence classes of all real-valued Lipschitz maps on $\mathbb{U}_r$. The space of all signed (real-valued) Borel measures on $\mathbb{U}_r$ is isometrically embedded in the dual space of $\mathrm{CFL}(\mathbb{U}_r)$ and it is shown that the image of the embedding is a proper weak* dense subspace of $\mathrm{CFL}(\mathbb{U}_r)^*$. Some special properties of the space $\mathrm{CFL}(\mathbb{U}_r)$ are established.

The unbounded Urysohn space was introduced in [13, 14]. Holmes [3] has proved that this space generates a unique (up to linear isometry) Banach space (for simpler proofs see [4], [8] or [10]). Such metric spaces are called *linearly rigid*. The Banach space generated by a linearly rigid metric space X is isometrically isomorphic to the predual of the space $\mathrm{Lip}_0(X)$ of real-valued Lipschitz maps on X vanishing at a fixed point of X (equipped with the “Lipschitz” norm). It turns out that linearly rigid spaces are necessarily unbounded, provided they have more than two points (see [10]). This means that bounded Urysohn spaces do not generate unique Banach spaces. However, as we shall show, the dual space of some Banach function space generated by a bounded Urysohn space $\mathbb{U}_r$ is isomorphic to the space $\mathrm{Lip}_0(\mathbb{U}_r)$. The fundamental properties of Urysohn spaces will also enable us to link Borel measures with Lipschitz maps by means of a linear isometric map (given by a simple formula). However, the correspondence is not one-to-one, i.e. there are Lipschitz maps which do not “come from” measures.

Notation and terminology. The sets of all nonnegative reals and positive integers are denoted by $\mathbb{R}_+$ and $\mathbb{N}_*$, respectively.

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For a separable complete metric space X, $\mathrm{Mes}(X)$ stands for the Banach space of all signed (real-valued) Borel measures on X, equipped with the standard total variation norm. It is well known that each nonnegative (finite) Borel measure μ on X is *regular*, i.e. $\mu(A) = \sup\{\mu(K) \mid K \subset A,\ K \text{ compact}\}$ for any Borel subset A of X. This implies that the subspace $\mathrm{Mes}_{\mathrm{c}}(X)$ of $\mathrm{Mes}(X)$ consisting of all measures supported on compact sets is dense (with respect to the norm topology) in the whole space.

A *Lipschitz map* between metric spaces (X, d) and (Y, ϱ) is any function $f : X \to Y$ for which there is a finite constant $M \geq 0$ such that $\varrho(f(x), f(y)) \leq Md(x, y)$ for every $x, y \in X$. We denote by $\mathrm{Lip}(X)$ the space of all real-valued Lipschitz maps on X. If X has more than one point, we equip the space $\mathrm{Lip}(X)$ with the following seminorm:

$$l(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)}, \qquad f \in \mathrm{Lip}(X).$$

This is not a norm, because $l(f) = 0$ if and only if f is constant. Therefore we take the quotient space $\mathrm{Lip}(X)/\mathrm{Const}(X)$, where $\mathrm{Const}(X)$ consists of all (real-valued) constant maps on X, and denote it by $\mathrm{LIP}(X)$. The space $\mathrm{LIP}(X)$ is a Banach space with respect to its norm:

$$L(f + \mathrm{Const}(X)) = l(f), \qquad f \in \mathrm{Lip}(X).$$

In what follows we shall write, for simplicity, $f \in \mathrm{LIP}(X)$ and $L(f)$ instead of $f \in \mathrm{Lip}(X)$ or $L(f+\mathrm{Const}(X))$. However, one has to remember that $\mathrm{LIP}(X)$ is **not** a function space. Nevertheless, if x and y are two points of X, the functional $\mathrm{LIP}(X) \ni f \mapsto f(x) - f(y) \in \mathbb{R}$ is well defined. Additionally, let $B_L(X)$ stand for the closed unit ball of $\mathrm{LIP}(X)$.

It is easy to see that $\mathrm{LIP}(X)$ is isometrically isomorphic to $\mathrm{Lip}_0(X, x)$, the subspace of $\mathrm{Lip}(X)$ constisting of the maps vanishing at x, where x is any fixed point of X. Spaces of type Lip_0 are well studied (see e.g. [15]). It is known that they are dual spaces, and the preduals are well described (the Arens–Eells spaces). For us, the two most important properties of $\mathrm{Lip}_0(X, x)$, after an adaptation to $\mathrm{LIP}(X)$, are (see also [11] for proofs):

- (L1) If X is separable, then the ball $B_L(X)$ is (compact and) metrizable in the weak* topology, and a sequence $(f_n)_{n\in\mathbb{N}_*}$ of elements of $B_L(X)$ is weak* convergent to $f \in B_L(X)$ if and only if $f_n(x) - f_n(y) \to f(x) - f(y)$ $(n \to \infty)$ for all $x, y \in X$ (this condition, in fact, defines the weak* topology on $B_L(X)$).
- (L2) Any weak* continuous functional $\psi : \mathrm{LIP}(X) \to \mathbb{R}$ has the form

$$\psi(f) = \sum_{n=1}^{\infty} a_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)},$$

where $\sum_{n=1}^{\infty}|a_n|<\infty$ and $(x_n)_{n\in\mathbb{N}_*}$ and $(y_n)_{n\in\mathbb{N}_*}$ are two sequences of elements of X such that $x_n \neq y_n$; what is more, the sequences $(a_n)_{n\in\mathbb{N}_*}$, $(x_n)_{n\in\mathbb{N}_*}$ and $(y_n)_{n\in\mathbb{N}_*}$ may be taken so that $\sum_{n=1}^{\infty}|a_n|$ is arbitrarily close to $\|\psi\|$.

Whenever we deal with spaces of real-valued maps, the symbol $\|\cdot\|$ denotes the supremum norm.

Now we pass to the main subject of the paper.

1. Definition. An *Urysohn space* is a separable complete metric space X such that every separable metric space of diameter no greater than $\operatorname{diam} X$ is isometrically embeddable in X, and each isometry between finite subsets of X is extendable to an isometry of X onto itself. An Urysohn space is *nontrivial* if it has more than one point.

