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Operator theoretic properties of semigroups in terms of their generators

by

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Abstract. Let (T_t) be a C_0 semigroup with generator A on a Banach space X and let $\mathcal{A}$ be an operator ideal, e.g. the class of compact, Hilbert–Schmidt or trace class operators. We show that the resolvent $R(\lambda, A)$ of A belongs to $\mathcal{A}$ if and only if the integrated semigroup $S_t := \int_0^t T_s\, ds$ belongs to $\mathcal{A}$. For analytic semigroups, $S_t \in \mathcal{A}$ implies $T_t \in \mathcal{A}$, and in this case we give precise estimates for the growth of the $\mathcal{A}$-norm of T_t (e.g. the trace of T_t) in terms of the resolvent growth and the imbedding $D(A) \hookrightarrow X$.

0. Introduction. In this paper we study how operator theoretic properties of the generator and the resolvent of a C_0 semigroup on a Banach space X are reflected in the properties of the semigroup.

Often operator theoretic properties of an operator T can be checked conveniently by showing that T belongs to a suitable operator ideal. If T belongs to the ideal of compact or strictly singular operators we know that its spectrum $\sigma(T)$ consists of a series of eigenvalues with possible limit point 0, that T is an admissible Fredholm perturbation, etc. We know about the summability of its eigenvalues and its trace if T belongs to the Hilbert–Schmidt class, the trace class or to one of the ideals extending the Schatten classes to the Banach space setting.

To have a unified approach to many of these topics, we phrase our question as follows: Given an operator ideal $\mathcal{A}$ and a C_0 semigroup (T_t) with generator A and resolvent $R(\lambda, A)$, how can we characterize "$R(\lambda, A) \in \mathcal{A}$" in terms of $\mathcal{A}$ and the semigroup T_t?

Since the semigroup and the resolvent are connected by the Laplace transform

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T_t\, dt \tag{1}$$

the resolvent is a "smoothing" of the semigroup and one would generally

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expect that $T_t \in \mathcal{A}$ implies $R(\lambda, A) \in \mathcal{A}$ (see e.g. [V] for details). But the inversion of the Laplace transform is very singular so that it is not surprising that the converse is false in general. For instance, a well known result of Pazy shows that the compactness of $R(\lambda, A)$ implies the compactness of the T_t only if the mapping $t \mapsto T_t$ is continuous in the operator norm for $t > 0$.

In Section 2 we show essentially that $R(\lambda, A)$ belongs to an ideal $\mathcal{A}$ if and only if the integrated semigroup $S_t := \int_0^t T_s\, ds$ belongs to $\mathcal{A}$. We also express "$T_t \in \mathcal{A}$" in terms of growth conditions on the resolvent.

In Section 3 we extend this result to the Phillips functional calculus: For a closed ideal $\mathcal{A}$ (e.g. if $\mathcal{A}$ is the ideal of compact or weakly compact operators) we have $R(\lambda, A) \in \mathcal{A}$ if and only if $\widehat{g}(-A) \in \mathcal{A}$ for one or all functions g considered in this calculus (with a mild restriction). We also generalize Pazy's result on "$T_t \in \mathcal{K}$".

As pointed out to us by the referee, one obtains a similar result for Lipschitz continuous integrated semigroups.

In Section 4 we consider analytic semigroups. Here $R(\lambda, A) \in \mathcal{A}$ always implies $T_t \in \mathcal{A}$, but if $\mathcal{A}$ is not closed the norm $\|T_t\|_{\mathcal{A}}$ usually blows up as $t \to 0$ (e.g. if $\mathcal{A}$ is the Hilbert–Schmidt or the trace class). So the question arises how this growth of $\|T_t\|_{\mathcal{A}}$ as $t \to 0$ is related to the growth of the resolvent. Our main result here is that for all $\alpha > 0$ and $\beta \in (0, 1]$ one has

$$\|T_t\|_{\mathcal{A}} \le C t^{\beta-\alpha} \text{ on } \mathbb{R}_+ \quad \Leftrightarrow \quad \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le D\lambda^{-\beta} \text{ on } \mathbb{R}_+.$$

This result allows one to obtain estimates on (T_t) directly from information on the imbeddings $J_\alpha(A) : D((-A)^\alpha) \hookrightarrow X$. Indeed, if $J_\alpha(A) \in \mathcal{A}$ then $\|T_t\|_{\mathcal{A}} \le C t^{-\alpha}$ for small t.

To illustrate the use of our general framework we discuss briefly some applications to ultracontractivity as well as to trace and Gaussian estimates for semigroups, all of which are related to (one-sided) operator ideals. Detailed expositions of these applications are given in [B2] and [B3].

1. Preliminaries and notations. Let X be a complex Banach space with unit ball B_X and identity operator I_X. With the convention $S_0 := \mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}$ we denote for all $\delta \in [0, \pi/2)$ by Σ_δ and S_δ the open sectors

$$\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \pi/2 + \delta\}, \qquad S_\delta := \{z \in \mathbb{C} : |\arg(z)| < \delta\}.$$

Let $(T_t)_{t\ge0}$ be a C_0 semigroup on X with generator A. The family $(S_t)_{t\ge0}$ given by

$$S_t x := \int_0^t T_s x\, ds$$

is called the *integrated semigroup* and provides an integrated semigroup in the sense of [A]. We will also consider fractional powers of the resolvent (cf. 4.8(b)):

$$R(\lambda, A)^\alpha = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda t} t^{\alpha - 1} T_t \, dt, \qquad \alpha > 0. \tag{2}$$

If A is the generator of a bounded holomorphic semigroup we denote by $\delta(A)$ its maximal angle of holomorphy and put for all $0 \le \delta < \delta(A)$,

$$R_\delta(A) := \sup_{\lambda \in \Sigma_\delta} \|\lambda R(\lambda, A)\|.$$

For a generator A the operator $-A$ is non-negative in the sense of [MSM], i.e. its fractional powers $(-A)^\alpha$ are well defined for $\alpha > 0$.

We denote by $J_\alpha(A)$ and (if $0 \in \varrho(A)$) $J^0_\alpha(A)$ the imbedding operators

$$D((-A)^\alpha) \hookrightarrow X \quad \text{and} \quad D^0((-A)^\alpha) \hookrightarrow X$$

where $D^0((-A)^\alpha)$ is the usual domain equipped with the norm $x \mapsto \|(-A)^\alpha x\|$.

We briefly summarize some relevant facts from the theory of operator ideals as developed in [P]. Suppose that for every pair of Banach spaces X and Y we are given a subspace $\mathcal{A}(X, Y)$ of the space $\mathcal{L}(X, Y)$ of all bounded linear operators that contains the finite-dimensional operators and has the following "ideal" property:

$$T \in \mathcal{L}(X_0, X),\ S \in \mathcal{A}(X, Y),\ R \in \mathcal{L}(Y, Y_0) \ \Rightarrow\ RST \in \mathcal{A}(X_0, Y_0).$$

Then the union $\mathcal{A}$ of all $\mathcal{A}(X, Y)$ is called an *operator ideal*.

Such an operator ideal is called a *normed operator ideal* if every component $\mathcal{A}(X, Y)$ carries a complete norm $\|\cdot\|_{\mathcal{A}}$ stronger than the operator norm with

$$\|x^* \otimes y\|_{\mathcal{A}} = \|x^*\| \, \|y\| \qquad \text{for all } x^* \in X^*,\ y \in Y$$

and

$$T \in \mathcal{L}(X_0, X),\ S \in \mathcal{A}(X, Y),\ R \in \mathcal{L}(Y, Y_0) \ \Rightarrow\ \|RST\|_{\mathcal{A}} \le \|R\| \, \|S\|_{\mathcal{A}} \, \|T\|.$$

An operator ideal is called *closed* if $\|\cdot\|_{\mathcal{A}} = \|\cdot\|$ and $\mathcal{A}(X, Y)$ is closed in $\mathcal{L}(X, Y)$. For example the compact, weakly compact and strictly singular operators form closed operator ideals $\mathcal{K}$, $\mathcal{W}$ and $\mathcal{S}$, respectively.

Other well known examples are the operator ideals $\mathcal{P}_r$ and $\mathcal{N}_r$ of all absolutely r-summing operators and r-nuclear operators which coincide for $1 < r < \infty$ and Hilbert spaces X, Y with the class $\mathcal{S}_2$ of Hilbert–Schmidt operators and therefore can be considered as extensions of $\mathcal{S}_2$ to the Banach space setting.

