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Convergence a.e. of spherical partial Fourier integrals on weighted spaces for radial functions: endpoint estimates

by

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Abstract. We prove some extrapolation results for operators bounded on radial L^p functions with $p \in (p_0, p_1)$ and deduce some endpoint estimates. We apply our results to prove the almost everywhere convergence of the spherical partial Fourier integrals and to obtain estimates on maximal Bochner–Riesz type operators acting on radial functions in several weighted spaces.

1. Introduction. For maximal spherical partial Fourier integrals as well as for maximal Bochner–Riesz means long standing open problems still remain to be solved concerning boundedness properties for general $f \in L^p(\mathbb{R}^n)$ while a lot is known in case f is radial. The situation is similar for other operators such as general maximal spherical operators and Kakeya maximal operators ([7], [18], [19], [14], [1], [15], [28], [29], [17], [22]). Setting

$$L^p_{\mathrm{rad}}(\mu) = \{f \in L^p(\mu) : f \text{ is radial}\}$$

where μ is a sigma-finite measure in $\mathbb{R}^n$, we are interested in operators T such that

$$T : L^p_{\mathrm{rad}}(\mu) \to L^p(\mu)$$

is bounded for every $p \in (p_0, p_1)$ and the operator norm satisfies

$$\|T\|_p \le \frac{C}{(p-p_0)^{\alpha_0}(p-p_1)^{\alpha_1}}, \tag{1.1}$$

with C a constant independent of p. This is the case for many of the examples in the above literature.

To obtain information at the endpoints p_0, p_1, one is led naturally from inequalities of the kind (1.1) to the theory of extrapolation of operators due to Yano (see [31]), which says that a sublinear operator satisfying an L^p estimate, with constant $(p-1)^{-m}$ as $p \to 1^+$, is bounded from $L(\log L)^m$ to

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$L^1 + L^\infty$. More recently the work by Antonov [2] has spurred new research on endpoint estimates ([30], [11], [10], [9]) and has brought new ideas to this old theory.

In Section 2 we obtain endpoint results for operators bounded on $L^p_{\mathrm{rad}}(\mu)$ with $p \in (p_0, p_1)$ using extrapolation techniques. We deal separately with the right and the left endpoint since the spaces involved are different.

In Section 3 we present some applications. First we prove the almost everywhere convergence of the spherical partial Fourier integrals of radial functions f in certain weighted spaces such as $L^{p_j}(\log L)^{\beta_j}(w)$, as well as in weighted Lorentz spaces $\Lambda^1_w(t^{1/p_0-1}(1+\log\frac{1}{t})^{\beta_0})$ and $\Lambda^1_w(t^{1/p_1-1}(1+\log t)^{\beta_1})$ where w is a radial weight satisfying an A_1 condition and β_j are positive numbers, $j = 0, 1$. The case $w = 1$ has been studied in [15] and [27]. In the case of weighted L^p spaces a result can be found in [26]. In the process we estimate the norm of the maximal Carleson operator in weighted L^p spaces for certain weights in the Muchkenhoupt class A_p ([24]). Our second application deals with maximal Bochner–Riesz type operators acting on radial functions introduced in [19]. For n odd we improve the main result of [19] at the endpoints.

We shall denote by C a constant depending possibly on p but uniformly bounded for all $p \in [p_0, p_1]$. Also C might not be the same in all instances. We write $A \lesssim B$ if there exists a universal constant C such that $A \le CB$, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We shall work in $\mathbb{R}^n$, μ will be a sigma-finite measure on $\mathbb{R}^n$, and w will be a radial weight in $\mathbb{R}^n$. As usual, $g^*_\mu(t) = \inf\{s : \lambda^\mu_g(s) \le t\}$ is the decreasing rearrangement of g, where $\lambda^\mu_g(y) = \mu\{x \in \mathbb{R}^n : |g(x)| > y\}$ is the distribution function of g with respect to the measure μ and $g^{**}_\mu(t) = t^{-1}\int_0^t g^*_\mu(s)\,ds$. When the measure μ is given by $w(x)dx$ we shall write g^*_w and λ^w_g instead of using the subscript μ (we refer the reader to [4] for further information about distribution functions and decreasing rearrangements). For a measurable set E, χ_E denotes the characteristic function of E, $|E|$ denotes the Lebesgue measure of E and $w(E) = \int_E w(x)\,dx$.

We mention that with obvious changes everything could be done for operators $T : L^p_{\mathrm{rad}}(\mu) \to L^p(\nu)$, but for simplicity of notation we shall consider $\mu = \nu$.

2. Endpoint estimates. Let us first recall that given a weight v in $\mathbb{R}^+$, the weighted Lorentz spaces $\Lambda^1_\mu(v)$ and $\Gamma^{1,\infty}_u(v)$ are defined to be the sets of measurable functions such that

$$\|f\|_{\Lambda^1_\mu(v)} = \int_0^\infty f^*_\mu(t)v(t)\,dt < \infty$$

and

$$\|f\|_{\Gamma^{1,\infty}_\mu(w)} = \sup_{t>0} V(t) f^{**}_\mu(t) < \infty$$

respectively, with $V(t) = \int_0^t v(s)\,ds$.

Since in this section the measure μ is fixed, we shall write $\|f\|_p = \|f\|_{L^p(\mu)}$, $f^*_\mu = f^*$, $\lambda_f = \lambda^\mu_f$, $\Lambda^1_\mu(v) = \Lambda^1(v)$ and so on.

2.1. Let us start with the left endpoint p_0; that is, we shall assume that T is a sublinear operator such that

$$\|Tf\|_{L^p} \lesssim \frac{1}{(p-p_0)^\alpha} \|f\|_{L^p_{\mathrm{rad}}(\mu)} \tag{2.1}$$

for every $p \in (p_0, p_1)$ with $\alpha > 0$.

Our first extrapolation result is the following:

THEOREM 2.1. *If T satisfies* (2.1) *and f is a radial function*, *then*

$$\sup_{t>0} \frac{t^{1/p_0}(Tf)^{**}(t)}{(1+\log^+ t)^\alpha} \lesssim \int_0^\infty f^*(t) t^{1/p_0} \left(1 + \log^+ \frac{1}{t}\right)^\alpha \frac{dt}{t},$$

that is, $T : \Lambda^1_{\mathrm{rad}}(v_0) \to \Gamma^{1,\infty}(v_1)$ *is bounded with*

$$v_0(t) = t^{1/p_0 - 1}\left(1 + \log^+ \frac{1}{t}\right)^\alpha \quad \textit{and} \quad v_1(t) = t^{1/p_0-1}(1+\log^+ t)^{-\alpha}.$$

REMARK 2.2. At this point, it is important to observe that, if $p_0 > 1$, the above endpoint estimate is "near" (except for the logarithmic factors) to the restricted weak type estimate for $T : L^{p_0,1}_{\mathrm{rad}}(\mu) \to L^{p_0,\infty}(\mu)$. However, we have to mention that, under our hypothesis, we cannot expect to get the restricted weak type estimate, since there are examples of operators satisfying (2.1) for which the restricted weak type inequality is known to be false. For instance (see [28]), the classical spherical maximal function was observed by Bourgain to be of restricted weak type at the endpoint $p = d/(d-1)$ when $d \ge 3$ but it is known to fail to be of restricted weak type at $p = 2$ when $d = 2$. We thank the referee for pointing out to us this concrete example.