For every $r\in[0,+\infty]$ there is a unique (up to isometry) Urysohn space of diameter r. We shall denote it by $\mathbb{U}_r$, and $\mathbb{U}$ will stand for the unbounded Urysohn space.

A *Katětov map* on a metric space (X,d) is any function $f: X\to\mathbb{R}_+$ such that

$$|f(x)-f(y)|\le d(x,y)\le f(x)+f(y)\quad \text{for all } x,y\in X.$$

A *common sphere* is a set of the form

$$S_X(A,f)=\{x\in X \mid \forall a\in A: d(x,a)=f(a)\}$$

with nonempty $A\subset X$ and any function $f: A\to\mathbb{R}_+$. If X is Urysohn, then $S_X(K,f)$ is nonempty for each nonempty compact subset K of X and every Katětov map f on K such that $f(K)\subset[0,\operatorname{diam} X]$. This is a consequence of the Huhunaišvili theorem [5]. For $r\in[0,+\infty]$, we denote by $E_r(X)$ the set of Katětov maps f on X such that $f(X)\subset[0,r]$. For more on Katětov maps, see [6], [8], [1]. The reader interested in Urysohn spaces is referred to [7, 8].

From now on, $r\in(0,+\infty]$, d is the metric of $\mathbb{U}_r$, and $B_L=B_L(\mathbb{U}_r)$. Let $\varrho:\mathbb{U}_r\to\mathrm{LIP}(\mathbb{U}_r)$ be defined as follows: $\varrho(x)$ is the equivalence class of the map e_x, where $e_x(y)=d(x,y)$. It is easy to see that $\varrho(\mathbb{U}_r)\subset B_L$, and ϱ is continuous when $\mathrm{LIP}(\mathbb{U}_r)$ is considered with the weak* topology.

First we shall establish the basic properties of the set $\varrho(\mathbb{U}_r)$.

2. Proposition. *The set $\varrho(\mathbb{U}_r)$ is linearly independent.*

Proof. Let $n\ge 2$. Suppose that $x_1,\dots,x_n$ are distinct points of $\mathbb{U}_r$ and $\alpha_1,\dots,\alpha_n$ are scalars such that the map $u=\alpha_1 e_{x_1}+\dots+\alpha_n e_{x_n}$ is constant. Let $M=\operatorname{diam}\{x_1,\dots,x_n\}$. For $j\in\{1,\dots,n\}$, put $p_j=\min_{k\neq j} d(x_j,x_k)>0$. Let $A=\{x_1,\dots,x_n\}$, let $f_0: A\to\mathbb{R}_+$ be the constant map with value M, and for $j=1,\dots,n$, let $f_j: A\to\mathbb{R}_+$ be defined as follows: $f_j(x_j)=M-p_j$ and $f_j(x_k)=M$ for $k\neq j$. It is easy to verify that $f_0,\dots,f_n$ are

Katětov maps and take values in $[0,r]$. There are points $z_0,\dots,z_n$ such that $e_{x_j}(z_k)=f_k(x_j)$. Since the map u is constant it follows that $u(z_j)=u(z_0)$, or equivalently $\sum_{m=1}^n \alpha_m(f_j(x_m)-f_0(x_m))=0$. But this yields $\alpha_j p_j=0$ and thus $\alpha_1=\dots=\alpha_n=0$. ∎

3. Theorem. *The weak* closure of $\varrho(\mathbb{U}_r)$ contains the ball $\frac{1}{2}B_L$. If $r=+\infty$, then $\varrho(\mathbb{U}_r)$ is weak* dense in B_L.*

Proof. First assume that $r=+\infty$. It is enough to show that for each $f\in B_L$ and any finite nonempty subset A of $\mathbb{U}$ there are $C\in\mathbb{R}$ and $x\in\mathbb{U}$ such that $f+C=e_x$ on A. Since A is finite, there is C such that $d(a,b)-f(a)-f(b)\le 2C$ for $a,b\in A$. This implies that $f+C$ is a Katětov map on A. So, there exists $x\in\mathbb{U}$ for which $f(a)+C=d(x,a)$ $(a\in A)$. But this means that $f+C=e_x$ on A.

Now assume that r is finite. Take $f\in B_L$ and a finite nonempty subset A of $\mathbb{U}_r$. We have to prove that there are $C\in\mathbb{R}$ and $x\in\mathbb{U}_r$ such that $\frac{1}{2}f+C=e_x$ on A. Observe that since $L(f)\le 1$ and $\operatorname{diam}\mathbb{U}_r=r$, there is a constant α such that the image of $f+\alpha$ is contained in $\left[-\frac{1}{2}r,\frac{1}{2}r\right]$. But then the image of $\frac{1}{2}f+C$, where $C=\frac{1}{2}\alpha+\frac{3}{4}r$, is a subset of $\left[\frac{1}{2}r,r\right]$ and thus $f+C$ is a Katětov map on A (because $\frac{1}{2}r\ge\frac{1}{2}\operatorname{diam}A$). So, as in the first case, it suffices to take $x\in\mathbb{U}_r$ such that $\frac{1}{2}f+C=e_x$ on A. ∎

4. Corollary. *If $\psi\in\mathrm{LIP}(\mathbb{U}_r)^*$ is weak* continuous, then*

$$\frac{1}{2}\,\|\psi\|\le\sup_{x\in\mathbb{U}_r}\psi(\varrho(x))\le\|\psi\|,$$

and if $r=+\infty$, then $\|\psi\|=\sup_{x\in\mathbb{U}_r}\psi(\varrho(x))$.

5. Corollary. *The set $\varrho(\mathbb{U})$ is metrizable in the weak* topology, but is not completely metrizable. In particular, $\varrho:\mathbb{U}\to\varrho(\mathbb{U})$ is not a homeomorphism.*

Proof. Suppose that, on the contrary, $\varrho(\mathbb{U})$ is completely metrizable. Then, by Theorem 3, it is a dense $\mathcal{G}_\delta$-subset of B_L and thus so is $-\varrho(\mathbb{U})$. But $\varrho(\mathbb{U})$ and $-\varrho(\mathbb{U})$ are disjoint (thanks to Proposition 2), contrary to the Baire theorem. ∎

Corollary 4 leads us to the following

6. Definition. The *canonical function linear space* (for short, the CFL space) of the Urysohn space $\mathbb{U}_r$ is the space $\mathrm{CFL}(\mathbb{U}_r)$ consisting of the maps $f:\mathbb{U}_r\to\mathbb{R}$ of type $f(x)=\psi(\varrho(x))$, where ψ is a weak* continuous functional on $\mathrm{LIP}(\mathbb{U}_r)$. Since $\varrho(\mathbb{U}_r)$ is a subset of B_L, $\mathrm{CFL}(\mathbb{U}_r)$ consists of bounded maps. The CFL space is equipped with the supremum norm.