The ideal $\mathcal{S}_1$ of operators with summable approximation numbers extends in this sense the trace class of Hilbert space operators, and there are further ideals extending the other Schatten classes $\mathcal{S}_p$ to the Banach space setting. For definitions and basic properties of these ideals see e.g. [P].

We will use the notation $\mathcal{A}(X) := \mathcal{A}(X, X)$ and the following useful fact which is immediate from a convexity theorem for the Bochner integral (see e.g. [DU], II.2.8).

REMARK 1.1. Let I be an interval and $F \in L_1(I, \mathcal{L}(X))$ satisfy $F(I) \subset B_{\mathcal{A}(X)}$. If $B_{\mathcal{A}(X)}$ is closed in the operator norm then $\int_I F(t)\,dt \in |I| B_{\mathcal{A}(X)}$.

This observation can be applied not only to closed ideals but to most popular ideals.

LEMMA 1.2. *The unit ball $B_{\mathcal{A}(X)}$ is closed with respect to the strong operator topology of $\mathcal{L}(X)$ if*

(a) *$\mathcal{A}$ is maximal* (*see* [P]; *e.g. if $\mathcal{A} = \mathcal{P}_r$*), *or*

(b) *$\mathcal{A} = \mathcal{N}_r$ for some $1 \le r < \infty$ and X is a dual space such that X^* has the Radon–Nikodym property* (*e.g. if X is reflexive*).

Proof. Let $\mathcal{A}^{\max}$ denote the maximal hull of $\mathcal{A}$ ([P], 8.7.1). Then

$$B_{\mathcal{A}(X)} \subset \mathcal{L}\text{-cl}(B_{\mathcal{A}(X)}) \subset s\text{-cl}(B_{\mathcal{A}(X)}) \subset B_{\mathcal{A}^{\max}(X)} \tag{3}$$

where s-cl denotes the closure in the strong operator topology. Here the last inclusion is shown similarly to [P], 10.3.4. The maximality of $\mathcal{A}$ means $\mathcal{A} = \mathcal{A}^{\max}$, hence in the case (a) the claim follows from (3). The maximality of $\mathcal{P}_r$ is shown in [P]. In the case (b) the assumptions on X yield $\mathcal{A}(X) = \mathcal{A}^{\max}(X)$ ([P], 19.2.1; [DF], 33.6.1) though $\mathcal{A} = \mathcal{N}_r$ is not maximal and we can conclude as before. ∎

2. General semigroups. Let A be the generator of a C_0 semigroup (T_t) on X. In this general situation we will show that the properties of the resolvent are reflected not necessarily by the semigroup (T_t) itself but rather in the properties of the integrated semigroup

$$S_t x := \int_0^t T_s x\,ds.$$

If the growth bound $\omega(T_t)$ is not negative we also need

$$\widetilde{S}_t x := \int_0^t e^{-bs} T_s x\,ds$$

for some $b > \omega(T_t)$. (Of course, if $\omega(T_t) < 0$ we might choose $\widetilde{S}_t = S_t$.)

THEOREM 2.1. *For a C_0 semigroup (T_t) with generator A and (S_t), $(\widetilde{S}_t)$ as above the following statements are equivalent for every normed operator ideal $\mathcal{A}$*:

(a) *$R(\lambda, A) \in \mathcal{A}(X)$ for one* (*all*) *$\lambda \in \varrho(A)$.*

(b) *$J_1(A) \in \mathcal{A}(D(A), X)$.*

(c) *$\widetilde{S}_t \in \mathcal{A}(X)$ for one* (*all*) *$t > 0$.*

If $B_{\mathcal{A}(X)}$ is closed in $\mathcal{L}(X)$ we may add

(d) $\{t \in \mathbb{R}_+ : S_t \in \mathcal{A}(X)\}$ *contains a set of positive Lebesgue measure.*

REMARK 2.2. (i) Condition (c) implies that $(\widetilde{S}_t)$ is $\|\cdot\|_{\mathcal{A}}$-bounded while condition (d) ensures that (S_t) is $\|\cdot\|_{\mathcal{A}}$-bounded on finite intervals and

$$\limsup_{t\to\infty} \frac{1}{t} \log \|S_t\|_{\mathcal{A}} \le \omega(T_t) \vee 0.$$

(ii) The proof will show that aside from the equivalence (a)⇔(b) all claims of Theorem 2.1 remain true if we assume $\mathcal{A}(X)$ only to be a one-sided ideal in the Banach algebra $\mathcal{L}(X)$ with norm estimate

$$\|ST\|_{\mathcal{A}} \le \|S\|\,\|T\|_{\mathcal{A}} \quad \text{or} \quad \|ST\|_{\mathcal{A}} \le \|T\|\|S\|_{\mathcal{A}}. \tag{4}$$

Contrary to what one might expect, one cannot use the "translation" $\widetilde{A} = A - b$ to get (c) if $S_t \in \mathcal{A}(X)$ for some $t \in \mathbb{R}_+$. This is shown by the following example where $\{t \in \mathbb{R}_+ : S_t \in \mathcal{A}(X)\}$ is infinite.

EXAMPLE 2.3. *The bounded translation semigroup $(T_t f)(s) := f(s - t)$ on $L_2(\mathbb{T})$ is unitarily equivalent to the semigroup of diagonal operators*

$$D_t x := (e^{-a_n t} x_n) \quad \text{on } l_2(\mathbb{Z}) \tag{5}$$

for the special choice $a_n := -in$ and has the following properties:

(a) $R(\lambda, A) \in \mathcal{S}_2$, *but* $R(\lambda, A) \notin \mathcal{S}_1$ *for all* $\lambda \in \varrho(A)$.
(b) $S_t \in \mathcal{S}_1$ *for* $t = 2\pi k$, $k \in \mathbb{N}$.

Proof. For $h_t(s) := \sum_n e^{-a_n t} e^{ins}$ a direct computation shows $h_t * f = T_t f$ for all $f \in L_2(\mathbb{T})$. Thus it is sufficient to establish the properties (a) and (b) for the semigroup (D_t) defined in (5) which admits the following representation:

$$R(\lambda, A)x = \left(\frac{1}{\lambda + a_n}\, x_n\right), \qquad S_t x = \left(\frac{1 - e^{-a_n t}}{a_n}\, x_n\right) \tag{6}$$

where expressions of the form $(1 - e^{-at})/a$ are read as t if $a = 0$.

Now (a) is obvious because $(1/(\lambda - in)) \notin l_1$ for all λ in the resolvent set $\varrho(A)$. Furthermore, in the case $t = 2\pi k$ the operator S_t has rank 1 since

$$S_t x = (\ldots, 0, tx_0, 0, \ldots). \ \blacksquare$$

Example 2.3 also shows how far (T_t) can be from $\mathcal{A}$ if we only have $R(\lambda, A) \in \mathcal{A}$. However, for a differentiable (in particular analytic) semigroup the T_t are bounded operators from X into $D(A)$ and hence we have

COROLLARY 2.4. *If (T_t) is a differentiable semigroup then $R(\lambda, A) \in \mathcal{A}(X)$ implies $T_t \in \mathcal{A}(X)$ for all $t > 0$. The converse holds if $B_{\mathcal{A}(X)}$ is closed in $\mathcal{L}(X)$.*

In general, we have the following connection between the resolvent and the semigroup.

PROPOSITION 2.5. *Let $B_{\mathcal{A}(X)}$ be closed with respect to the strong operator topology of $\mathcal{L}(X)$* (cf. Lemma 1.2). *Then $T_{t_0} \in \mathcal{A}(X)$ for a fixed $t_0 > 0$ if and only if there are constants C, $\lambda_0 \geq 0$ such that*

$$\|T_{t_0} R(\lambda, A)\|_{\mathcal{A}} \leq C\lambda^{-1} \quad \text{for all } \lambda \geq \lambda_0.$$

REMARK 2.6. (a) If $T_{t_0} \in \mathcal{A}(X)$ we trivially have, for every $\omega > \omega(T_t)$,

$$\|T_{t_0+t}\|_{\mathcal{A}} \leq Me^{\omega t}\|T_{t_0}\|_{\mathcal{A}} \quad \text{for all } t \geq 0.$$

(b) If $I_X \notin \mathcal{A}(X)$ then the strong convergence $T_t \to I_X$ implies $\|T_t\|_{\mathcal{A}} \to \infty$ as $t \to 0$. In Section 4 we will relate this growth to the growth of $\|R(\lambda, A)\|_{\mathcal{A}}$ as $\lambda \to \infty$ in the case of analytic semigroups.