Proof of Theorem 2.1. If $\|g\|_\infty \le 1$ and g is radial then

$$(Tg)^{**}(t) t^{1/p} \le \|Tg\|_p \lesssim \frac{1}{(p-p_0)^\alpha} \|g\|_p \lesssim \frac{1}{(p-p_0)^\alpha} \|g\|_1^{1/p}$$

and hence

$$(Tg)^{**}(t) \lesssim \frac{1}{(p-p_0)^\alpha} \left(\frac{\|g\|_1}{t}\right)^{1/p}.$$

Taking the infimum over $p_0 < p < p_1$, we get

$$(Tg)^{**}(t) \lesssim \left(\frac{\|g\|_1}{t}\right)^{1/p_0} \left(1 + \log^+ \frac{t}{\|g\|_1}\right)^\alpha.$$

Let now f be a general radial function and let us write $f = \sum_{i\in\mathbb{Z}} 2^i f_i$ where $f_i = (f/2^i)\chi_{\{2^{i-1}\le|f|<2^i\}}$ is also radial and $\|f_i\|_\infty \le 1$. Hence,

$$(Tf)^{**}(t) \le \sum_{i\in\mathbb{Z}} 2^i (Tf_i)^{**}(t) \lesssim \sum_{i\in\mathbb{Z}} 2^i \left(\frac{\|f_i\|_1}{t}\right)^{1/p_0} \left(1+\log^+ \frac{t}{\|f_i\|_1}\right)^{\alpha},$$

and since $\|f_i\|_1 \le \lambda_f(2^{i-1})$, we obtain

$$\sup_{t>0} \frac{t^{1/p_0}(Tf)^{**}(t)}{(1+\log^+ t)^{\alpha}} \lesssim \sum_{i\in\mathbb{Z}} 2^i D(\lambda_f(2^i)) \approx \int_0^\infty D(\lambda_f(y))\,dy$$

with $D(x) = x^{1/p_0}\big(1+\log^+ \frac{1}{x}\big)^{\alpha}$. The result now follows since

$$\int_0^\infty D(\lambda_f(y))\,dy \approx \int_0^\infty f^*(t) t^{1/p_0} \left(1+\log^+ \frac{1}{t}\right)^{\alpha} \frac{dt}{t}. \ \blacksquare$$

This result is the best known for the case $p_0 = 1$. However, if $p_0 > 1$, it can be improved using some technical results of interpolation theory plus a modification adapted to radial functions of some extrapolation techniques developed in [11].

Let us start with a technical lemma.

LEMMA 2.3. *Let G be a concave function such that*

$$G(t) \approx \Big(\int_0^{t^{p_0}} g^*(s)^{p_0}\,ds\Big)^{1/p_0}$$

and let, for every $i \in \mathbb{Z}$,

$$E_i = \{s \in (0,\infty) : G'(s) > 2^i\}.$$

Then there exist $(g_i)_{i\in\mathbb{Z}}$ such that $g = \sum_i 2^i g_i$ and

$$\Big(\int_0^{t^{p_0}} g_i^*(s)^{p_0}\,ds\Big)^{1/p_0} \lesssim \min(t, |E_i|).$$

Proof. Let us first mention that since the function

$$H(t) = \Big(\int_0^{t^{p_0}} g^*(s)^{p_0}\,ds\Big)^{1/p_0}$$

is increasing and $H(s)/s$ is decreasing, H is quasi-concave and hence equivalent to a concave function; so the existence of G is clear. Since G' is a decreasing function we have $G'(s) \le 4\sum_i 2^i \chi_{E_i}(s)$ and $G(t) \lesssim \sum_i 2^i \min(t, |E_i|)$. On the other hand, $H(t) \approx K(g, t; L^{p_0}, L^\infty)$ and thus we can use the K-divisibility theorem of interpolation theory ([5, p. 325]) to prove the result. $\blacksquare$

Our next theorem is an improvement of Theorem 2.1.

THEOREM 2.4. *If T satisfies* (2.1), *then for every radial function f*,

$$\sup_{t>0}\frac{(\int_0^t[(Tf)^*(s)]^{p_0}\,ds)^{1/p_0}}{(1+\log^+t)^\alpha}\lesssim\|f\|_{p_0}+\int_0^1\frac{(\int_0^t f^*(s)^{p_0}\,ds)^{1/p_0}}{t}\left(\log\frac{1}{t}\right)^{\alpha-1}dt.$$

Proof. Let f be a radial function on $\mathbb{R}^n$ and let g be defined in $(0,\infty)$ by $g(u)=f(u^{1/n})$. Then simple computations show that $f^*(t)=g^*(c_nt)$ for a certain constant c_n depending only on n.

Given g we can apply Lemma 2.3 to deduce that $g=\sum_{i\in\mathbb{Z}}2^ig_i$, where

$$\Big(\int_0^{t^{p_0}}g_i^*(s)^{p_0}\,ds\Big)^{1/p_0}\lesssim\min(t,|E_i|).$$

Taking f_i radial such that $f_i(u^{1/n})=g_i(u)$ we find that $f=\sum_{i\in\mathbb{Z}}2^if_i$ and

$$\Big(\int_0^{t^{p_0}}f_i^*(s)^{p_0}\,ds\Big)^{1/p_0}\lesssim\min(t,|E_i|),$$

or equivalently

$$\int_0^t f_i^*(s)^{p_0}\,ds\lesssim\min(t,|E_i|^{p_0}).$$

From this it follows (see [4, p. 61]) that, for every $p\ge p_0$,

$$\|f_i\|_p\lesssim|E_i|^{p_0/p}.$$

Then, for every $i\in\mathbb{Z}$ and every $p>p_0$,

$$\begin{aligned}\left(\frac{1}{t}\int_0^t[(Tf_i)^*(s)]^{p_0}\,ds\right)^{1/p_0}&\le\left(\frac{1}{t}\int_0^t[(Tf_i)^*(s)]^{p}\,ds\right)^{1/p}\le t^{-1/p}\|Tf_i\|_p\\&\lesssim t^{-1/p}\,\frac{1}{(p-p_0)^\alpha}\,\|f_i\|_p\le\frac{1}{(p-p_0)^\alpha}\left(\frac{|E_i|^{p_0}}{t}\right)^{1/p}.\end{aligned}$$

Taking the infimum over $p_0<p\le q<p_1$ we obtain

$$\left(\frac{1}{t}\int_0^t[(Tf_i)^*(s)]^{p_0}\,ds\right)^{1/p_0}\lesssim\frac{|E_i|}{t^{1/p_0}}\left(1+\log^+\frac{t^{1/p_0}}{|E_i|}\right)^\alpha$$

and therefore

$$\sup_{t>0}\frac{(\int_0^t[(Tf_i)^*(s)]^{p_0}\,ds)^{1/p_0}}{(1+\log^+t)^\alpha}\lesssim|E_i|\left(1+\log^+\frac{1}{|E_i|}\right)^\alpha=:D(|E_i|).$$

Summing over $i\in\mathbb{Z}$ we obtain

$$\sup_{t>0}\frac{(\int_0^t[(Tf)^*(s)]^{p_0}\,ds)^{1/p_0}}{(1+\log^+t)^\alpha}\lesssim\sum_{i\in\mathbb{Z}}2^iD(|E_i|)\approx\int_0^\infty D(\lambda_{G'}(y))\,dy,$$

where G is defined as in Lemma 2.3,

$$G(t) \approx \Big(\int_0^{t^{p_0}} g^*(s)^{p_0}\,ds\Big)^{1/p_0} = \bigg(\int_0^{t^{p_0}} f^*\Big(\frac{s}{c_n}\Big)^{p_0}\,ds\bigg)^{1/p_0}$$
$$= \Big(c_n \int_0^{t^{p_0}/c_n} f^*(s)^{p_0}\,ds\Big)^{1/p_0} \approx \Big(\int_0^{t^{p_0}} f^*(s)^{p_0}\,ds\Big)^{1/p_0}.$$