As an immediate consequence of Corollary 4 we obtain

7. Theorem. *The CFL space of the* [*unbounded*] *Urysohn space* $\mathbb{U}_r$ *is* [*isometrically*] *isomorphic to the predual of* $\mathrm{LIP}(\mathbb{U}_r)$ *and therefore it is a Banach space. The canonical* [*isometric*] *isomorphism* $J : \mathrm{LIP}(\mathbb{U}_r)_* \to \mathrm{CFL}(\mathbb{U}_r)$ *has the form*

$$(J(\psi))(x) = \psi(\varrho(x)) \qquad (\psi \in \mathrm{LIP}(\mathbb{U}_r)_*,\ x \in \mathbb{U}_r),$$

and $\max(\|J\|, \|J^{-1}\|) \le 2$.

Now we will characterize the maps belonging to $\mathrm{CFL}(\mathbb{U}_r)$.

8. Theorem. *A function* $f : \mathbb{U}_r \to \mathbb{R}$ *is a member of* $\mathrm{CFL}(\mathbb{U}_r)$ *if and only if for any* $\varepsilon > 0$ *there are* $u_1, \ldots, u_m \in \mathbb{U}_r$ *and* $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ *such that* $\sum_{j=1}^m \alpha_j = 0$ *and for each* $x \in \mathbb{U}_r$,

$$\Big| f(x) - \sum_{j=1}^m \alpha_j d(u_j, x) \Big| \le \varepsilon. \tag{1}$$

Proof. If $f \in \mathrm{CFL}(\mathbb{U}_r)$, then, by (L2), there are sequences $(a_n)_{n=1}^\infty \in l^1$ and $(x_n, y_n)_{n=1}^\infty \in (\mathbb{U}_r \times \mathbb{U}_r)^{\mathbb{N}_*}$ such that $x_n \neq y_n$ and

$$f(x) = \sum_{n=1}^\infty a_n \frac{d(x_n, x) - d(y_n, x)}{d(x_n, y_n)}.$$

So, it is enough to take $N \ge 1$ such that $\sum_{n=N+1}^\infty |a_n| < \varepsilon$ and to express the map $x \mapsto \sum_{n=1}^N a_n (d(x_n, x) - d(y_n, x))/d(x_n, y_n)$ in the form $\sum_{j=1}^m \alpha_j e_{u_j}$.

Now assume that f is the uniform limit of a sequence of maps of the form $\sum_{j=1}^m \alpha_j e_{u_j}$ with $\sum_{j=1}^m \alpha_j = 0$. It is easy to check that

$$\Big(\sum_{j=1}^m \alpha_j e_{u_j}\Big)(x) = \psi(\varrho(x)),$$

where $\psi : \mathrm{LIP}(\mathbb{U}_r) \to \mathbb{R}$ is a weak* continuous functional given by $\psi(g) = \sum_{j=1}^m \alpha_j g(u_j)$ $(g \in \mathrm{LIP}(\mathbb{U}_r))$. So, $\sum_{j=1}^m \alpha_j e_{u_j} \in \mathrm{CFL}(\mathbb{U}_r)$ and thus the completeness of $\mathrm{CFL}(\mathbb{U}_r)$ finishes the proof. ∎

9. Corollary. *Let* f *be a nonzero member of the CFL space of* $\mathbb{U}$. *Then the image* R *of* f *is a bounded interval such that* $(-\|f\|, \|f\|) \subset R \subset [-\|f\|, \|f\|]$.

Proof. By Theorem 8, f is continuous and thus R is an interval. Further, thanks to Corollary 4, $\|f\|$ belongs to the closure of R and, similarly, $\|-f\|$ is in the closure of $-R$. This implies that the closure of R coincides with $[-\|f\|, \|f\|]$. Now the assertion is clear. ∎

In contrast to the case of the unbounded Urysohn space, for $r < +\infty$ the set $\varrho(\mathbb{U}_r)$ is not weak* dense in B_L. What is more, the canonical isomorphism $J : \mathrm{LIP}(\mathbb{U}_r)_* \to \mathrm{CFL}(\mathbb{U}_r)$ is nonisometric, as shown by

10. Proposition. *For $r < +\infty$, the convex hull V of the set $\varrho(\mathbb{U}_r) \cup (-\varrho(\mathbb{U}_r))$ is not weak* dense in B_L. In particular, the canonical isomorphism J is nonisometric.*

Proof. Take four points p_1, p_2, p_3, p_4 of $\mathbb{U}_r$ such that $d(p_j, p_k) = r$ for distinct $j, k \in \{1, 2, 3, 4\}$. Let $g \in B_L$ be any nonexpansive map such that $g(p_j) = 0$ and $g(p_k) = r$ for $j = 1, 2$ and $k = 3, 4$. We claim that g does not belong to the weak* closure W of V. Suppose, for contradiction, that $g \in W$. This implies that there are numbers $t_1, \ldots, t_n$ and points $x_1, \ldots, x_n$ of $\mathbb{U}_r$ such that $\sum_{j=1}^n |t_j| = 1$ and the map $g - \sum_{j=1}^n t_j e_{x_j}$ is constant on $A = \{p_1, p_2, p_3, p_4\}$. Since $e_x \in E_r(\mathbb{U}_r)$ for each $x \in \mathbb{U}_r$ and $E_r(A)$ is convex we infer that there is a $c \in \mathbb{R}$ such that

$$(g + c)|_A \in \operatorname{conv}[E_r(A) \cup (-E_r(A))] \tag{2}$$

("conv" stands for convex hull). It is easily seen that $g|_A$, as an element of $\mathrm{LIP}(A)$, is an extreme point of $B_L(A)$. Thus, by (2), $g|_A + c = \pm f$ for some $f \in E_r(A)$. This shows that $f(p_1) = f(p_2)$, $f(p_3) = f(p_4)$ and $|f(p_1) - f(p_3)| = r$. But $f(A) \subset [0, r]$ and therefore $f(p_1) = f(p_2) = 0$ and $f(p_3) = f(p_4) = r$ or conversely. So, $\operatorname{card} f^{-1}(\{0\}) > 1$, which contradicts the relation $f \in E_r(A)$.