(c) For closed ideals $\mathcal{A}$ (for which $B_{\mathcal{A}(X)}$ is in general not closed with respect to the strong operator topology) we refer to Theorem 3.2.

Proof of Proposition 2.5. The "only if" part follows from the strong convergence of $\lambda R(\lambda, A)T_{t_0}$ to T_{t_0} to $\lambda \to \infty$. The "if" part is a consequence of 2.6(a) and

$$T_{t_0} R(\lambda, A) = \int_0^\infty e^{-\lambda t} T_{t+t_0}\, dt. \ ■$$

For the proof of Theorem 2.1 we need

LEMMA 2.7. *Let $f : [0, T] \to \mathcal{L}(X)$ be strongly continuous and bounded. For all $\omega \in \mathbb{R}$ consider the continuous function $F_\omega : [0, T] \to \mathcal{L}(X)$ defined by*

$$F_\omega(t)x := \int_0^t e^{-\omega s} f(s)x\, ds.$$

Assume that $B_{\mathcal{A}(X)}$ is closed in $\mathcal{L}(X)$. If F_{ω_0} is $\mathcal{A}(X)$-valued and $\|\cdot\|_{\mathcal{A}}$-bounded for some $\omega_0 \in \mathbb{R}$ then the same holds for all $\omega \in \mathbb{R}$.

Proof. Fix $\omega \in \mathbb{R}$. For $g_t : [0, T] \to \mathbb{R}$, $s \mapsto (\omega_0 - \omega)e^{-(\omega_0-\omega)(t-s)}$, a straightforward computation using Fubini's theorem shows

$$F_\omega(t) = e^{(\omega_0-\omega)t}\Big(F_{\omega_0}(t) - \int_0^t F_{\omega_0}(s) g_t(s)\, ds\Big). \tag{7}$$

Hence the assertion follows from 1.1 and the assumption on $\mathcal{A}$. ■

Proof of Theorem 2.1. The resolvent identity shows the equivalence of the two versions of (a). For (a)⇔(b) we just have to observe that $J_1(A) = (R(\lambda, A) : X \to X) \circ (A - \lambda I : D(A) \to X)$ and $(R(\lambda, A) : X \to X) = J_1(A) \circ (R(\lambda, A) : X \to D(A))$ for all $\lambda \in \varrho(A)$.

(a)⇒(c). Let $\widetilde{M} := (1 + M)\|R(b, A)\|_{\mathcal{A}}$ and $t \geq 0$. From

$$(I - \widetilde{T}_t)R(b, A) = \int_0^\infty \widetilde{T}_s\, ds - \int_t^\infty \widetilde{T}_s\, ds = \widetilde{S}_t$$

we get $\widetilde{S}_t = (I-\widetilde{T}_t)R(b,A) \in \mathcal{A}(X)$ and $\|\widetilde{S}_t\|_{\mathcal{A}} \le \|I-\widetilde{T}_t\| \cdot \|R(b,A)\|_{\mathcal{A}} \le \widetilde{M}$.

(c)⇒(a). We can assume that $\|\widetilde{T}_t\| < 1$ and the representation

$$R(b,A) = \int_0^\infty \widetilde{T}_s\,ds = \sum_{k=0}^\infty \widetilde{T}_{kt} \int_0^t \widetilde{T}_s\,ds = \Big(\sum_{k=0}^\infty (\widetilde{T}_t)^k\Big)\widetilde{S}_t$$

reveals that $R(b,A) = (\sum_{k=0}^\infty (\widetilde{T}_t)^k) S_t \in \mathcal{A}(X)$.

(c)⇒(d). By the proof of (a)⇒(c) we can assume in addition that $(\widetilde{S}_t)$ is $\|\cdot\|_{\mathcal{A}}$-bounded on $[0,T]$. Now we can apply Lemma 2.7 to the function $f(t) := T_t$.

(d)⇒(c). If we knew that (S_t) is $\|\cdot\|_{\mathcal{A}}$-bounded on $[0,T]$ for some $T>0$ we could apply Lemma 2.7 again. As a first step towards this boundedness we show that there exists a non-empty open subset E of $\mathbb{R}_+$ with the property that (S_t) is $\|\cdot\|_{\mathcal{A}}$-bounded on E.

Indeed, since $\{t \in \mathbb{R}_+ : S_t \in \mathcal{A}(X)\}$ contains a set of positive Lebesgue measure it has a compact subset K still of positive Lebesgue measure. Now put $K_n := \{t \in K : S_t \in nB_{\mathcal{A}(X)}\}$. If $K_n \ni t_m \to t$ then $S_{t_m} \to S_t$ in $\mathcal{L}(X)$, and since $B_{\mathcal{A}(X)}$ is closed in $\mathcal{L}(X)$ it follows that $t \in K_n$, i.e. K_n is closed. Since $K = \bigcup K_n$ some K_n has positive Lebesgue measure and thus contains a non-empty open set E.

Now we deduce the $\|\cdot\|_{\mathcal{A}}$-boundedness of (S_t) on some interval $[0,T]$ as follows. By Corollary 20.17 in [HR] we have $[0,T] \subset E-E$ for some $T>0$. An integrated semigroup satisfies the identity

$$S_t = T_h S_{t-h} + S_h, \qquad t \ge h \ge 0.$$

If we choose for every $h \in [0,T]$ some $t,s \in E$ such that $h = t-s$ then $S_h = S_t - T_h S_s$ and

$$\|S_h\|_{\mathcal{A}} \le \|S_t\|_{\mathcal{A}} + \|T_h\| \cdot \|S_s\|_{\mathcal{A}} \le 2 \sup_{t\le T} \|T_t\| C. \ \blacksquare$$

Proof of Remark 2.2. (i) The $\mathcal{A}$-boundedness of $(\widetilde{S}_t)$ is already shown in (a)⇒(c) above. The claim for (S_t) follows inductively from the identity

$$S_{nT} = T_{(n-1)T} S_T + S_{(n-1)T}.$$

Indeed, we get $\|S_{nT}\|_{\mathcal{A}} \le n\|T_{nT}\| \cdot \|S_T\|_{\mathcal{A}}$.

(ii) The above proof of Theorem 2.1—aside from the equivalence (a)⇔(b)—only uses the fact that $\mathcal{A}$ is a left ideal. Since the T_t, $R(\lambda,A)$ and S_t commute the proof is also valid for right ideals. ■

3. Application to the Phillips calculus. In this section we give a refinement of Theorem 2.1 in terms of the Phillips functional calculus for a C_0 semigroup (T_t) with generator A.

Let $L_1^+(a)$ denote the space of all measurable $\mathbb{C}$-valued functions g on $\mathbb{R}_{\geq 0}$ satisfying

$$\|g\|_{1,a} := \int_0^\infty |g(s)|e^{as}\,ds < \infty.$$

If $a > \omega(T_t)$ the Phillips functional calculus ([HP], §15) is defined by

$$\widehat{g}(-A)x := \int_0^\infty T_s x g(s)\,ds, \qquad g \in L_1^+(a).$$

EXAMPLE 3.1. (a) Let $\mathrm{Re}(\lambda) > a$, $\alpha > 0$. Then $g_{\lambda,\alpha}(s) := (s^{\alpha-1}/\Gamma(\alpha))e^{-\lambda s} \in L_1^+(a)$ and

$$\widehat{g}_{\lambda,\alpha}(-A) = R(\lambda, A)^\alpha. \tag{8}$$

(b) Let $t > 0$. Then $\chi_t := \chi_{[0,t]} \in L_1^+(a)$ and

$$\widehat{\chi}_t(-A) = S_t. \tag{9}$$

(c) For all $t > 0$ and $\delta \in (0,1)$ let $f_{t,\delta}$ denote the inverse Laplace transform of e^{-tz^δ} on $\mathbb{C}_+$. Then $f_{t,\delta} \in L_1^+(a)$ for $a \leq 0$, and the fractional power $-(-A)^\delta$ generates the holomorphic semigroup $T_{t,\delta}x := \int_0^\infty T_s x f_{t,\delta}(s)\,ds$ ([Y], §IX.11), i.e.

$$\widehat{f}_{t,\delta}(-A) = T_{t,\delta}. \tag{10}$$

After a glance at (8) and (9) the equivalent statements of Theorem 2.1 are of the form "$\widehat{g}(-A) \in \mathcal{A}(X)$" for certain $g \in L_1^+(a)$. For instance, for the ideal $\mathcal{A} = \mathcal{K}$ of compact operators this can be generalized.