Thus, integrating by parts gives

$$\sup_{t>0} \frac{(\int_0^t [(Tf)^*(s)]^{p_0}\,ds)^{1/p_0}}{(1+\log^+ t)^\alpha} \lesssim \int_0^\infty G'(t)\,dD(t)$$
$$= \int_0^1 G'(t)\Big(\log\frac{1}{t}\Big)^\alpha dt + \int_1^\infty G'(t)\,dt \lesssim \|f\|_{p_0} + \int_0^1 \frac{G(t)}{t}\Big(\log\frac{1}{t}\Big)^{\alpha-1} dt$$
$$\lesssim \|f\|_{p_0} + \int_0^1 \frac{(\int_0^{t^{p_0}} f^*(s)^{p_0}\,ds)^{1/p_0}}{t}\Big(\log\frac{1}{t}\Big)^{\alpha-1} dt,$$

as we wanted to prove. ■

Definition 2.5. We define the spaces $D^+_{p,\alpha}$ and $R^+_{p,\alpha}$ by the norms

$$\|f\|_{D^+_{p,\alpha}} = \|f\|_p + \int_0^1 \frac{(\int_0^t f^*(s)^p\,ds)^{1/p}}{t}\Big(\log\frac{1}{t}\Big)^{\alpha-1} dt$$

and

$$\|f\|_{R^+_{p,\alpha}} = \sup_{t>0} \frac{(\int_0^t f^*(s)^p\,ds)^{1/p}}{(1+\log^+ t)^\alpha}.$$

Also,

$$D^+_{p,\alpha,\mathrm{rad}} = \{f \in D^+_{p,\alpha} : f \text{ is radial}\}.$$

With these definitions, Theorem 2.4 can be stated as follows:

Theorem 2.6. *If T satisfies* (2.1) *then $T : D^+_{p_0,\alpha,\mathrm{rad}} \to R^+_{p_0,\alpha}$ is bounded.*

In the following remark, we compare our spaces above with classical Orlicz and Lorentz spaces.

Remark 2.7. 1. Theorem 2.4 is an improvement of Theorem 2.1: to see this, let us just recall that $L^{p_0,1} \subset L^{p_0}$ ([4]) and, in fact (see [13]),

$$\sup_f \frac{(\int_0^t f^*(s)^{p_0}\,ds)^{1/p_0}}{\int_0^t f^*(s)s^{1/p_0-1}\,dt} = \sup_{r>0} \frac{(\int_0^t \chi_{(0,r)}\,ds)^{1/p_0}}{\int_0^t \chi_{(0,r)}s^{1/p_0-1}\,dt} = p_0 < \infty.$$

Hence

$$
\begin{aligned}
\|f\|_{D^+_{p_0,\alpha}} &= \|f\|_{p_0} + \int_0^1 \frac{(\int_0^t f^*(s)^{p_0}\,ds)^{1/p_0}}{t}\Big(\log\frac{1}{t}\Big)^{\alpha-1}\,dt \\
&\le \|f\|_{p_0} + \int_0^1 \frac{\int_0^t f^*(s)s^{1/p_0-1}\,ds}{t}\Big(\log\frac{1}{t}\Big)^{\alpha-1}\,dt \\
&\lesssim \int_0^\infty f^*(s)s^{1/p_0-1}\Big(1+\log\frac{1}{s}\Big)^{\alpha}\,ds = \|f\|_{\Lambda^1(v_0)},
\end{aligned}
$$

with v_0 as in the statement of Theorem 2.1. Therefore

$$\Lambda^1(v_0) \subset D^+_{p_0,\alpha}.$$

In fact, the above inclusion is strict since if we take f such that

$$f^*(s) = s^{-1/p_0}\Big(1+\log^+\frac{1}{s}\Big)^{-\beta} \qquad \text{with } 1/p_0+\alpha<\beta<1+\alpha$$

then we have $\|f\|_{\Lambda^1(v_0)} = \infty$ while $\|f\|_{D^+_{p_0,\alpha}} < \infty$.

Also, if v_1 is as in the statement of Theorem 2.1,

$$\|f\|_{\Gamma^{1,\infty}(v_1)} = \sup_{t>0}\frac{t^{1/p_0}f^{**}(t)}{(1+\log^+ t)^\alpha} \le \sup_{t>0}\frac{(\int_0^t f^*(s)^{p_0}\,ds)^{1/p_0}}{(1+\log^+ t)^\alpha} = \|f\|_{R^+_{p_0,\alpha}}$$

and hence

$$R^+_{p_0,\alpha} \subset \Gamma^{1,\infty}(v_1).$$

Moreover, this inclusion is strict since taking f such that $f^*(s) = s^{-1/p_0}$ we have $\|f\|_{R^+_{p_0,\alpha}} = \infty$ while $\|f\|_{\Gamma^{1,\infty}(v_1)} < \infty$.

2. $D^+_{p_0,\alpha}$ is not comparable with $L^{p_0,1}$ for any α. To see this, we observe first that taking

$$f^*(t) = \chi_{(0,1)}(t) + \frac{1}{t^{1/p_0}\big(\log\frac{1}{t}\big)}\,\chi_{(1,\infty)}(t)$$

we have $f \in D^+_{p_0,\alpha}\backslash L^{p_0,1}$. For the converse, we have to apply Theorem 4.1(ii) of [12] to deduce that, for every $\alpha>0$,

$$\sup_f \frac{\int_0^1 (\int_0^t f^*(s)^{p_0}\,ds)^{1/p_0}\big(\log\frac{1}{t}\big)^{\alpha-1}\,\frac{dt}{t}}{\int_0^1 f^*(s)s^{1/p_0-1}\,dt} = \infty.$$

3. If $\beta/p_0 > \alpha$, then

$$\int_0^1 \frac{(\int_0^{t^{p_0}} f^*(s)^{p_0}\,ds)^{1/p_0}}{t}\left(\log\frac{1}{t}\right)^{\alpha-1} dt \le \int_0^1 \frac{(\int_0^{t^{p_0}} f^*(s)^{p_0}(1+\log\frac{1}{s})^{\beta}\,ds)^{1/p_0}}{t(1+\log\frac{1}{t})^{\beta/p_0+1-\alpha}}\,dt$$
$$\le \left(\int_0^\infty f^*(s)^{p_0}\left(1+\log\frac{1}{s}\right)^{\beta} ds\right)^{1/p_0}$$
$$\lesssim \|f\|_{L^{p_0}(\log L)^{\beta}},$$

that is, $L^{p_0}(\log L)^{\beta} \subset D^+_{p_0,\alpha}$.

2.2. Now we consider the right endpoint p_1. Let T be a sublinear operator such that, for every $p_0 \le p < p_1$,

$$\|Tf\|_{L^p(\mu)} \lesssim \frac{1}{(p_1-p)^{\alpha}}\,\|f\|_{L^p_{\mathrm{rad}}(\mu)}. \tag{2.2}$$

Theorem 2.8. *If T satisfies* (2.2) *then*, *for every radial function f*,

$$\sup_{t>0}\frac{t^{1/p_1}(Tf)^{**}(t)}{(1+\log^+\frac{1}{t})^{\alpha}} \lesssim \int_0^\infty f^*(t)t^{1/p_1}(1+\log^+ t)^{\alpha}\,\frac{dt}{t};$$

that is, $T : \Lambda^1(v_0) \to \Gamma^{1,\infty}(v_1)$ *is bounded with* $v_0(t) = t^{1/p_1-1}(1+\log^+ t)^{\alpha}$ *and* $v_1(t) = t^{1/p_1-1}(1+\log^+\frac{1}{t})^{-\alpha}$.