We have shown that $g \notin W$. By the separation theorem, there is a weak* continuous functional $\psi : \mathrm{LIP}(\mathbb{U}_r) \to \mathbb{R}$ such that $|\psi(u)| \le 1$ for each $u \in W$ and $\psi(g) > 1$. Hence $\|\psi\| > 1$ and $\|J(\psi)\| \le 1$, which finishes the proof. ∎

Theorem 7 says that the dual of $\mathrm{CFL}(\mathbb{U}_r)$ may be identified with $\mathrm{LIP}(\mathbb{U}_r)$ (at least in the unbounded case). This identification has the following form: a functional $\varphi \in \mathrm{CFL}(\mathbb{U}_r)^*$ corresponds to a map $g \in \mathrm{LIP}(\mathbb{U}_r)$ such that $g(x) - g(y) = \psi(e_x - e_y)$ $(x, y \in \mathbb{U}_r)$. For $f \in \mathrm{CFL}(\mathbb{U}_r)$ and $g \in \mathrm{LIP}(\mathbb{U}_r)$, we shall write $\int f \, dg$ or $\int f(x) \, dg(x)$ for the value at f of the functional corresponding to g. Thus for each $a, b \in \mathbb{U}_r$,

$$\int (e_a - e_b) \, dg = \int (d(a, x) - d(b, x)) \, dg(x) = g(a) - g(b). \tag{3}$$

The next result will enable us to link measures with Lipschitz maps on Urysohn spaces.

11. Theorem. *Let K be a (nonempty) compact subset of $\mathbb{U}_r$ and let $f : K \to \mathbb{R}$ be continuous. Then there is $F \in \mathrm{CFL}(\mathbb{U}_r)$ such that $F|_K = f$ and $\|F\| = \|f\|$. What is more, for a given element z of the common sphere $S_{\mathbb{U}_r}(K, s)$, where $s \in (0, +\infty)$ is such that $\frac{2}{3} \operatorname{diam} K \le s \le \frac{4}{5} r$, there are sequences $(x_n)_{n \in \mathbb{N}_*}$ and $(t_n)_{n \in \mathbb{N}_*}$ of elements of $\mathbb{U}_r$ and of positive numbers, respectively, such that $\|F\| = \sum_{n \in \mathbb{N}_*} t_n d(x_n, z)$ and $F = \sum_{n \in \mathbb{N}_*} t_n (e_{x_n} - e_z)$.*

Proof. It is easily seen that $S_{\mathbb{U}_r}(K, s)$ is nonempty and $z \notin K$.

First assume that $f \in \mathrm{Lip}(K)$, $l(f) \le 1$ and $\|f\| \le \frac{1}{4} s$. Define a map $g : K \cup \{z\} \to \mathbb{R}$ by $g(x) = f(x) + s$ for $x \in K$ and $g(z) = \|f\|$. The map g

is clearly nonexpansive on K (i.e. $l(g|_K) \leq 1$). What is more, for $x, y \in K$ we have $d(x,y) \leq \frac{3}{2}s \leq g(x) + g(y)$. Further,

$$|g(x) - g(z)| = s - \|f\| + f(x) \leq s = d(x,z) \leq s + f(x) + \|f\| = g(x) + g(z).$$

So, g is a Katětov map and $g(K \cup \{z\}) \subset [0,r]$. This implies that there is $u \in \mathbb{U}_r$ such that $g(y) = d(u,y)$ for $y \in K \cup \{z\}$. But then $f = (e_u - e_z)|_K$ and $d(u,z) = g(z) = \|f\|$, and thus in that case the proof is finished.

Now consider an arbitrary map f. By [12], there are sequences $(t_n)_{n\in\mathbb{N}_*}$ and $(f_n)_{n\in\mathbb{N}_*}$ of positive numbers and real-valued nonexpansive maps on K (respectively) such that $\|f\| = \sum_{n\in\mathbb{N}_*} t_n\|f_n\|$ and $f = \sum_{n\in\mathbb{N}_*} t_n f_n$. Replacing, if necessary, the pair (t_n, f_n) by a suitable pair $(t_n/s_n, s_n f_n)$ with $s_n \in (0,1)$, we may assume that $\|f_n\| \leq \frac{1}{4}s$ for every n. We infer from the first part of the proof that there is a sequence $(x_n)_{n\in\mathbb{N}_*}$ of elements of $\mathbb{U}_r$ for which $(e_{x_n} - e_z)|_K = f_n$ and $d(x_n, z) = \|f_n\|$. This implies that $\|f\| = \sum_{n\in\mathbb{N}_*} t_n d(x_n, z)$ and therefore the series $\sum_{n\in\mathbb{N}_*} t_n(e_{x_n} - e_z)$ is uniformly convergent. Let F be its uniform limit. By Theorem 7, $F \in \mathrm{CFL}(\mathbb{U}_r)$. Furthermore, $F|_K = \sum_{n\in\mathbb{N}_*} t_n f_n = f$ and thus $\|f\| \leq \|F\|$. On the other hand, $\|F\| \leq \sum_{n\in\mathbb{N}_*} t_n d(x_n, z) = \|f\|$, which finishes the proof. ∎

12. Corollary. *Let K be a (nonempty) compact subset of $\mathbb{U}_r$ and*

$$\Phi_K : \mathrm{CFL}(\mathbb{U}_r) \ni f \mapsto f|_K \in \mathcal{C}(K),$$

where $\mathcal{C}(K)$ is the algebra of all real-valued continuous functions on K. Then Φ_K sends the closed unit ball onto the closed unit ball and therefore the adjoint operator $\Phi_K^* : \mathrm{Mes}(K) \to \mathrm{LIP}(\mathbb{U}_r)$ is a weak continuous isomorphic (and isometric if $r = +\infty$) embedding such that $\max(\|\Phi_K^*\|, \|(\Phi_K^*)^{-1}\|) \leq 2$. If $\mu \in \mathrm{Mes}(K)$ and $g = \Phi_K^*(\mu)$, then for each $f \in \mathrm{CFL}(\mathbb{U}_r)$,*

$$\int_K f\,\mathrm{d}\mu = \int f\,\mathrm{d}g. \tag{4}$$

In particular, $\Phi_K^*(\delta_a) = \varrho(a)$ for $a \in K$, where δ_a is the Dirac measure at a.