THEOREM 3.2. *If $\mathcal{A}$ is closed then the following statements are equivalent*:

(a) $R(\lambda, A) \in \mathcal{A}(X)$ *for one* (*all*) $\lambda \in \varrho(A)$.
(b) $S_t \in \mathcal{A}(X)$ *on* $[0, T]$ *for some* $T > 0$.
(c) $\widehat{g}(-A) \in \mathcal{A}(X)$ *for all* $g \in L_1^+(a)$.
(d) $\widehat{g}_0(-A) \in \mathcal{A}(X)$ *for some* $g_0 \in L_1^+(a)$ *satisfying*

$$\widehat{g}_0(z) \neq 0 \quad \textit{for all } \mathrm{Re}(z) \geq -a \quad \textit{and} \quad 0 \in \mathrm{supp}(g_0). \tag{11}$$

If in addition $a \leq 0$ then another equivalent statement is:

(e) $T_{t,\delta} \in \mathcal{A}(X)$ *for one* (*all*) $t > 0$ *and one* (*all*) $0 < \delta < 1$.

Notice that $\widehat{\chi}_t(z) = (1 - e^{-tz})/z \neq 0$ for $\mathrm{Re}(z) > 0$, i.e. if $a < 0$ and $b = 0$ then the choice $g_0 := \chi_t$ in (d) reproduces the condition (d) of 2.1. Plugging $g_0 := g_{\lambda,\alpha}$ in statement (d) leads to

COROLLARY 3.3. *The resolvent is compact if and only if it is power compact.*

This can also be seen from the formula [De]

$$R(\lambda, A) = (m-1)\int_0^\infty \mu^{m-2} R(\lambda+\mu, A)^m \, d\mu, \qquad \lambda > a.$$

The corollary has some interesting consequences.

REMARK 3.4. (a) Let X be a space $C(K)$ or $L_1(\Omega, \mu)$ (or more generally, a space with the so-called Dunford–Pettis property [DU]). Then the product of two weakly compact operators is compact, hence Corollary 3.3 implies that a weakly compact resolvent is already compact. In other words, if A is a closed operator such that $R(\lambda, A)$ is weakly compact but not compact, then A is not the generator of a C_0 semigroup.

(b) Let X be a space $L_p(\Omega, \mu)$, $1 < p < \infty$. Then in a similar way we deduce from [M] and [W] that if $R(\lambda, A)$ is strictly singular (or an admissible Fredholm perturbation) then $R(\lambda, A)$ is already compact.

From (10) we derive

COROLLARY 3.5. *If $\mathcal{A}$ is a closed ideal and $R(\lambda, A) \in \mathcal{A}(X)$ then the subordinated semigroups $(T_{t,\delta})$ and the δ-times integrated semigroups*

$$S_{t,\delta} = \Gamma(\delta)^{-1}\int_0^t (t-s)^{\delta-1} T_s \, ds$$

belong to $\mathcal{A}$ for all $t > 0$ and all $\delta \in (0, 1)$.

In view of these consequences it is not surprising that the closedness of $\mathcal{A}$ is essential.

EXAMPLE 3.6. For the semigroup in 2.3 we see from (6) that

$$R(\lambda, A)^2 \in \mathcal{S}_1 \quad \text{and} \quad R(\lambda, A) \notin \mathcal{S}_1.$$

For the proof of Theorem 3.2 we need

LEMMA 3.7. (a) *$\{\chi_t : t > 0\}$ is total in $L_1^+(a)$.*

(b) *Let $g_0 \in L_1^+(a)$. Then $g_0 * L_1^+(a)$ is dense in $L_1^+(a)$ if and only if* (11) *holds.*

Proof. (a) is the density of the step functions in $L_1^+(a)$.

(b) We can assume $a = 0$ by applying this special case to $e^{a(\cdot)} g_0 \in L_1^+ := L_1^+(0)$ otherwise. According to Nyman's lemma ([D], 6.1), (11) holds if and only if the right translates $\{\tau_t g_0 : t > 0\}$ of g_0 are total in L_1^+. This is equivalent to the density of $g_0 * L_1^+$ in L_1^+ (see e.g. [K], 2.3). ∎

Proof of Theorem 3.2. In view of (8)–(10) all statements are implied by (c).

(b)⇒(c). Due to 3.7(a) every $g \in L_1^+(a)$ is approximable in $L_1^+(a)$ by linear combinations g_n of suitable χ_t. But (b) implies $\widehat{\chi}_t(-A) = S_t \in \mathcal{A}(X)$

for all $t \geq 0$. Hence $\widehat{g}(-A) \in \mathcal{A}(X)$ follows from the closedness of $\mathcal{A}$ and the continuity of the functional calculus:

$$\mathcal{A}(X) \ni \widehat{g}_n(-A) \to \widehat{g}(-A) \quad \text{in } \mathcal{L}(X).$$

(d)⇒(c). By means of 3.7(b) we find some $h \in L_1^+(a)$ such that $\|g - g_0 * h\|_{1,a} \leq \varepsilon$. Recalling that $\widehat{g}_0(-A)\widehat{h}(-A) \in \mathcal{A}(X)$ yields the assertion because

$$\begin{aligned}\|\widehat{g}(-A) - \widehat{g}_0(-A)\widehat{h}(-A)\| &= \|\widehat{g}(-A) - \widehat{g_0 * h}(-A)\| \\ &\leq M\|g - g_0 * h\|_{1,a} \leq M\varepsilon.\end{aligned}$$

Now (a)⇒(c) and (e)⇒(c) follow from $\widehat{g}_{\lambda,1}(z) = (\lambda + z)^{-1} \neq 0$ and $\widehat{f}_{t,\delta}(z) = e^{-tz^\delta} \neq 0$ for $\mathrm{Re}(z) \geq -a$. ∎

Next we present an extension of Pazy's result on compact semigroups ([Pa], §I.2.3).

THEOREM 3.8. *Let $\mathcal{A}$ be closed and (T_t) be norm-continuous for $t > 0$. Then we may add in Theorem* 3.2 *the following equivalent statements*:

(f) $T_t \in \mathcal{A}(X)$ *for all* $t > 0$.
(g) $\widehat{\mu}(-A) \in \mathcal{A}(X)$ *for all* $\mu \in M_1^+(a)$.

Here we denote by $M_1^+(a)$ the space of all measures μ on $\mathbb{R}_+$ satisfying

$$\|\mu\|_{1,a} := \int_0^\infty e^{as}\, d|\mu|(s) < \infty.$$

REMARK 3.9. If X is a Hilbert space the norm-continuity of (T_t) is equivalent to

$$\|R(a + ir, A)\| \to 0 \quad \text{for } |r| \to \infty \text{ and some } a > \omega(T_t).$$

The same is true for positive semigroups on $L_p(\Omega, \mu)$ (see [GW] for references). So in these cases statement (f) can be characterized by conditions on the resolvent only.

Proof of Theorem 3.8. (a)⇒(f). For a fixed $t > 0$ and all $\lambda > 0$ we have

$$\lambda R(\lambda, A)T_t - T_t = \lambda \int_0^\infty e^{-\lambda s}(T_{t+s} - T_t)\, ds$$

and thus for every $\delta > 0$ and $a > \omega(T_t)$,

$$\|\lambda R(\lambda, A)T_t - T_t\| \leq \sup_{s\in[0,\delta]} \|T_{t+s} - T_t\| + C\frac{\lambda}{\lambda - \omega}e^{a(t+\delta)}e^{-\delta\lambda}.$$

The first term tends to zero as $\delta \to 0$ while for every fixed δ the second term tends to zero as $\lambda \to \infty$. Since $\lambda R(\lambda, A)T_t \in \mathcal{A}(X)$ and $\mathcal{A}$ is closed it follows that $T_t \in \mathcal{A}(X)$.

(f)⇒(g). Since $\mathbb{R}_+ \ni t \mapsto T_t \in \mathcal{L}(X)$ is measurable the integral

$$\widehat{\mu}(-A) = \int_0^\infty T_t \, d\mu(t)$$

exists in $\mathcal{L}(X)$ for every $\mu \in M_1^+(a)$ with $a > \omega(T_t)$. Since $\mathcal{A}$ is closed we deduce $\widehat{\mu}(-A) \in \mathcal{A}(X)$.