Proof. Let g be a radial function such that $\|g\|_\infty \le 1$. Then

$$(Tg)^{**}(t) \lesssim \left(\frac{1}{p_1-p}\right)^{\alpha}\left(\frac{\|g\|_1}{t}\right)^{1/p}$$

and taking the infimum over $p_0 \le p < p_1$ we obtain

$$(Tg)^{**}(t) \lesssim \left(\frac{\|g\|_1}{t}\right)^{1/p_1}\left(1+\log^+\frac{\|g\|_1}{t}\right)^{\alpha}.$$

Let now f be a general radial function and let us write $f = \sum_{i\in\mathbb{Z}} 2^i f_i$ where $f_i = (f/2^i)\chi_{\{2^{i-1}\le|f|<2^i\}}$. As in Theorem 2.1 we obtain

$$\sup_{t>0}\frac{t^{1/p_1}(Tf)^{**}(t)}{(1+\log^+\frac{1}{t})^{\alpha}} \lesssim \int_0^\infty D(\lambda_f(y))\,dy$$

with $D(x) = x^{1/p_1}(1+\log^+ x)^{\alpha}$. The result now follows since

$$\int_0^\infty D(\lambda_f(y))\,dy \approx \int_0^\infty f^*(t)t^{1/p_1}(1+\log^+ t)^{\alpha}\,\frac{dt}{t}. \ \blacksquare$$

To improve the above result in the case $p_1 < \infty$, we have to proceed as in Theorem 2.4.

LEMMA 2.9. *Let G be a concave function such that*

$$G(t) \approx t\Big(\int_{t^{p_1'}}^{\infty} g^{**}(s)^{p_1}\,ds\Big)^{1/p_1}$$

and let, for every $i \in \mathbb{Z}$,

$$E_i = \{s \in (0,\infty) : G'(s) > 2^i\}.$$

Then there exist $(g_i)_{i\in\mathbb{Z}}$ such that $g = \sum_i 2^i g_i$ and

$$t\Big(\int_{t^{p_1'}}^{\infty} g_i^{**}(s)^{p_1}\,ds\Big)^{1/p_1} \lesssim \min(t, |E_i|).$$

Proof. It suffices to observe that $G(t) \approx K(g,t;L^1,L^{p_1})$ (see [4]) and proceed as in Lemma 2.3. ∎

THEOREM 2.10. *If T satisfies (2.2) then, for every f radial,*

$$\sup_{t>0} \frac{(\int_t^\infty (Tf)^{**}(s)^{p_1}\,ds)^{1/p_1}}{(1+\log^+\frac{1}{t})^\alpha} \lesssim \|f\|_{p_1} + \int_1^\infty \frac{(\int_t^\infty f^{**}(s)^{p_1}\,ds)^{1/p_1}}{t}(1+\log^+ t)^{\alpha-1}dt.$$

Proof. Let $f \in L^{p_1}_{\mathrm{rad}}(\mathbb{R}^n)$ be such that $\|f\|_{p_1} = c$ with c to be chosen later, and let g be defined in $(0,\infty)$ by $g(u) = f(u^{1/n})$.

Now given g we can apply Lemma 2.9 to deduce that $g = \sum_{i\in\mathbb{Z}} 2^i g_i$, where

$$t\Big(\int_{t^{p_1'}}^{\infty} g_i^{**}(s)^{p_1}\,ds\Big)^{1/p_1} \lesssim \min(t, |E_i|).$$

with $E_i = \{s : G'(s) > 2^i\}$ and G defined as in Lemma 2.9.

Since G' is decreasing,

$$G'(t) \le \frac{1}{t}\int_0^t G'(s)\,ds = \frac{G(t)}{t} \lesssim \|f\|_{p_1} \approx c,$$

and hence choosing c such that $G'(t) \le 1$ we find that $E_i = \emptyset$ whenever $i > 0$. Taking f_i radial such that $f_i(u^{1/n}) = g_i(u)$ we have $f = \sum_{i=-\infty}^0 2^i f_i$ and

$$t\Big(\int_{t^{p_1'}}^{\infty} f_i^{**}(s)^{p_1}\,ds\Big)^{1/p_1} \lesssim \min(t, |E_i|).$$

From this it follows that $\|f_i\|_{p_1} \lesssim 1$ and since

$$\int_0^t f^*(s)\,ds \approx t^{1/p_1'}\Big(\int_t^\infty \frac{1}{s^{p_1}}\,ds\Big)^{1/p_1}\int_0^t f^*(s)\,ds$$
$$\lesssim t^{1/p_1'}\Big(\int_t^\infty \frac{(\int_0^s f^*(u)\,du)^{p_1}}{s^{p_1}}\,ds\Big)^{1/p_1} \lesssim \min(t^{1/p_1'}, |E_i|)$$

we see that $\|f_i\|_1 \lesssim |E_i|$ and hence, for every $p < p_1$,

$$\|f_i\|_p \lesssim |E_i|^{1-p_1'+p_1'/p}.$$

Then, for every $i \in \mathbb{Z}^-$ and every $p < p_1$,

$$\Big(\int_{t^{p_1'}}^\infty [(Tf_i)^{**}(s)]^{p_1}\,ds\Big)^{1/p_1} \lesssim \Big(\int_{t^{p_1'}}^\infty [(Tf_i)^{**}(s)]^{p} s^{p/p_1-1}\,ds\Big)^{1/p}$$
$$\lesssim t^{p_1'(1/p_1-1/p)}\|Tf_i\|_p \lesssim t^{p_1'(1/p_1-1/p)}\Big(\frac{1}{p_1-p}\Big)^\alpha \|f_i\|_p$$
$$\le \Big(\frac{1}{p_1-p}\Big)^\alpha \Big(\frac{|E_i|}{t}\Big)^{1-p_1'}\Big(\frac{|E_i|}{t}\Big)^{p_1'/p}.$$

Taking the infimum over $p_0 \le p < p_1$ we obtain

$$\Big(\int_{t^{p_1'}}^\infty [(Tf_i)^{**}(s)]^{p_1}\,ds\Big)^{1/p_1} \lesssim \Big(1 + \log^+ \frac{|E_i|}{t}\Big)^\alpha$$

and therefore

$$\sup_{t>0} \frac{(\int_{t^{p_1'}}^\infty [(Tf_i)^{**}(s)]^{p_1}\,ds)^{1/p_1}}{\big(1+\log^+ \frac{1}{t}\big)^\alpha} \lesssim (1+\log^+|E_i|)^\alpha =: D(|E_i|).$$

Summing over $i \in \mathbb{Z}^-$ we obtain

$$\sup_{t>0} \frac{(\int_t^\infty [(Tf)^{**}(s)]^{p_1}\,ds)^{1/p_1}}{\big(1+\log^+ \frac{1}{t}\big)^\alpha} \lesssim \sum_{i\in\mathbb{Z}^-} 2^i D(|E_i|) \approx \int_0^1 D(\lambda_{G'}(y))\,dy$$
$$\approx 1 + \int_0^\infty (\log^+(\lambda_{G'}(y)))^\alpha\,dy.$$

Integrating by parts gives

$$\sup_{t>0} \frac{(\int_t^\infty [(Tf)^{**}(s)]^{p_1}\,ds)^{1/p_1}}{\big(1+\log^+ \frac{1}{t}\big)^\alpha} \lesssim 1 + \int_1^\infty G'(t)(\log t)^{\alpha-1}\,\frac{dt}{t}$$
$$\lesssim 1 + \int_1^\infty G(t)(\log t)^{\alpha-1}\,\frac{dt}{t^2} \le 1 + \int_1^\infty \frac{(\int_t^\infty f^{**}(s)^{p_1}\,ds)^{1/p_1}}{t}(\log t)^{\alpha-1}\,dt,$$

as we wanted to prove. ∎

Definition 2.11. We define the spaces $D^-_{p,\alpha}$ and $R^-_{p,\alpha}$ by the norms

$$\|f\|_{D^-_{p,\alpha}} = \|f\|_p + \int_1^\infty \frac{(\int_t^\infty f^{**}(s)^p\,ds)^{1/p}}{t}(1+\log^+ t)^{\alpha-1}\,dt$$

and

$$\|f\|_{R^-_{p,\alpha}} = \sup_{t>0}\frac{(\int_t^\infty f^{**}(s)^p\,ds)^{1/p}}{\left(1+\log^+\frac{1}{t}\right)^\alpha}.$$

Also

$$D^-_{p,\alpha,\mathrm{rad}} = \{f\in D^-_{p,\alpha} : f \text{ is radial}\}.$$

With these definitions, Theorem 2.10 reads as follows:

Theorem 2.12. *Under the hypothesis of Theorem* 2.10,

$$T : D^-_{p_1,\alpha,\mathrm{rad}} \to R^-_{p_1,\alpha}$$

is bounded.