13. Lemma. *Let K and L be two (nonempty) compact subsets of $\mathbb{U}_r$. If $\mu \in \mathrm{Mes}(\mathbb{U}_r)$ is a measure supported on $K \cap L$ (and therefore μ may be seen as a member of $\mathrm{Mes}(K)$ and $\mathrm{Mes}(L)$), then $\Phi_K^*(\mu) = \Phi_L^*(\mu)$.*

Proof. Let $g = \Phi_K^*(\mu)$ and $h = \Phi_L^*(\mu)$. Then for any $f \in \mathrm{CFL}(\mathbb{U}_r)$ we have

$$\int f\,\mathrm{d}g = \int_K f\,\mathrm{d}\mu = \int_{K\cap L} f\,\mathrm{d}\mu = \int_L f\,\mathrm{d}\mu = \int f\,\mathrm{d}h,$$

which implies that $g = h$. ∎

The above lemma enables us to define an operator $j_0 : \mathrm{Mes}_c(\mathbb{U}_r) \to \mathrm{LIP}(\mathbb{U}_r)$ by $j_0(\mu) = \Phi_K^*(\mu)$, where K is a compact subset of $\mathbb{U}_r$ such that μ is supported on K. Since the definition is independent of the choice of K, the map j_0 is a linear embedding such that $\max(\|j_0\|, \|j_0^{-1}\|) \leq 2$ and thus it is uniquely extendable to an isomorphic embedding of $\mathrm{Mes}(\mathbb{U}_r)$ in $\mathrm{LIP}(\mathbb{U}_r)$. We introduce the following

14. Definition. The *canonical embedding* of $\mathrm{Mes}(\mathbb{U}_r)$ in $\mathrm{LIP}(\mathbb{U}_r)$ is a unique continuous extension $j : \mathrm{Mes}(\mathbb{U}_r) \to \mathrm{LIP}(\mathbb{U}_r)$ of j_0. The canonical embedding is an isomorphism between its domain and range which sends Dirac's measure δ_x to $\varrho(x)$ for each $x \in \mathbb{U}_r$. What is more, the formula (4) is satisfied for any $\mu \in \mathrm{Mes}(\mathbb{U}_r)$ with $g = j(\mu)$ and K replaced by $\mathbb{U}_r$. If $r = +\infty$, then j is isometric.

The next result can be easily obtained from (4) by substituting $f = e_x - e_y$.

15. Theorem. *Let $\mu \in \mathrm{Mes}(\mathbb{U}_r)$ and $g = j(\mu)$. Then for each $x, y \in \mathbb{U}_r$,*

$$g(x) - g(y) = \int_{\mathbb{U}_r} (d(x,z) - d(y,z))\,\mathrm{d}\mu(z). \tag{5}$$

It is rather surprising that j is isometric also for bounded Urysohn spaces. We shall prove this in the following

16. Proposition. *For $r < +\infty$, the canonical embedding $j : \mathrm{Mes}(\mathbb{U}_r) \to \mathrm{LIP}(\mathbb{U}_r)$ is isometric.*

Proof. Let K be a compact nonempty subset of $\mathbb{U}_r$ and let $\mu \in \mathrm{Mes}(\mathbb{U}_r)$. Put $g = j(\mu)$. It is enough to check that $\|g\| \geq \|\mu\| - \varepsilon$ for $\varepsilon > 0$. Since the space $\mathrm{Lip}(K)$ is dense in $\mathcal{C}(K)$, there is $u \in \mathrm{Lip}(K)$ such that $\|u\| = 1$ and $\int_K u\,\mathrm{d}\mu \geq \|\mu\| - \varepsilon$. Take $t > 0$ such that $l(tu) \leq 1$ and $\|tu\| \leq \frac{3}{16}r$. It follows from the proof of Theorem 11 that there are $x, z \in \mathbb{U}_r$ for which $u = \frac{1}{t}(e_x - e_z)|_K$. This yields $t = \|e_x - e_z\| = d(x,z)$ and thus

$$\|\mu\| - \varepsilon \leq \int_K u\,\mathrm{d}\mu = \int_K \frac{e_x - e_z}{d(x,z)}\,\mathrm{d}\mu = \frac{g(x) - g(z)}{d(x,z)} \leq \|g\|. \ \blacksquare$$

17. Corollary. *The norm closure of the linear span of $\varrho(\mathbb{U}_r)$ in $\mathrm{LIP}(\mathbb{U}_r)$ is (naturally) isometrically isomorphic to $l^1(\varrho(\mathbb{U}_r))$.*

18. Corollary. *For any $\mu \in \mathrm{Mes}(\mathbb{U}_r)$, the total variation $|\mu|(\mathbb{U}_r)$ of the measure μ satisfies the condition*

$$|\mu|(\mathbb{U}_r) = \sup\left\{\left|\int_{\mathbb{U}_r} \frac{d(x,z) - d(y,z)}{d(x,y)}\,\mathrm{d}\mu(z)\right| : x, y \in \mathbb{U}_r,\ x \neq y\right\}.$$

Our next aim is to prove that the canonical embedding j is nonsurjective. We shall do this using different methods for bounded and unbounded Urysohn spaces.