(g)⇒(a) is clear. ∎

As pointed out to us by the referee, our Theorem 3.2 can be extended to bounded integrated semigroups $(S_t)_{t\geq 0}$ (cf. [A]) satisfying the Lipschitz condition

$$\limsup_{h\searrow 0} h^{-1} \|S_{t+h} - S_t\| \leq M \quad \text{for all } t \geq 0.$$

By the integrated version of Widder's Theorem [A, Cor. 1.2], the operators

$$R(\lambda) := \lambda \int_0^\infty e^{-\lambda t} S_t \, dt, \quad \lambda > 0,$$

provide a pseudo-resolvent $(R(\lambda))_{\lambda>0}$ which is tempered at infinity in the sense of [CK]. Due to [CK, Thm. 2], there exists a continuous Banach algebra homomorphism $\mathcal{H} : L_1(\mathbb{R}_+) \to \mathcal{L}(X)$ such that

$$\mathcal{H}(e^{-\lambda \cdot}) = R(\lambda) \quad \text{and} \quad \mathcal{H}(\chi_{[0,t]}) = S_t \quad \text{for all } \lambda > 0,\ t \geq 0.$$

In this setting, the proof of Theorem 3.2 (for $a = 0$) shows the following version for integrated semigroups. By $\mathcal{A}$ we still denote a normed operator ideal.

THEOREM 3.10. *If $\mathcal{A}$ is closed then the following statements are equivalent*:

(a) $R(\lambda) \in \mathcal{A}(X)$ *for one* (*all*) $\lambda > 0$.
(b) $S_t \in \mathcal{A}(X)$ *for all* $t \geq 0$.
(c) $\mathcal{H}(g) \in \mathcal{A}(X)$ *for all* $g \in L_1(\mathbb{R}_+)$.
(d) $\mathcal{H}(g_0) \in \mathcal{A}(X)$ *for some* $g_0 \in L_1(\mathbb{R}_+)$ *satisfying*

$$\widehat{g}_0(z) \neq 0 \quad \textit{for all } \operatorname{Re}(z) \geq 0 \quad \textit{and} \quad 0 \in \operatorname{supp}(g_0).$$

4. Analytic semigroups. In this section (T_t) is always a bounded holomorphic semigroup of angle $\delta(A)$ with generator A, and $\mathcal{A}$ is a one-sided operator ideal in the sense of (4).

From Corollary 2.4 we already know that $R(\lambda, A) \in \mathcal{A}$ if and only if $T_t \in \mathcal{A}$ but this qualitative information can be improved by using precise growth estimates on the $\mathcal{A}$-norms of $R(\lambda, A)$ and T_t, using among other things estimates by Prüss on the inverse Laplace transform [Pr].

If $B_{\mathcal{A}(X)}$ is closed with respect to the strong operator topology of $\mathcal{L}(X)$ (as shown in Lemma 1.2 for maximal ideals, e.g. the Hilbert–Schmidt and the trace class) and $I_X \notin \mathcal{A}(X)$ then, of course, $\|T_t\|_{\mathcal{A}} \to \infty$ blows up as $t \to 0$. The essential information of the next theorem is then how this growth is related to the growth of $\|R(\lambda, A)\|_{\mathcal{A}}$ as $|\lambda| \to \infty$.

In our growth conditions we use the symbol $\preceq$ to indicate that the left hand side is dominated by a positive multiple of the right hand side.

THEOREM 4.1. *Let $\beta \in [0,1]$, $\alpha \geq \beta$ and $\delta \in [0, \delta(A))$.*

(a) *If $\beta > 0$ and $\|T_t\|_{\mathcal{A}} \preceq t^{\beta-\alpha}$ on $\mathbb{R}_+$ then*

$$\|R(\lambda, A)^\alpha\|_{\mathcal{A}} \preceq |\lambda|^{-\beta} \quad \text{on } \mathbb{C}_+.$$

(b) *If $\|R(\lambda, A)^\alpha\|_{\mathcal{A}} \leq D_\delta |\lambda|^{-\beta}$ on Σ_δ then*

$$\|T_z\|_{\mathcal{A}} \leq \mathcal{C}_\beta \Gamma(\alpha+2) R_\delta(A)^2 D_\delta |z|^{\beta-\alpha} \quad \text{on } S_\delta.$$

Here we let $\mathbb{C}_+ := \Sigma_0 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Simple examples of diagonal operators of the type (5) show that the assertion (a) is not true for $\beta = 0$ (cf. [B1]).

We postpone the proof of Theorem 4.1 to the end of the section and discuss first some consequences. Combining (a) and (b) of Theorem 4.1 leads to the following characterization.

COROLLARY 4.2. (a) *If $\beta \in (0,1]$ then the following equivalences hold*:

$$\begin{array}{ccc} \|T_t\|_{\mathcal{A}} \preceq t^{\beta-\alpha} \text{ on } \mathbb{R}_+ & \Leftrightarrow & \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \preceq \lambda^{-\beta} \text{ on } \mathbb{R}_+ \\ \Updownarrow & & \Updownarrow \\ \forall \delta : \|T_z\|_{\mathcal{A}} \preceq |z|^{\beta-\alpha} \text{ on } S_\delta & \Leftrightarrow & \forall \delta : \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \preceq |\lambda|^{-\beta} \text{ on } \Sigma_\delta \end{array}$$

All implications except for the two "$\Rightarrow$" also hold for $\beta = 0$.

(b) *Let $\varepsilon > 0$. If $0 \in \varrho(A)$ then the following implications hold*:

$$(-A)^{-\alpha} \in \mathcal{A}(X) \Rightarrow \|T_z\|_{\mathcal{A}} \preceq |z|^{-\alpha} \text{ on } S_\delta \Rightarrow (-A)^{-(\alpha+\varepsilon)} \in \mathcal{A}(X).$$

EXAMPLE 4.3. Let (T_t) be a bounded holomorphic semigroup on $L_p(G)$ for a measure space G. By considering the right-sided operator ideal $\mathcal{A}$ defined via $\mathcal{A}(X, Y) := \mathcal{L}(X, L_q(G))$ and using the notation $\|\cdot\|_{p,q} := \|\cdot\|_{\mathcal{L}(L_p, L_q)}$ this section establishes the following connections:

$$\|T_t\|_{p,q} \preceq t^{-\alpha} \rightleftharpoons (-A)^{-\alpha} \in \mathcal{L}(L_p, L_q) \rightleftharpoons \|R(\lambda, A)^{\alpha+\beta}\|_{p,q} \preceq |\lambda|^{-\beta}.$$

(The symbol "$\rightleftharpoons$" is used in order to indicate that, if necessary, the exponents α and β have to be interpreted "up to an ε".) Characterizations of the above type are studied in the setting of *ultracontractivity* of semigroups, i.e. of $\|T_t\|_{1,\infty}$-estimates (cf. e.g. [C]; [VSC], §2).

In the following applications (mostly to differential operators) we use information on the imbedding $J_1(A) : D(A) \to X$, provided by Sobolev imbeddings or the integral kernel of the resolvent, to obtain directly estimates for the semigroup.

EXAMPLE 4.4. Let Ω be a bounded domain in $\mathbb{R}^n$ and fix $\varepsilon > 0$. Assume that A is a differential operator in $L_p(\Omega)$ with $D((-A)^\alpha) \subset W_p^{1+\varepsilon}(\Omega)$. If $p = n$ the Sobolev imbedding theorem implies that $R(\lambda, A)^\alpha(L_p(\Omega)) \subset C_{\mathrm{b}}(\Omega) \subset L_p(\Omega)$ and $R(\lambda, A)$ is a *Hille–Tamarkin operator* (these provide a one-sided operator ideal $\mathcal{A}$ in $\mathcal{L}(L_p(\Omega))$; cf. [Pi]) with $\|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le C$. Corollary 4.2 implies

$$\|T_t\|_{\mathcal{A}} \preceq t^{-\alpha}.$$

In particular, the T_t are integral operators with p-summable eigenvalues. Indeed, since operators in $\mathcal{A}$ are absolutely p-summing with $p = n$ we deduce from [Pi], Thm. 3.7.3, that T_t^n has summable eigenvalues $\lambda_k(T_t^n)$. Hence $T_t = (T_{t/n})^n$ yields

$$\sum |\lambda_k(T)| \le C_1 \|T_{t/n}\|_{\mathcal{A}}^n \le C_2 (n/t)^{\alpha n} = C_3 t^{-\alpha n}.$$

For a Schrödinger operator A one can choose $\alpha > 1/2 + \varepsilon$ and we almost get the trace estimate $O(t^{-n/2})$ obtained by direct estimation of the integral kernel of the T_t. But our approach applies to more general operators A also in L_p-spaces and it shows how the exponent in the Sobolev imbedding theorem determines the exponent in the trace estimate.