As before, let us now make some comparison with classical spaces.

Remark 2.13. 1. Let us see that Theorem 2.10 improves Theorem 2.8: first of all, by a discretization argument, it is easy to see that

$$\Big(\int_t^\infty f^{**}(s)^{p_1}\,ds\Big)^{1/p_1} \lesssim \int_t^\infty f^{**}(s)s^{1/p_1-1}\,ds$$

and hence

$$\begin{aligned}
\|f\|_{D^-_{p_1,\alpha}} &= \|f\|_{p_1} + \int_1^\infty \frac{(\int_t^\infty f^{**}(s)^{p_1}\,ds)^{1/p_1}}{t}(1+\log^+ t)^{\alpha-1}\,dt\\
&\lesssim \|f\|_{p_1} + \int_1^\infty \frac{\int_t^\infty f^{**}(s)s^{1/p_1-1}\,ds}{t}(1+\log^+ t)^{\alpha-1}\,dt\\
&= \|f\|_{p_1} + \int_1^\infty f^{**}(s)s^{1/p_1-1}\int_1^s \frac{(1+\log^+ t)^{\alpha-1}}{t}\,dt\\
&\approx \|f\|_{p_1} + \int_1^\infty f^{**}(s)s^{1/p_1-1}(1+\log^+ s)^\alpha\,ds\\
&\lesssim \int_0^\infty f^*(u)\int_{\max(1,u)}^\infty s^{1/p_1-2}(1+\log^+ s)^\alpha\,ds\\
&\approx \int_0^\infty f^*(u)u^{1/p_1-1}(1+\log^+ u)^\alpha\,du\\
&= \|f\|_{\Lambda^1(v_0)}
\end{aligned}$$

with v_0 as in Theorem 2.8. Hence

$$\Lambda^1(v_0) \subset D^-_{p_1,\alpha}.$$

The inclusion is strict since taking f such that $f^*(s) = s^{-1/p_1}(1+\log^+ u)^{-\beta}$ with $\alpha + 1/p_1 < \beta \le 1+\alpha$ we obtain $\|f\|_{\Lambda^1(v_0)} = \infty$ and $\|f\|_{D^-_{p_1,\alpha}} < \infty$. Also taking $h^*(t) = t^{1/p_1}$ one can imediately see that $R^-_{p_1,\alpha} \subset \Gamma^{1,\infty}(v_1)$ with v_1 as in Theorem 2.8 and the embedding is strict.

2. In particular, if $f \in L^{p_1}(\log 1/L)^\beta$ with $\beta > \alpha p_1$, that is,

$$\int_{\mathbb{R}^n} |f(x)|^{p_1}\Big(1+\log^+\frac{1}{|f(x)|}\Big)^\beta\,dx < \infty$$

then

$$\begin{aligned}
&\int_1^\infty \frac{(\int_t^\infty f^{**}(s)^{p_1}\,ds)^{1/p_1}}{t}(\log t)^{\alpha-1}\,dt \\
&\quad\le \Big(\int_0^\infty f^{**}(s)^{p_1}(1+\log^+ s)^\beta\,ds\Big)^{1/p_1}\int_1^\infty \frac{(\log t)^{\alpha-1}}{t(1+\log t)^{\beta/p_1}}\,dt \\
&\quad\lesssim \Big(\int_0^\infty f^{**}(s)^{p_1}(1+\log^+ s)^\beta\,ds\Big)^{1/p_1} \approx \Big(\int_0^\infty f^{*}(s)^{p_1}(1+\log^+ s)^\beta\,ds\Big)^{1/p_1}
\end{aligned}$$

where the last inequality follows since $(1+\log^+ s)^\beta \in B_{p_1}$ (see [3]). Finally, since $\sup_s f^*(s)s^{1/p_1} < \infty$ we have

$$\begin{aligned}
\Big(\int_0^\infty f^{*}(s)^{p_1}(1+\log^+ s)^\beta\,ds\Big)^{1/p_1} &\lesssim \Big(\int_0^\infty f^*(s)^{p_1}\Big(1+\log\frac{1}{f^*(s)}\Big)^\beta ds\Big)^{1/p_1} \\
&\approx \Big(\int_{\mathbb{R}^n} |f(x)|^{p_1}\Big(1+\log^+\frac{1}{|f(x)|}\Big)^\beta dx\Big)^{1/p_1}
\end{aligned}$$

and therefore $L^{p_1}(\log 1/L)^\beta \subset D^-_{p_1,\alpha}$.

3. Applications

3.1. *Almost everywhere convergence of spherical partial Fourier integrals for radial functions in weighted spaces.* In [25] one of the authors proved that if f is a radial function belonging to $L^p(\mathbb{R}^n)$, $2n/(n+1) < p < 2n/(n-1)$, then $S_R f(x)$ converges a.e. to $f(x)$ whenever R tends to ∞, where

$$S_R f(x) = \int_{B(0,R)} \hat f(\xi)e^{2\pi i x\xi}\,d\xi$$

is the spherical partial Fourier integral. To do this it was shown that, for radial functions f,

$$\tilde{S}f(x) = \sup_R |S_R f(x)| \leq \frac{C(n)}{s^{(n-1)/2}} (M + L + \tilde{H} + \tilde{C})(g)(s) \tag{3.1}$$

where $s = |x|$, $g(r) = f(r)r^{(n-1)/2}\chi_{(0,\infty)}(r)$, M is the Hardy–Littlewood maximal operator, $\tilde{H}$ is the maximal Hilbert transform, $\tilde{C}$ is the maximal Carleson operator defined by

$$\tilde{C}f(x) = \sup_{y\in\mathbb{R}} \sup_{\varepsilon>0} \bigg| \int\limits_{\varepsilon<|x-t|} \frac{e^{-iyt}f(t)}{x-t}\, dt \bigg|$$

and L is the Hilbert integral

$$Lf(s) = \int\limits_0^\infty \frac{f(t)}{s+t}\, dt.$$

Using (3.1) it is proved in [27] and [15] that

$$\tilde{S} : L^{p_j,1}_{\mathrm{rad}} \to L^{p_j,\infty}, \qquad j = 0, 1,$$

is bounded with

$$p_0 = \frac{2n}{n+1} \quad \text{and} \quad p_1 = \frac{2n}{n-1}.$$

From this the almost everywhere convergence of $S_R f(x)$ in $L^{p_j,1}_{\mathrm{rad}}$ follows.

Again (3.1) is used in [26] to prove that if v is a radial weight such that $u(s) = v(s)s^{(n-1)(1-p/2)} \in A_p$ then

$$\|\tilde{S}f\|_{L^p(v)} \leq C_{v,p}\|f\|_{L^p_{\mathrm{rad}}(v)} \tag{3.2}$$

but no information is given about the behaviour of the constant $C_{v,p}$.