It is folklore that $\varrho(\mathbb{U}_r)$ consists of extreme points of the ball B_L. However, $\varrho(\mathbb{U}_r) \cup (-\varrho(\mathbb{U}_r))$ is a proper subset of the set of all extreme points of B_L, as we shall see below. In fact, for $r = +\infty$, this is a consequence of the following

19. Lemma. *Let (Z, λ) be a metric space and A its nonempty subset. If $f \in \mathrm{LIP}(A)$ is an extreme point of $B_L(A)$, then the Katětov extension $\widehat{f}$ of f is an extreme point of $B_L(Z)$, where $\widehat{f}(z) = \inf_{a \in A}(f(a) + \lambda(a, z))$.*

Proof. It is easily checked that $\widehat{f} \in B_L(Z)$ and $\widehat{f}$ extends f. What is more, $\widehat{f}$ is the greatest element (with respect to the pointwise order) among nonexpansive extensions of f. So, if $g_1, g_2 \in B_L(Z)$ are such that $\widehat{f} = (g_1 + g_2)/2 + C$ for some constant C, then $f = (g_1|_A + g_2|_A)/2 + C$ and thus $f = g_1|_A + C_1 = g_2|_A + C_2$, where C_1 and C_2 are constants with $C_1 + C_2 = 2C$. Thus $g_j + C_j \le \widehat{f}$ $(j = 1, 2)$. But $g_1 + C_1 + g_2 + C_2 = 2\widehat{f}$ and therefore $\widehat{f} = g_1 + C_1 = g_2 + C_2$. ∎

20. Proposition. *There are extreme points of B_L which do not belong to $\varrho(\mathbb{U}) \cup (-\varrho(\mathbb{U}))$. In particular, the canonical embedding $j : \mathrm{Mes}(\mathbb{U}) \to \mathrm{LIP}(\mathbb{U})$ is not surjective.*

Proof. Let A be a subset of $\mathbb{U}$ which is isometric to $\mathbb{R}$ and let $\varphi : A \to \mathbb{R}$ be an isometry. Since the operator $\mathrm{LIP}(\mathbb{R}) \ni u \mapsto u \circ \varphi \in \mathrm{LIP}(A)$ is an isometric isomorphism and the map $f : \mathbb{R} \ni t \mapsto t \in \mathbb{R}$ is an extreme point of $B_L(\mathbb{R})$, it follows that $\varphi = f \circ \varphi$ is an extreme point of $B_L(A)$. So, by Lemma 19, $v = \widehat{\varphi}$ is extreme in B_L. Observe that $v(\mathbb{U}) = \mathbb{R}$, from which we infer that $v \notin \varrho(\mathbb{U}) \cup (-\varrho(\mathbb{U}))$. Finally, since the set $j^{-1}(\varrho(\mathbb{U}) \cup (-\varrho(\mathbb{U})))$ consists of all extreme points of the closed unit ball of $\mathrm{Mes}(\mathbb{U})$ (and j is isometric), it follows that v is not the value of j. ∎

Now we have to show the same for a bounded Urysohn space. In order to do that, we need the following

21. Lemma. *Let $r < +\infty$ and let $\{a_n : n \ge 1\}$ be a dense subset of $\mathbb{U}_r$. If $g \in \mathrm{LIP}(\mathbb{U}_r)$ is an element of the image of j, then for any $\varepsilon > 0$ there exists $N \ge 1$ such that*

$$|g(x) - g(y)| \le \|g\| \cdot \|(e_x - e_y)|_A\| + \varepsilon \tag{6}$$

for all $x, y \in \mathbb{U}_r$, where $A = \{a_1, \ldots, a_N\}$.

Proof. We may assume that $g \ne 0$. Let $\mu = j^{-1}(g)$. There is a compact nonempty subset K of $\mathbb{U}_r$ such that $|\mu|(\mathbb{U}_r \setminus K) \le \varepsilon/3r$. Since K is compact, there are $x_1, \ldots, x_p \in K$ such that $K \subset \bigcup_{j=1}^{p} B(x_j, \varepsilon/6\|g\|)$. Finally, there are positive integers $m_1, \ldots, m_p$ such that $d(x_j, a_{m_j}) \le \varepsilon/6\|g\|$ for $j = 1, \ldots, p$. Put $N = \max(m_1, \ldots, m_p)$ and $A = \{a_1, \ldots, a_N\}$. Take $x, y \in \mathbb{U}_r$. By the triangle inequality, for each $z \in K$ there is $a \in A$ for which $|d(x, z) - d(y, z)| \le |d(x, a) - d(y, a)| + 2\varepsilon/3\|g\|$ and thus $\|(e_x - e_y)|_K\| \le \|(e_x - e_y)|_A\| + 2\varepsilon/3\|g\|$. This yields

$$\begin{aligned}|g(x)-g(y)| &= \Big|\int\limits_{\mathbb{U}_r} (d(x,z)-d(y,z))\,\mathrm{d}\mu(z)\Big| \\ &\le \int\limits_K |e_x-e_y|\,\mathrm{d}|\mu| + \int\limits_{\mathbb{U}_r\setminus K} |e_x-e_y|\,\mathrm{d}|\mu| \\ &\le \Big(\|(e_x-e_y)|_A\| + \frac{2\varepsilon}{3\|g\|}\Big)|\mu|(\mathbb{U}_r) + r\cdot|\mu|(\mathbb{U}_r\setminus K) \\ &\le \|g\|\cdot\|(e_x-e_y)|_A\| + \varepsilon. \ \blacksquare\end{aligned}$$

And now the announced result:

22. Proposition. *For $r<+\infty$, the canonical embedding j is nonsurjective. There are extreme points of B_L which do not belong to $\varrho(\mathbb{U}_r)\cup(-\varrho(\mathbb{U}_r))$.*

Proof. Let $(U_n)_{n\ge1}$ be a sequence of nonempty open subsets of $\mathbb{U}_r$ which form a basis of the topology of $\mathbb{U}_r$. Take any $x_1\in U_1$ and put $g(x_1)=0$. Now suppose that we have found points $x_1,\dots,x_{3k-2}$ of $\mathbb{U}_r$ and have defined $g(x_1),\dots,g(x_{3k-2})$ (for some $k\ge1$) in such a way that for any $j\in\{1,\dots,k\}$:

$(1)_j$ $\{x_1,\dots,x_{3j-2}\}\cap U_j\ne\emptyset$,
$(2)_j$ $|g(x_p)-g(x_q)|\le d(x_p,x_q)$ and $g(x_p)\in[0,r]$ for $p,q=1,\dots,3j-2$,
$(3)_j$ if $j>1$, then $|g(x_{3j-4})-g(x_{3j-3})|=r$ and $e_{x_{3j-4}}=e_{x_{3j-3}}$ on the set $\{x_1,\dots,x_{3j-5}\}$.