In [B2] this method is refined to give precise estimates for elliptic operators on Hilbert spaces, even in the much more complicated case of unbounded domains.

EXAMPLE 4.5. In [B3] we present an alternative approach to *Gaussian heat kernel estimates*. Instead of applying the standard method using Nash-type inequalities and Moser iteration we apply the combination of the corresponding Sobolev inequality and the ellipticity of A to

$$D((-\widetilde{A}_\varrho)^{1/2}) \hookrightarrow \mathring{W}_2^m(\Omega) \hookrightarrow L_q(\Omega) \tag{12}$$

where the $\widetilde{A}_\varrho$ are suitable translations of the generators A_ϱ of the perturbations (T_t^ϱ) corresponding to Davies' perturbation method [Da]. From (12) it follows that $J_{1/2}(\widetilde{A}_\varrho) \in \mathcal{A}$ for the operator ideal $\mathcal{A}$ defined in Example 4.3 for $p = 2$. Hence one obtains $\|\widetilde{T}_t^\varrho\|_{2,q} \preceq t^{-1/2}$ uniformly in all perturbations from our perturbation result 4.6 and then $\|\widetilde{T}_t^\varrho\|_{1,\infty} \preceq t^{-N/2m}$ from an extrapolation result due to Coulhon [C]. Ultracontractive estimates of this type are equivalent to the desired Gaussian estimates (cf. [AE], 3.3).

In Corollary 4.2 we omitted explicit constants in the characterization

$$\|T_z\|_{\mathcal{A}} \le C_1 |z|^{\beta-\alpha} \;\Leftrightarrow\; \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le C_2 |\lambda|^{-\beta}. \tag{13}$$

Here we analyse the implication "$\Leftarrow$" and state explicitly which parameters of the semigroup enter in the constant C_1. This allows uniform $\|\cdot\|_{\mathcal{A}}$-estimates for such perturbations $(\widehat{T}_t)$ of the given semigroup (T_t) whose perturbative effect on these parameters is uniformly bounded.

This is of crucial importance for both announced applications [B2], [B3] to partial differential operators A. Hence we will express the right hand side of (13) in terms of the imbedding operator $J_\alpha(A)$, i.e. in terms of the Sobolev imbedding corresponding to A.

As in Theorem 2.1 we need a *right-sided* ideal property of $\|\cdot\|_{\mathcal{A}}$ here.

Proposition 4.6. *If* $J_\alpha(A) \in \mathcal{A}$ *and* $0 \in \varrho(A)$ *then*

$$\|T_t\|_{\mathcal{A}} \le C_\alpha R_0(A)^{\alpha+5} \|J^0_\alpha(A)\|_{\mathcal{A}} t^{-\alpha} \quad \text{on } \mathbb{R}_+ \tag{14}$$

for a constant C_α *of the type* $C_\alpha = C' C^\alpha \Gamma(\alpha+2)$.

Now we come to the proofs for this section. Recall that (T_t) is always a bounded holomorphic semigroup. In the proof of Theorem 4.1 we will use the ideal property to obtain $\mathcal{A}$-norm estimates for resolvent powers $R(\lambda, A)^\alpha$ from the corresponding estimates for the operator norm. These resolvent estimates are collected in the following two lemmas.

Lemma 4.7. (a) *For all* $\lambda \in \overline{\Sigma}_\delta$, $\eta > 0$ *we have* $\|\eta(\eta+\lambda-A)^{-1}\| \le (1+\tan\delta) R_\delta(A)$. *In particular*, $\lambda - A$ *is non-negative for all such* λ.

(b) $D((\lambda-A)^\alpha)$ *is independent of* $\lambda \in \overline{\Sigma}_\delta$ *with equivalent graph-norms.*

(c) *Let* $0 < \alpha < 1$. *Then, for all* $\lambda_1, \lambda_2 \in \overline{\Sigma}_\delta$ *and* $x \in D((-A)^\alpha)$,

$$\|(\lambda_1 - A)^\alpha x - (\lambda_2 - A)^\alpha x\| \le C_\delta R_\delta(A)^2 |\lambda_1 - \lambda_2|^\alpha \|x\|. \tag{15}$$

If $\lambda_1 = 0$ *we can replace* C_δ *by*

$$\mathcal{E}_\alpha := \sqrt{\pi}\,\Gamma\Big(1 - \frac{\alpha}{2}\Big)^{-1} \Gamma\Big(\frac{1+\alpha}{2}\Big)^{-1}.$$

Proof. (a) is trivial since $\eta + \lambda \in \Sigma_\delta$ and $\eta/|\eta+\lambda| \le 1 + \tan\delta =: C_\delta$.

(b) $D((\lambda-A)^\alpha) = D((\lambda+\varepsilon-A)^\alpha)$ holds for all $\lambda \in \overline{\Sigma}_\delta$, $\varepsilon > 0$ due to [MSM], Theorem 2.1. Applying this to rotations $e^{i\gamma}A$ of A yields the first statement.

Now let $\|\cdot\|_\lambda$ denote the graph-norm of $(\lambda - A)^\alpha$. Then $D := D((-A)^\alpha) = D((\lambda-A)^\alpha)$ is a Banach space with respect to $\|\cdot\|_0$ and $\|\cdot\|_\lambda$ because $(-A)^\alpha$ and $(\lambda-A)^\alpha$ are closed operators. Hence D is also a Banach space with respect to $\|\cdot\|_0 + \|\cdot\|_\lambda$ which is stronger than and thus equivalent to $\|\cdot\|_0$ and $\|\cdot\|_\lambda$.

(c) The standard representation for fractional powers of non-negative operators B,

$$B^\alpha x = \frac{\sin\alpha\pi}{\pi} \int_0^\infty \eta^{\alpha-1}(\eta+B)^{-1} Bx\, d\eta, \quad x \in D(B) \tag{16}$$

([MSM], Remarks 2.4, 2.5), admits the following estimate for all $x \in D(A)$:

$$\begin{aligned}
&\|x\|^{-1}\frac{\pi}{\sin\alpha\pi}\|(\lambda_1-A)^\alpha x-(\lambda_2-A)^\alpha x\| \\
&\quad\le\int_0^\infty \eta^\alpha\|(\eta+\lambda_1-A)^{-1}-(\eta+\lambda_2-A)^{-1}\|\,d\eta \\
&\quad\le 2C_\delta R_\delta(A)\int_0^{|\lambda_1-\lambda_2|}\eta^{\alpha-1}\,d\eta+C_\delta^2R_\delta(A)^2|\lambda_1-\lambda_2|\int_{|\lambda_1-\lambda_2|}^\infty \eta^{\alpha-2}\,d\eta \\
&\quad\le (2C_\delta\alpha^{-1}+C_\delta^2(1-\alpha)^{-1})R_\delta(A)^2|\lambda_1-\lambda_2|^\alpha.
\end{aligned}$$

Thus (15) is shown for $x\in D(A)$. But $D(A)$ is a core for $(-A)^\alpha$ and the general statement follows from (b). If $\lambda_1=0$ and $\lambda_2=\lambda\in\mathbb{C}_+$ the estimate reads as follows:

$$\begin{aligned}
\|x\|^{-1}\frac{\pi}{\sin\alpha\pi}\|(-A)^\alpha x-(\lambda-A)^\alpha x\| &\le\int_0^\infty \eta^\alpha\|\lambda R(\eta+\lambda,A)R(\eta,A)\|\,d\eta \\
&\le R_0(A)^2\int_0^\infty\frac{\eta^{\alpha-1}|\lambda|}{\sqrt{\eta^2+|\lambda|^2}}\,d\eta \\
&= R_0(A)^2|\lambda|^\alpha\int_0^\infty\frac{s^{\alpha-1}}{\sqrt{1+s^2}}\,ds.
\end{aligned}$$

The assertion follows from the formula

$$\int_0^\infty\frac{s^{\alpha-1}}{\sqrt{1+s^2}}\,ds=\frac{\pi}{\sin\alpha\pi}\cdot\frac{\sqrt{\pi}}{\Gamma(1-\alpha/2)\Gamma((1+\alpha)/2)}\tag{17}$$

and is transferred to $\lambda_2\in\overline{\Sigma}_\delta$ by means of the above rotation argument. ■

LEMMA 4.8. (a) *Let* $\lambda\in\Sigma_\delta$, $\mu\in\overline{\Sigma}_\delta$. *Then* $R(\lambda,A)$ *and* $(\mu-A)R(\lambda,A)$ *are non-negative*, $R(\lambda,A)^\alpha\in\mathcal{L}(X)$ *has range* $D((-A)^\alpha)$, *and*

$$(\mu-A)^\alpha R(\lambda,A)^\alpha=((\mu-A)R(\lambda,A))^\alpha\in\mathcal{L}(X).\tag{18}$$