In this section, we shall give an estimate of the constant $C_{w,p}$ for any radial weight w on $\mathbb{R}^n$ provided $w_0 \in A_1$, where $w_0(r) = w(|x|)$ for $|x| = r > 0$ and $w_0(r) = w_0(-r)$ for $r < 0$. Recall $w_0 \in A_1$ if

$$Mw_0(s) \leq Cw_0(s) \quad \text{ a.e. } s \in \mathbb{R},$$

and $\|w_0\|_{A_1}$ is the infimum of all the above constants C.

When $w : \mathbb{R}^n \to \mathbb{R}^+$ is a radial function such that $w_0 \in A_1$, we shall write $w \in A_1(\mathbb{R})$ and $\|w\|_{A_1(\mathbb{R})} = \|w_0\|_{A_1}$.

With this notation we shall prove that, for every $p_0 < p \leq 2$,

$$C_{w,p} \lesssim \|w\|_{A_1(\mathbb{R})} \bigg(\frac{1}{p-p_0}\bigg)^3.$$

Then we can apply Theorem 2.6 and deduce the convergence almost everywhere of the spherical partial Fourier integrals for radial functions in

$D^+_{p_0,3,\mathrm{rad}}(w)$, which is $D^+_{p_0,3,\mathrm{rad}}$ with $\mu = w(x)\,dx$; similarly for $R^+_{p_0,3}(w)$. Precisely, we shall prove the following:

THEOREM 3.1. *If w is a radial function in $\mathbb{R}^n$ such that $w \in A_1(\mathbb{R})$ then*

$$\tilde{S} : D^+_{p_0,3,\mathrm{rad}}(w) \to R^+_{p_0,3}(w)$$

is bounded. Hence if $f \in D^+_{p_0,3,\mathrm{rad}}(w)$, then

$$S_R f(x) \to f(x) \tag{3.3}$$

as $R \to \infty$ for almost every $x \in \mathbb{R}^n$. In particular, (3.3) *holds for every radial function f satisfying*

$$\int_0^\infty f^*_w(t) t^{1/p_0} \left(1 + \log^+ \frac{1}{t}\right)^3 \frac{dt}{t} < \infty$$

and every radial function in $L^{p_0}(\log L)^\beta(w)$ with $\beta > 3p_0$.

Proof. From (3.1) we have

$$\|\tilde{S} f\|_{L^p(w)} \lesssim \|Tg\|_{L^p(\mathbb{R}^+;\, w(s)s^{(n-1)(1-p/2)})} \le \|Tg\|_{L^p(\mathbb{R};u)}$$

where $u(s) = w(s)|s|^{(n-1)(1-p/2)}$ and, for $s \in \mathbb{R}$,

$$Tg(s) = (M + L + \tilde{H} + \tilde{C})(g)(|s|).$$

To give an estimate of the constant $C(w,p)$ in the inequality

$$\|Tg\|_{L^p(u)} \le C(w,p)\|g\|_{L^p(u)}$$

we will go through the following steps:

STEP 1. If $-1 < \alpha \le 0$ and $v_\alpha(s) = |s|^\alpha$, with $s \in \mathbb{R}$, then

$$\|v_\alpha\|_{A_1} \le \frac{2}{1+\alpha}.$$

Then recalling that $u(s) = w(s)|s|^{(n-1)(1-p/2)}$ and $w \in A_1(\mathbb{R})$, we shall prove that, for every $p_0 < p \le 2$, the following four steps hold:

STEP 2.

$$\|u\|_{A_p} \lesssim \|w\|_{A_1(\mathbb{R})} \left(\frac{1}{p-p_0}\right)^{p-1}.$$

STEP 3.

$$\|Mg\|_{L^p(u)} \le \frac{\|w\|^{(n+1)/(n-1)}_{A_1(\mathbb{R})}}{p-p_0} \|g\|_{L^p(u)}.$$

STEP 4.

$$\|Lg\|_{L^p(u)} \le \frac{\|w\|^{(n+1)/(n-1)}_{A_1(\mathbb{R})}}{p-p_0} \|g\|_{L^p(u)}.$$

Step 5.

$$\|(\tilde{H}+\tilde{C})g\|_{L^p(u)} \lesssim \|w\|_{A_1(\mathbb{R})}^{(n+1)/(n-1)}\left(\frac{1}{p-p_0}\right)^3 \|g\|_{L^p(u)}.$$

From Steps 3–5 it follows that

$$\|\tilde{S}f\|_{L^p(w)} \lesssim \|w\|_{A_1(\mathbb{R})}^{(n+1)/(n-1)}\left(\frac{1}{p-p_0}\right)^3 \|g\|_{L^p(u)}.$$

Then the result follows by Theorem 2.4, since trivially $\|g\|_{L^p(u)} \approx \|f\|_{L^p(w)}$.

Proof of Step 1. Since v_α is even, so also is

$$M(v_\alpha)(x) = \sup_{a<x<b} \frac{1}{b-a}\int_a^b |s|^\alpha\, ds,$$

and hence we can assume $x>0$. If $a>0$ then from $-1<\alpha<0$ it follows that $b^\alpha < x^\alpha < a^\alpha$ and

$$\frac{1}{b-a}\int_a^b |s|^\alpha\, ds = \frac{1}{1+\alpha}\,\frac{b^{1+\alpha}-a^{1+\alpha}}{b-a} \le \frac{1}{1+\alpha}\,\frac{x^\alpha b - x^\alpha a}{b-a} = \frac{x^\alpha}{1+\alpha}.$$

If $a<0$ and $-a>b$ then

$$\frac{1}{b-a}\int_a^b |s|^\alpha\, ds = \frac{1}{1+\alpha}\,\frac{b^{1+\alpha}+(-a)^{1+\alpha}}{b-a} \le \frac{2}{1+\alpha}\, b^\alpha \le \frac{2x^\alpha}{1+\alpha}.$$

If $a<0$ and $-a<b$, $0<B=-a/b<1$ then

$$\frac{1}{b-a}\int_a^b |s|^\alpha\, ds = \frac{1}{1+\alpha}\,\frac{b^{1+\alpha}+(-a)^{1+\alpha}}{b-a} \le \frac{1}{1+\alpha}\, b^\alpha\, \frac{1+B^{1+\alpha}}{1+B} \le \frac{2x^\alpha}{1+\alpha}.$$

Therefore

$$M(v_\alpha)(x) \le \frac{2x^\alpha}{1+\alpha}.$$

Proof of Step 2. This result follows from the fact (see [16]) that if $w_0, w_1 \in A_1$, then $w_0 w_1^{1-p} \in A_p$ and

$$\|w_0 w_1^{1-p}\|_{A_p} \le \|w_0\|_{A_1}\|w_1\|_{A_1}^{p-1}. \tag{3.4}$$

Hence writing

$$w_0 = w \quad\text{and}\quad w_1(s) = |s|^{\frac{(n-1)(1-p/2)}{1-p}}$$

we see that $u = w_0 w_1^{1-p}$ and thus by Step 1,

$$\begin{aligned}\|u\|_{A_p} \le \|w\|_{A_1}\|w_1\|_{A_1}^{p-1} &\lesssim \|w\|_{A_1}\left(\frac{1}{1+\frac{(n-1)(1-p/2)}{1-p}}\right)^{p-1}\\ &\lesssim \|w\|_{A_1(\mathbb{R})}\left(\frac{1}{p-p_0}\right)^{p-1}.\end{aligned}$$

Proof of Step 3. By [6] (see also [23]) we know that

$$\|Mg\|_{L^p(u)} \le \|u\|_{A_p}^{1/(p-1)} \|g\|_{L^p(u)}, \tag{3.5}$$

and hence the result follows by Step 2 since $1/(p-1) \le (n+1)/(n-1)$.