Take $x_{3k-1},x_{3k}\in\mathbb{U}_r$ such that $d(x_p,x_q)=r$ for $p=1,\dots,3k-2$ and $q=3k-1,3k$ and $d(x_{3k-1},x_{3k})=r$. Put $g(x_{3k-1})=r$ and $g(x_{3k})=0$. Now pick any $x_{3k+1}\in U_{k+1}\setminus\{x_1,\dots,x_{3k}\}$ and define $g(x_{3k+1})=\min\{g(x_j) + d(x_j,x_{3k+1}) : j\in\{1,\dots,3k\}\}$. There is no difficulty in checking that $g(x_{3k+1})\in[0,r]$ and that the conditions $(1)_{k+1}$–$(3)_{k+1}$ hold. Thus we have obtained sequences $(x_n)_{n\in\mathbb{N}_*}$ and $(g(x_n))_{n\in\mathbb{N}_*}$ such that the set $D=\{x_n : n\ge1\}$ is dense in $\mathbb{U}_r$, the map $g:D\to\mathbb{R}$ is nonexpansive and for any finite subset C of D there are $x,y\in D$ such that $e_x=e_y$ on C and $|g(x)-g(y)|=r$. Let $h\in B_L$ be the unique nonexpansive extension of g. The properties of g and Lemma 21 imply that $h\notin j(\mathrm{Mes}(\mathbb{U}_r))$.

Now suppose that the set of all extreme points of B_L coincides with $M=\varrho(\mathbb{U}_r)\cup(-\varrho(\mathbb{U}_r))$. As B_L is metrizable in the weak* topology, the Choquet theorem yields a Borel probability measure λ on M such that

$$\int\limits_M u\,\mathrm{d}\lambda(u)=h. \tag{7}$$

Further, since $\mathbb{U}_r\ni x\mapsto\varrho(x)\in\varrho(\mathbb{U}_r)$ is a continuous bijection, $\varrho(\mathbb{U}_r)$ is a Borel subset of M and the inverse function is Borel. Let $\mu_1,\mu_2\in\mathrm{Mes}(\mathbb{U}_r)$ be defined by $\mu_1(A)=\lambda(\varrho(A))$ and $\mu_2(A)=\lambda(-\varrho(A))$ for a Borel subset A of $\mathbb{U}_r$, and let $\mu=\mu_1-\mu_2$. Fix $x,y\in\mathbb{U}_r$. Since the functional $\mathrm{LIP}(\mathbb{U}_r)\ni$

$u \mapsto u(x) - u(y) \in \mathbb{R}$ is weak* continuous, by (7) and the measure transport theorem,

$$\begin{aligned} h(x) - h(y) &= \int_M (u(x) - u(y))\,\mathrm{d}\lambda(u) \\ &= \int_{\varrho(\mathbb{U}_r)} (u(x) - u(y))\,\mathrm{d}\lambda(u) + \int_{-\varrho(\mathbb{U}_r)} (u(x) - u(y))\,\mathrm{d}\lambda(u) \\ &= \int_{\mathbb{U}_r} (\varrho(z)(x) - \varrho(z)(y))\,\mathrm{d}\mu_1(z) + \int_{\mathbb{U}_r} (-\varrho(z)(x) + \varrho(z)(y))\,\mathrm{d}\mu_2(z) \\ &= \int_{\mathbb{U}_r} (d(x,z) - d(y,z))\,\mathrm{d}\mu(z), \end{aligned}$$

which means that $h = j(\mu)$. But this contradicts the first part of the proof. ∎

23. Remark. The nonsurjectivity of j in the case of a bounded Urysohn space may be immediately deduced from Propositions 10 and 16. Indeed, it is easy to check that if $\Psi : \mathrm{Mes}(\mathbb{U}_r) \to \mathrm{CFL}(\mathbb{U}_r)^*$ is an operator defined by $\Psi(\mu)(f) = \int_{\mathbb{U}_r} f\,\mathrm{d}\mu$, then Ψ is isometric (by Theorem 11 or Corollary 12) and $j = J \circ \Psi$. The same argument shows that $J|_E$ is an isometry between $E = \Psi(\mathrm{Mes}(\mathbb{U}_r))$ and $F = j(\mathrm{Mes}(\mathbb{U}_r))$. What is more, J, as a dual operator, is a weak* homeomorphism and the spaces E and F are weak* dense in $\mathrm{CFL}(\mathbb{U}_r)^*$ and $\mathrm{LIP}(\mathbb{U}_r)$, respectively. So, we have obtained an interesting example of a weak* homeomorphism which is isometric on a weak* dense subspace of the domain, but not isometric on the whole domain.

Our last aim is to establish some geometric properties of the space $\mathrm{CFL}(\mathbb{U}_r)$. The property (L2) implies that B_L is the closed convex hull of the set $\mathrm{CFL}_0(\mathbb{U}_r) = \{(e_x - e_y)/d(x,y) : x, y \in \mathbb{U}_r,\ x \neq y\}$. The next result shows that the set $\mathrm{CFL}_0(\mathbb{U})$ is transitive with respect to isometric isomorphisms of $\mathrm{CFL}(\mathbb{U})$.