(b) *For all* $|\gamma|<\delta(A)$ *and all* $\lambda\in\mathbb{C}_+$ *we have*

$$R(e^{i\gamma}\lambda,A)^\alpha=\Gamma(\alpha)^{-1}e^{-i\gamma\alpha}\int_0^\infty e^{-\lambda t}t^{\alpha-1}T_{e^{-i\gamma}t}\,dt.\tag{19}$$

(c) *For all* $\lambda\in\Sigma_\delta$ *we have*

$$\|(\lambda R(\lambda,A))^\alpha\|\le\mathcal{E}_\alpha R_\delta(A),\qquad 0<\alpha<1,\tag{20}$$

$$\|(-A)^\alpha R(\lambda,A)^\alpha\|\le C'C^\alpha R_\delta(A)^{\alpha+3},\tag{21}$$

$$\|(|\lambda|-A)^\alpha R(\lambda,A)^\alpha\|\le C_\delta C^\alpha R_\delta(A)^{\alpha+3}.\tag{22}$$

Proof. (a) Since $\eta(\eta+R(\lambda,A))^{-1}=I-\eta^{-1}(\eta^{-1}+\lambda-A)^{-1}$ and $\lambda-A$ is non-negative this is also true for $R(\lambda,A)$. So $R(\lambda,A)^\alpha\in\mathcal{L}(X)$ follows from [MSM], Rem. 2.3. Moreover, $(\lambda - A)^\alpha = [R(\lambda, A)^\alpha]^{-1}$ *per definitionem* ([MSM], 2.1).

To show the non-negativity of $(\mu - A)R(\lambda, A)$ and the formula (18), one follows (the proof of) Theorem 2.1 of [MSM], and uses the above rotation argument.

(b) By means of the rotation argument we can assume $\gamma = 0$. For $\alpha \in \mathbb{N}$, (19) follows inductively from (1), and for $\alpha \in (0,1)$ it is obtained by using the integral representation (16). These two cases combine to the general statement by applying the additivity of the fractional powers [MSM] and Fubini's theorem.

(c) Again we can assume $\delta = 0$, i.e. $\lambda \in \mathbb{C}_+$. Then (20) follows from (17) and

$$\begin{aligned}\|x\|^{-1} \frac{\pi}{\sin \alpha\pi} \|R(\lambda, A)^\alpha x\| &= \|x\|^{-1} \Big\| \int_0^\infty \eta^{\alpha-2} R(\eta^{-1} + \lambda, A)x \, d\eta \Big\| \\ &\le R_0(A) \int_0^\infty \frac{\eta^{\alpha-2}}{\sqrt{\eta^{-2} + |\lambda|^2}} \, d\eta \\ &= R_0(A)|\lambda|^{-\alpha} \int_0^\infty \frac{s^{-\alpha}}{\sqrt{1+s^2}} \, ds\end{aligned}$$

where we used (16) again. Now we verify (21). For $\beta := \alpha - \lfloor \alpha \rfloor \in [0,1)$ we have

$$\|(-A)^\alpha R(\lambda, A)^\alpha\| \le (1 + R_\delta(A))^{\lfloor \alpha \rfloor} \|(-A)^\beta R(\lambda, A)^\beta\|.$$

Thus we can assume $\alpha \in (0,1)$. Using 4.7(c) for $\lambda_1 := 0$ and $\lambda_2 := \lambda$ as well as (20) we deduce for all $x \in X$ that

$$\|(-A)^\alpha R(\lambda, A)^\alpha x - x\| \le \mathcal{E}_\alpha R_\delta(A)^2 |\lambda|^\alpha \|R(\lambda, A)^\alpha x\| \le \mathcal{E}_\alpha^2 R_\delta(A)^3 \|x\|.$$

The proof of (2.2) is completely analogous. ∎

Remark 4.9. If $0 \in \varrho(A)$ one deduces analogously that 4.8(a)–(b) also hold for $\lambda = 0$. In this case we define the negative powers of $-A$ by $(-A)^{-\alpha} := R(0, A)^\alpha$.

In order to interpret the representations of $R(\lambda, A)^\alpha$ in (2), (19) as an $\mathcal{A}(X)$-valued Lebesgue integral (provided that $T_t \in \mathcal{A}(X)$ on $\mathbb{R}_+$ with a suitable $\|\cdot\|_{\mathcal{A}}$-behaviour) we have to check that the $\|\cdot\|$-smoothness of the semigroup implies at least its $\|\cdot\|_{\mathcal{A}}$-measurability. Similarly, if the resolvent belongs to $\mathcal{A}$, it will be necessary to transfer its $\|\cdot\|$-holomorphy into $\|\cdot\|_{\mathcal{A}}$-holomorphy.

Lemma 4.10. (a) *If $T_t \in \mathcal{A}(X)$ for all $t > 0$ then $T_{(\cdot)} : \mathbb{R}_+ \to \mathcal{A}(X)$ is holomorphic.*

(b) *Let $R(\lambda_0, A)^\alpha \in \mathcal{A}(X)$ for some $\lambda_0 \in \Sigma_\delta$. Then $R(\lambda, A)^\alpha \in \mathcal{A}(X)$ for all $\lambda \in \Sigma_\delta$ and $F_\alpha : \Sigma_\delta \to \mathcal{A}(X)$, $\lambda \mapsto R(\lambda, A)^\alpha$, is holomorphic with* *derivative*

$$F'_\alpha(\lambda) = -\alpha R(\lambda, A)^{\alpha+1}.$$

Proof. (a) follows by letting $h \to 0$ in

$$\|h^{-1}(T_{t+h} - T_t) - AT_t\|_{\mathcal{A}} \le \|T_{t/2}\|_{\mathcal{A}}\|h^{-1}(T_{t/2+h} - T_{t/2}) - AT_{t/2}\|.$$

(b) The first part is trivial in view of the "resolvent power identity"

$$R(\lambda, A)^\alpha - R(\mu, A)^\alpha = ((\mu - A)^\alpha - (\lambda - A)^\alpha)R(\lambda, A)^\alpha R(\mu, A)^\alpha.$$

Now denote by $\|\cdot\|_\lambda$ the graph-norm of $(\lambda - A)^\alpha$. We see that F_α is holomorphic into $\mathcal{L}(X)$ via differentiating under the integral in (19). Let us assume for the moment that F_α is continuous into $L := \mathcal{L}(X, D((-A)^\alpha))$. Then F_α is even holomorphic into L since $L \hookrightarrow \mathcal{L}(X)$ (cf. [Wr]). Because of $R(\lambda, A)^{\alpha+1} \in \mathcal{A}(X)$ and $\|\cdot\|_\lambda \preceq \|\cdot\|_0$ we can estimate as follows:

$$\begin{aligned}
&\left\|\frac{R(\lambda+h, A)^\alpha - R(\lambda, A)^\alpha}{h} + \alpha R(\lambda, A)^{\alpha+1}\right\| \\
&\quad = \left\|(\lambda - A)^\alpha\left(\frac{R(\lambda+h, A)^\alpha - R(\lambda, A)^\alpha}{h} + \alpha R(\lambda, A)^{\alpha+1}\right)R(\lambda, A)^\alpha\right\|_{\mathcal{A}} \\
&\quad \preceq \left\|\frac{R(\lambda+h, A)^\alpha - R(\lambda, A)^\alpha}{h} + \alpha R(\lambda, A)^{\alpha+1}\right\|_L \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \\
&\quad \to 0 \qquad \text{as } h \to 0.
\end{aligned}$$

Therefore we will show inductively that for all $n \in \mathbb{N}$ and all $\alpha \in (0, n)$ the map $F_\alpha : \Sigma_\delta \to L$ is continuous. For $n = 1$ this follows from 4.7(b)–(c):

$$\begin{aligned}
\|R(\lambda+h, A)^\alpha x - R(\lambda, A)^\alpha x\|_0 &\preceq \|R(\lambda+h, A)^\alpha x - R(\lambda, A)^\alpha x\|_\lambda \\
&\preceq \|((\lambda - A)^\alpha - (\lambda+h-A)^\alpha)R(\lambda+h, A)^\alpha x\| \\
&\preceq |h|^\alpha \|R(\lambda+h, A)^\alpha x\| \le |h|^\alpha C\|x\|
\end{aligned}$$

for $|h|$ small. For $n > 1$ the commutativity of the $(\lambda - A)^\alpha$ guarantees

$$\begin{aligned}
&\|R(\lambda+h, A)^\alpha x - R(\lambda, A)^\alpha x\|_0 \\
&\quad \preceq \|((\lambda - A)^\alpha - (\lambda+h-A)^\alpha)R(\lambda+h, A)^\alpha x\| \\
&\quad = \Big\|((\lambda - A)^{\alpha/n} - (\lambda+h-A)^{\alpha/n}) \\
&\qquad\quad \circ \sum_{j=0}^{n-1} (\lambda - A)^{\alpha(n-1-j)/n} R(\lambda+h, A)^{\alpha(n-j)/n} x\Big\| \\
&\quad \preceq |h|^{\alpha/n} \sum_{j=0}^{n-1} \|(\lambda - A)^{\alpha(n-1-j)/n} R(\lambda+h, A)^{\alpha(n-j)/n} x\| \le |h|^{\alpha/n} C\|x\|
\end{aligned}$$

for $|h|$ small, where we used the induction hypothesis in the last step. ∎

Now the preparations for the proofs of Section 4 are finished.