Proof of Step 4. First we observe that

$$\begin{aligned} Lg(|s|) = \int_0^\infty \frac{g(t)}{t+|s|}\,dt &\le \frac{1}{|s|}\int_0^{2|s|} g(t)\,dt + \int_{|s|}^\infty \frac{g(t)}{t}\,dt \\ &\lesssim Mg(|s|) + \int_{|s|}^\infty \frac{g(t)}{t}\,dt = Mg(|s|) + Rg(s). \end{aligned}$$

For the first term, we can apply Step 3, and for the second one we shall proceed by duality:

$$\begin{aligned} \|Rg\|_{L^p(u)} &= \sup_{\|h\|_{L^{p'}(u^{-p'/p})}\le 1} \left| \int \left(\int_{|s|}^\infty \frac{g(t)}{t}\,dt \right) h(s)\,ds \right| \\ &= \sup_{\|h\|_{L^{p'}(u^{-p'/p})}\le 1} \left| \int g(t) \left(\frac{1}{t} \int_{|s|\le t} h(s)\,ds \right) dt \right| \\ &\lesssim \sup_{\|h\|_{L^{p'}(u^{-p'/p})}\le 1} \int |g(t)| Mh(t)\,dt \le \|g\|_{L^p(u)} \|Mh\|_{L^{p'}(u^{-p'/p})}. \end{aligned}$$

Using (3.5), we obtain

$$\|Mh\|_{L^{p'}(u^{-p'/p})} \le \|u^{-p'/p}\|_{A_{p'}}^{1/(p'-1)} \|h\|_{L^{p'}(u^{-p'/p})}$$

and hence

$$\|Rg\|_{L^p(u)} \le \|u^{-p'/p}\|_{A_{p'}}^{1/(p'-1)} \|g\|_{L^p(u)}.$$

Since (see [16]) $u^{-p'/p} \in A_{p'}$ if and only if $u \in A_p$ and, in fact,

$$\|u^{-p'/p}\|_{A_{p'}} = \|u\|_{A_p}^{p'-1},$$

applying Step 2 for $p \le 2$, we obtain

$$\begin{aligned} \|Rg\|_{L^p(u)} \le \|u\|_{A_p} \|g\|_{L^p(u)} &\lesssim \|w\|_{A_1} \left(\frac{1}{p-p_0} \right)^{p-1} \|g\|_{L^p(u)} \\ &\le \frac{\|w\|_{A_1}}{p-p_0} \|g\|_{L^p(u)}. \end{aligned}$$

Proof of Step 5. For this step we refer to [21] where the following good-lambda inequality is proved, for a decomposition of $\{\tilde{C}f(x) > \lambda\}$ into pairwise disjoint intervals $(I_j)_j$:

$$|\{x \in I_j : \tilde{C}f(x) > 3\lambda,\ M_r f(x) \le \gamma\lambda\}| \le C_r \gamma^r |I_j|,$$

where $M_r f(x) = \big(M(|f|^r)(x)\big)^{1/r}$, $1 < r < p$ and $\gamma > 0$ is small enough. We are interested in the behaviour of C_r as $r \to 1$. The proof shows that $C_r \lesssim 1/(r-1)^2$ since $\|\tilde{C}\|_{L^r \to L^{r,\infty}} \lesssim 1/(r-1)^2$ ([8], [20]). Hence, for $r > 1$,

$$|\{x \in I_j : \tilde{C}f(x) > 3\lambda,\ M_r f(x) \le \gamma\lambda\}| \lesssim \frac{\gamma^r}{(r-1)^2} |I_j|.$$

Now we observe that even though our weight u depends on p, by using Step 1 and (3.4), it follows that, for every $2n/(n+1) < p \le 2$,

$$\begin{aligned}\|u\|_{A_2} &\le \|w\|_{A_1} \| \, |s|^{(n-1)(p/2-1)} \|_{A_1} \lesssim \|w\|_{A_1} \frac{1}{1+(n-1)(p/2-1)} \\ &\le \frac{n+1}{2} \, \|w\|_{A_1}.\end{aligned}$$

Hence the norm of u as a weight in A_∞ is uniformly bounded on p. Since $u \in A_\infty$ implies that there exist C and δ depending on $\|u\|_{A_\infty}$ such that, for every subset S of an interval I,

$$\frac{u(S)}{u(I)} \le C\bigg(\frac{|S|}{|I|}\bigg)^{\delta},$$

we obtain

$$u(\{x \in I_j : \tilde{C}f(x) > 3\lambda,\ M_r f(x) \le \gamma\lambda\}) \lesssim \bigg(\frac{\gamma^r}{(r-1)^2}\bigg)^{\delta} u(I_j).$$

Summing over j we have

$$u(\{x : \tilde{C}f(x) > 3\lambda,\ M_r f(x) \le \gamma\lambda\}) \lesssim \bigg(\frac{\gamma^r}{(r-1)^2}\bigg)^{\delta} u(\{\tilde{C}f(x) > \lambda\}).$$

From this inequality and using standard techniques we deduce that, for $r>1$,

$$\|\tilde{C}g\|_{L^p(u)} \lesssim \frac{1}{(r-1)^2} \, \|M_r g\|_{L^p(u)}$$

and thus

$$\|\tilde{C}g\|_{L^p(u)} \lesssim \frac{1}{(r-1)^2} \|u\|_{A_{p/r}}^{\frac{r}{p/r-1}} \|g\|_{L^p(u)} \lesssim \frac{1}{(r-1)^2} \, \frac{\|w\|_{A_1}^{\frac{r}{p/r-1}}}{1 + \frac{(n-1)(1-p/2)}{1-p/r}} \|g\|_{L^p(u)}.$$

Let us choose r such that

$$1 - \frac{1}{r} = c\bigg(p - \frac{2n}{n+1}\bigg),$$

with c small enough. Then it is easy to see that if p is near $2n/(n+1)$, then $1 < r < p$ and

$$\frac{1}{(r-1)^2} \frac{1}{1 + \frac{(n-1)(1-p/2)}{1-p/r}} \approx \bigg(\frac{1}{p-p_0}\bigg)^3,$$

and the result follows.

The particular cases are consequences of Remark 2.7. ■

At the right endpoint p_1, it is not true that $u(s) = w(s)|s|^{(n-1)(1-p/2)} \in A_p$ with p near p_1 for every $w \in A_1$ and thus we have to impose another condition on w:

THEOREM 3.2. *If* $w(s)|s|^{-2n/(n-1)} \in A_1(\mathbb{R})$ *then*

$$\tilde{S} : D^+_{p_1,3,\mathrm{rad}}(w) \to R^+_{p_1,3}(w)$$

is bounded. Hence if $f \in D^+_{p_1,3,\mathrm{rad}}(w)$ *then* $S_R f(x) \to f(x)$ *as* $R \to \infty$ *for almost every* $x \in \mathbb{R}^n$. *In particular, the almost everywhere convergence holds for every radial function satisfying*

$$\int_0^\infty f^*(u) u^{1/p_1 - 1}(1 + \log^+(1/u))^3\, du < \infty$$

or $f \in L^{p_1}(\log L)^\beta$ *with* $\beta > 3p_1$.