24. Theorem. *For any $f, g \in \mathrm{CFL}_0(\mathbb{U})$ there exists an isometric isomorphism $V : \mathrm{CFL}(\mathbb{U}) \to \mathrm{CFL}(\mathbb{U})$ such that $V(f) = g$.*

Proof. Let (p,q) and (a,b) be pairs of distinct points of $\mathbb{U}$ such that

$$f = \frac{e_p - e_q}{d(p,q)} \quad \text{and} \quad g = \frac{e_a - e_b}{d(a,b)}.$$

There is a bijection $\varphi : \mathbb{U} \to \mathbb{U}$ such that $\varphi(p) = a$, $\varphi(q) = b$ and $d(\varphi(x), \varphi(y)) = \lambda d(x,y)$ for any $x, y \in \mathbb{U}$, where $\lambda = d(a,b)/d(p,q)$. Now let $V : \mathrm{CFL}(\mathbb{U}) \to \mathrm{CFL}(\mathbb{U})$ be the linear operator defined by $V(h) = h \circ \varphi^{-1}$ ($h \in \mathrm{CFL}(\mathbb{U})$). The map V is well defined, because

$$V(e_x - e_y) = \frac{e_{\varphi(x)} - e_{\varphi(y)}}{\lambda}. \tag{8}$$

It is clearly a bijective isometric map and (8) shows that $V(f) = g$. ∎

Now let $\omega \in \mathbb{U}_r$ and $A_\omega = \{e_x - e_\omega : x \in \mathbb{U}_r\} \subset \mathrm{CFL}(\mathbb{U}_r)$. Note that $0 \in A_\omega$ and the map $m_\omega : \mathbb{U}_r \ni x \mapsto e_x - e_\omega \in A_\omega$ is isometric, so the set A_ω is closed. It is also a total subset of $\mathrm{CFL}(\mathbb{U}_r)$. First we shall prove the following

25. Proposition. *The set $A_\omega \setminus \{0\}$ is linearly independent.*

Proof. Let $x_1, \ldots, x_n$ be distinct elements of $\mathbb{U}_r \setminus \{\omega\}$ and $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\sum_{j=1}^n \alpha_j (e_{x_j} - e_\omega) = 0$. This implies that $\sum_{j=1}^n \alpha_j \varrho(x_j) = (\sum_{j=1}^n \alpha_j)\varrho(\omega)$. So, Proposition 2 finishes the proof. ∎

To state the next result, we need an auxiliary notion. For a number $\lambda \in (0, +\infty)$, we say that a function $w : P \to Q$ between metric spaces (P, p) and (Q, q) is *λ-isometric* if

$$q(w(x), w(y)) = \lambda p(x, y) \quad \text{for each } x, y \in P.$$

Additionally, set $\Lambda(P) = \{1\}$ if P is bounded, and $\Lambda(P) = (0, +\infty)$ otherwise. Now we are ready to present

26. Theorem. *Let $\omega, \tau \in \mathbb{U}_r$. Let K be a nonempty compact subset of A_ω and let $v : K \to A_\tau$ be λ-isometric with $\lambda \in \Lambda(\mathbb{U}_r)$. Then there is an isometric isomorphism $V : \mathrm{CFL}(\mathbb{U}_r) \to \mathrm{CFL}(\mathbb{U}_r)$ and $f_0 \in A_\tau$ such that $v(f) = \lambda V(f) + f_0$ for every $f \in K$.*

Proof. Let $K_0 = m_\omega^{-1}(K)$ and $u : K_0 \ni x \mapsto m_\tau^{-1}(v(m_\omega(x))) \in \mathbb{U}_r$. The set K_0 is compact and u is λ-isometric, so there is a bijective λ-isometric map $U : \mathbb{U}_r \to \mathbb{U}_r$ which extends u. Now put $V : \mathrm{CFL}(\mathbb{U}_r) \ni f \mapsto f \circ U^{-1} \in \mathrm{CFL}(\mathbb{U}_r)$ and $f_0 = e_{U(\omega)} - e_\tau \in A_\tau$. As in the proof of Theorem 24, V is an isometric isomorphism such that $V(e_x - e_y) = (e_{U(x)} - e_{U(y)})/\lambda$. So, if $x \in \mathbb{U}_r$ and $f = e_x - e_\omega$, then

$$\begin{aligned} v(f) &= v(m_\omega(x)) = m_\tau(u(x)) = m_\tau(U(x)) = e_{U(x)} - e_\tau \\ &= \lambda V(e_x - e_\omega) + e_{U(\omega)} - e_\tau = \lambda V(f) + f_0. \ ∎ \end{aligned}$$

27. Remark. As mentioned at the beginning of the paper, the fact that the dual of $\mathrm{CFL}(\mathbb{U})$ is linearly isometric to $\mathrm{LIP}(\mathbb{U})$ is a consequence of the Holmes theorem [3, 4]. Namely, he has shown that if $(E, \|\cdot\|)$ is any Banach space such that $\mathbb{U} \subset E$ and $\|x - y\| = d(x, y)$ for all $x, y \in \mathbb{U}$, then for any $x_1, \ldots, x_n \in \mathbb{U}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{j=1}^n \alpha_j = 0$ one has

$$\Big\|\sum_{j=1}^n \alpha_j x_j\Big\| = \sup\Big\{\Big|\sum_{j=1}^n \alpha_j f(x_j)\Big| : f \in B_L(\{x_1, \ldots, x_n\})\Big\}.$$

However, other properties of $\mathrm{CFL}(\mathbb{U})$ cannot be deduced from the above fact, and Holmes' theorem applies only to the unbounded Urysohn space.

We end the paper with the following two questions. In both of them, r is finite.

QUESTION 1. The universality of an unbounded Urysohn space $\mathbb{U}$ and the results of Godefroy and Kalton [2] imply that the space $\mathrm{CFL}(\mathbb{U})$ is universal for separable Banach spaces (this was observed by Melleray [9]). These arguments do not work in the case of a bounded Urysohn space. Is the space $\mathcal{C}([0,1])$ isometrically or isomorphically embeddable in $\mathrm{CFL}(\mathbb{U}_r)$?

QUESTION 2. Suppose that $(E, \|\cdot\|)$ is a Banach space such that $\mathbb{U}_r \subset E$ and $\|x-y\| = d(x,y)$ for all $x, y \in \mathbb{U}_r$. Does there exist a constant $c > 0$ such that whenever $x_1, \ldots, x_n$ are points of $\mathbb{U}_r$ and $\alpha_1, \ldots, \alpha_n$ are real numbers with $\sum_{j=1} \alpha_j = 0$, then

$$\Big\|\sum_{j=1}^{n} \alpha_j x_j\Big\| \geq c \sup\Big\{\Big|\sum_{j=1}^{n} \alpha_j g(x_j)\Big| : g \in B_L(\{x_1, \ldots, x_n\})\Big\}?$$

Does there exist a universal constant $c > 0$ for which the above estimate holds (independently of the space E)?

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