Proof of Theorem 4.1. (a) By hypothesis we have $\|T_t\|_{\mathcal{A}} \preceq t^{\beta-\alpha}$ on $\mathbb{R}_+$, which extends to $\|T_z\|_{\mathcal{A}}$-estimates on sectors. Indeed, let $\delta', \delta'' \in (\delta, \delta(A))$ such that $\delta' < \delta''$ and $r := \sin(\delta'' - \delta')$. Since $z \in S_{\delta'}$ implies $z - r|z| \in S_{\delta''}$ we have

$$\|T_z\|_{\mathcal{A}} \le \|T_{r|z|}\|_{\mathcal{A}} \|T_{z-r|z|}\| \preceq |z|^{\beta-\alpha} \quad \text{on } S_{\delta'}. \tag{23}$$

Now we continue with a slightly modified rotation argument. For $|\gamma| \le \delta'$ the semigroup $(T_{e^{i\gamma}t})$ has the generator $A_\gamma := e^{i\gamma}A$ and satisfies, by Lemma 4.10(a), the representation (19) and the line (23), the following estimate on $\mathbb{C}_+$:

$$\|R(\lambda, A_\gamma)^\alpha\|_{\mathcal{A}} \le \Gamma(\alpha)^{-1} D_0 \int_0^\infty e^{-\operatorname{Re}(\lambda)t} t^{\beta-1}\, dt = D \operatorname{Re}(\lambda)^{-\beta}$$

with D independent of γ and λ. Thus we only have to show the estimate on $\mathcal{S} := \Sigma_\delta \setminus S_{\pi/2-\varepsilon}$, where $\varepsilon := \delta' - \delta$. Now fix $\gamma := -\delta'$. For all $\lambda \in \mathcal{S}$ with $\operatorname{Im}(\lambda) > 0$ we have $e^{i\gamma}\lambda \in S_{\pi/2-\varepsilon}$ and therefore

$$\|R(\lambda, A)^\alpha\|_{\mathcal{A}} = \|R(e^{i\gamma}\lambda, A_\gamma)^\alpha\|_{\mathcal{A}} \le D \operatorname{Re}(e^{-i\gamma}\lambda)^{-\beta} \preceq |\lambda|^{-\beta}.$$

On $\mathcal{S} \cap \{z \mid \operatorname{Im}(z) < 0\}$ we proceed analogously using $\gamma := \delta'$.

(b) Suppose first that $\delta = 0$. By 4.10(b) the function $F : \mathbb{C}_+ \to \mathcal{A}(X)$, $\lambda \mapsto R(\lambda, A)^\alpha$, is holomorphic and satisfies the following estimate on $\mathbb{C}_+$:

$$\|F(\lambda)\|_{\mathcal{A}}, \|\lambda F'(\lambda)\|_{\mathcal{A}} \le \max(1, \alpha R_0(A)) D_0 |\lambda|^{-\beta}. \tag{24}$$

This can be seen from

$$\|\lambda F'(\lambda)\|_{\mathcal{A}} \le \alpha \|\lambda R(\lambda, A)\| \cdot \|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le \alpha R_0(A) D_0 |\lambda|^{-\beta}.$$

So if $\beta > 0$ we are in a position to apply Theorems 1 and 2 of [Pr] and obtain a continuous function $u : \mathbb{R}_+ \to \mathcal{A}(X)$ with the properties

$$F(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt \quad \text{on } \mathbb{C}_+, \tag{25}$$

$$\|u(t)\|_{\mathcal{A}} \le \mathcal{C}_\beta \max(1, \alpha R_0(A)) D_0 t^{\beta-1} \quad \text{on } \mathbb{R}_+.$$

In view of (19) the identity (25) holds for the function

$$u_0(t) = \Gamma(\alpha)^{-1} t^{\alpha-1} T_t$$

so that the uniqueness of the Laplace transform yields the assertion. Now we turn to the case $\beta = 0$ and set $\widetilde{\alpha} := \alpha + 1$, $\widetilde{\beta} := 1$, $\widetilde{F}(\lambda) := R(\lambda, A)^{\widetilde{\alpha}}$ and $\widetilde{D}_0 := R_0(A) D_0$. Then (24) holds for these $\widetilde{\ }$-quantities and our earlier conclusion shows

$$\|T_t\|_{\mathcal{A}} \le \mathcal{C}_1 \Gamma(\widetilde{\alpha}) \widetilde{\alpha} R_0(A) \widetilde{D}_0 t^{\widetilde{\beta}-\widetilde{\alpha}} = \mathcal{C}_1 \Gamma(\alpha+2) R_0(A)^2 D_0 t^{\beta-\alpha}.$$

For arbitrary $\delta \in [0, \delta(A))$ we use the rotation argument again. Indeed, for $|\gamma| < \delta$ we get from the first part

$$\|T_{e^{i\gamma}t}\|_{\mathcal{A}} \le \mathcal{C}_\beta \Gamma(\alpha+2) R_0(e^{i\gamma}A)^2 D_\delta t^{\beta-\alpha} \le \mathcal{C}_\beta \Gamma(\alpha+2) R_\delta(A)^2 D_\delta |e^{i\gamma}t|^{\beta-\alpha}. \ \blacksquare$$

Proof of Corollary 4.2. (a) The $\|\cdot\|_{\mathcal{A}}$-estimates for the semigroup on the halfline imply estimates on sectors analogous to those shown in the proof of Theorem 4.1(a). For the resolvent this can be seen as follows. If $\|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le D|\lambda|^{-\beta}$ on $\mathbb{R}_+$ then from (18) and (22) we obtain, for all $\lambda \in \Sigma_\delta$,

$$\begin{aligned}\|R(\lambda, A)^\alpha\|_{\mathcal{A}} &= \|((|\lambda| - A)^\alpha R(\lambda, A)^\alpha) R(|\lambda|, A)^\alpha\|_{\mathcal{A}} \\ &\le \|(|\lambda| - A)^\alpha R(\lambda, A)^\alpha\| \|R(|\lambda|, A)^\alpha\|_{\mathcal{A}} \\ &\le C_\delta C^\alpha R_\delta(A)^{\alpha+3} D|\lambda|^{-\beta}.\end{aligned}$$

All other implications are trivial or proved in Theorem 4.1.

(b) According to (21) we can estimate on Σ_δ as follows if $(-A)^{-\alpha} \in \mathcal{A}(X)$:

$$\|R(\lambda, A)^\alpha\|_{\mathcal{A}} \le \|(-A)^\alpha R(\lambda, A)^\alpha\| \|(-A)^{-\alpha}\|_{\mathcal{A}} \preceq 1.$$

Hence 4.1 yields the first implication. The second follows directly from (19) for $\gamma = 0$ and $\lambda = 0$ (cf. 4.9!) if we keep in mind that (T_t) has negative type. ■

Proof of Proposition 4.6. Because of (18) and (21), on $\mathbb{C}_+$ we have

$$\begin{aligned}\|R(\lambda, A)^\alpha\|_{\mathcal{A}} &\le \|(-A)^\alpha R(\lambda, A)^\alpha\| \|J^0_\alpha(A)\|_{\mathcal{A}} \\ &\le C' C^\alpha R_0(A)^{\alpha+3} \|J^0_\alpha(A)\|_{\mathcal{A}},\end{aligned} \tag{26}$$

and we can apply Theorem 4.1(b) for $\beta = 0$. ■

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