Proof. The proof follows the same steps as that of Theorem 3.1 as soon as we prove that $u \in A_p$. We shall use the fact that $u \in A_p$ if and only if $u^{1-p'} \in A_{p'}$ and that $\|u\|_{A_p} = \|u^{1-p'}\|^{1/(p'-1)}_{A_{p'}}$ and write

$$u(s)^{1-p'} = (w(s)|s|^{-2n/(n-1)})^{1-p'}|s|^{((n-1)(1-p/2)+2n/(n-1))(1-p')}.$$

To see that $u^{1-p'} \in A_{p'}$ it is enough to check $|s|^{((n-1)(1-p/2)+2n/(n-1))(1-p')} \in A_1$ and hence we need

$$-1 < \left((n-1)\left(1 - \frac{p}{2}\right) + \frac{2n}{n-1}\right)(1 - p') \le 0,$$

which trivially holds for p near p_1, $p > p_1$, and by Step 1 in the proof of Theorem 3.1 and (3.4),

$$\begin{aligned}
\|u\|_{A_p} &= \|u^{1-p'}\|^{1/(p'-1)}_{A_{p'}} \\
&\lesssim \left(\|w(s)|s|^{-2n/(n-1)}\|_1^{p'-1} \frac{1}{1 + (1-p')\big((n-1)(1-p/2) + \frac{2n}{n-1}\big)}\right)^{1/(p'-1)} \\
&\lesssim \|w(s)|s|^{-2n/(n-1)}\|_1 \left(\frac{1}{p - 1 - (n-1)(1-p/2) - \frac{2n}{n-1}}\right)^{p-1} \\
&\lesssim \|w(s)|s|^{-2n/(n-1)}\|_1 \left(\frac{1}{p - \frac{2n}{n-1}}\right)^{p-1}.
\end{aligned}$$

Following now the same steps as in the proof of Theorem 3.1, it remains only to prove that $u \in A_\infty$; but this follows easily since, in fact, for $n > 1$, $u \in A_3$ using again (3.4) and Step 1.

The particular cases follow from Remark 2.13. ∎

3.2. *Endpoint estimates for maximal Bochner–Riesz type operators acting on radial functions.* In [19] the following theorem for maximal Bochner–Riesz type operators was proved:

THEOREM 3.3. *Let* $Tf(x) = \sup_{t>0} |(K_t * f)(x)|$, *where* $K : \mathbb{R}^n \to \mathbb{C}$ *is a radial, bounded measurable function, and let* $0 < \delta < (n-1)/2$. *Suppose that* $K(x) = a(|x|)e^{i\varphi(|x|)}$ *for* $|x| > \varrho$, *where*

$$\left|\frac{d^m}{ds^m}\,a(s)\right| \le cs^{-((n+1)/2+\delta+m)}, \qquad \left|\frac{d^m}{ds^m}(\varphi'(s))^{-1}\right| \le cs^{-m}, \tag{3.6}$$

for $s > \varrho$ *and* $m = 0, \dots, [n/2]$. *Then* T *is bounded on* $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ *for every* $2n/(n+1+2\delta) < p < 2n/(n-1-2\delta)$.

REMARK 3.4. In [19] it is also proved that, for n even, T is of weak type on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for $p = 2n/(n+1+2\delta)$ and restricted weak type for $p = 2n/(n-1-2\delta)$. If n is odd the same is proved under the additional assumption that (3.6) also holds for $m = (n+1)/2$.

Set

$$p_0(\delta) = \frac{2n}{n+1+2\delta} \quad \text{and} \quad p_1(\delta) = \frac{2n}{n-1-2\delta}.$$

CLAIM. *If* T *satisfies the hypothesis of Theorem* 3.3 *then*

$$\|Tf\|_{p,\infty} \lesssim \frac{1}{p-p_0(\delta)}\left(\frac{1}{p_1(\delta)-p}\right)^{(3n+1+2\delta)/2n} \|f\|_{L^p_{\mathrm{rad}}}.$$

Proof. This follows by carefully checking Epperson's calculations. First of all, it turns out that the constant c in the statement of Lemma 1.4 in [19] satisfies

$$c = c(\eta) \lesssim 1/\eta.$$

Indeed, the computations involve $\int_{k_1}^{\pi/2} \theta^{-(1+\eta)}\,d\eta$ and $\int_{\pi/2}^{k_2} (\pi-\theta)^{-(1+\eta)}\,d\eta$ (Case 4, p. 114), which are both bounded by C/η. The other cases give better constants.

Now, the proof of Theorem 1.1 in [19] shows that

$$\|Tf\|_{L^{p,\infty}} \lesssim (\max_{i=1,\dots,6} a_i)\|f\|_p$$

where the a_i are given by

$$\int\limits_{C^i_\lambda} s^{n-1}\,ds \le a_i^p\left(\frac{\|f\|_p}{\lambda}\right)^p.$$

Following Epperson's calculations we see that for $i = 1, 3, 5, 6$,

$$a_i \lesssim c(\eta).$$

For $i = 2, 4$,

$$a_i \lesssim \frac{c(\eta)}{\eta^{1/p'}}$$

if p is close to $p_1(\delta)$, and $a_i \lesssim c(\eta)$ if p is close to $p_0(\delta)$.

From this our claim follows. Indeed, if $p < p_1(\delta)$, we choose $\eta > 0$ such that $p_1(\bar{\delta}) < p$ with $\bar{\delta} = \delta - \eta$ and so we have

$$p_1(\delta) - p_1(\bar{\delta}) \lesssim \eta$$

and consequently

$$\frac{1}{\eta} \lesssim \frac{1}{p_1(\delta) - p}.$$

Hence, for p close to $p_1(\delta)$, we obtain

$$\|Tf\|_{L^{p,\infty}} \lesssim \left(\frac{1}{p_1(\delta) - p}\right)^{1+1/p'} \|f\|_{L^p_{\mathrm{rad}}},$$

which implies

$$\|Tf\|_{L^{p,\infty}} \lesssim \left(\frac{1}{p_1(\delta) - p}\right)^{(3n+1+2\delta)/2n} \|f\|_{L^p_{\mathrm{rad}}},$$

and similarly, for p close to $p_0(\delta)$,

$$\|Tf\|_{L^{p,\infty}} \lesssim \frac{1}{p - p_0(\delta)} \|f\|_{L^p_{\mathrm{rad}}}. \blacksquare$$

THEOREM 3.5. *Under the hypothesis of the previous theorem*,

$$T : \Lambda^1_{\mathrm{rad}}(v_0) \to \Gamma^{1,\infty}(v_1)$$

is bounded with

(i) $v_0(t) = t^{1/p_0(\delta)-1}\left(1 + \log^+ \frac{1}{t}\right)$ *and* $v_1(t) = t^{1/p_0(\delta)-1}(1 + \log^+ t)^{-1}$

and also

(ii) $v_0(t) = t^{1/p_1(\delta)-1}(1+\log^+ t)^{\alpha}$ *and* $v_1(t) = t^{1/p_1(\delta)-1}\left(1+\log^+ \frac{1}{t}\right)^{-\alpha}$

with $\alpha = (3n + 1 + 2\delta)/2n$.

Proof. (i) Near $p_0(\delta)$ we have

$$\|Tf\|_{p,\infty} \lesssim \frac{1}{p - p_0(\delta)} \|f\|_p. \tag{3.7}$$

Now, from the proof of Theorem 2.1, one can easily see that, in fact, the only condition on T that we use is that

$$\sup_{t>0} (Tf)^{**}(t)t^{1/p} \lesssim \frac{1}{p - p_0} \|f\|_p,$$

and this condition follows immediately from (3.7) since $p > p_\delta > 1$.

(ii) Near $p_1(\delta)$, we have

$$\|Tf\|_{p,\infty} \lesssim \left(\frac{1}{p_1(\delta)-p}\right)^{(3n+1+2\delta)/2n} \|f\|_p, \tag{3.8}$$

but from the proof of Theorem 2.8, one can easily see that, in fact, the only condition on T that we use is that

$$\sup_{t>0} (Tf)^{**}(t)t^{1/p} \lesssim \left(\frac{1}{p_1(\delta)-p}\right)^{(3n+1+2\delta)/2n} \|f\|_p,$$

and this condition follows immediately from (3.8). ∎

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