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A new characterization of Eberlein compacta

by

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Abstract. We give a sufficient and necessary condition for a Radon–Nikodým compact space to be Eberlein compact in terms of a separable fibre connecting weak-* and norm approximation.

Introduction. A compact topological space is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space, and *Radon–Nikodým compact* if it is homeomorphic to a weak-* compact subset of the dual of an Asplund space. By the factorization result of [1], every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space, therefore an Eberlein compact space is Radon–Nikodým compact. However, these two classes are different; indeed any scattered compact space is Radon–Nikodým but no separable, non-metrizable scattered compact space can be an Eberlein compact since for the class of Eberlein compacta, separability and metrizability are equivalent.

The class of Radon–Nikodým compacta has been investigated by several authors [14, 15, 19, 22] after the systematic study made by I. Namioka in [14]. In that paper the following question was asked:

Problem. Find conditions for a Radon–Nikodým compact space to be Eberlein compact.

An answer to this problem was given in [19] and [22] by showing that a necessary and sufficient condition for a Radon–Nikodým compact space to be Eberlein compact is that it is Corson compact. Recall that a compact space is called *Corson compact* if it is homeomorphic to a subset of the Σ-product space

$$\Sigma(\Gamma) = \{x \in [-1,1]^{\Gamma} : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

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It is our aim here to give another necessary and sufficient condition on a Radon–Nikodým compact space for it to be Eberlein compact.

If a Radon–Nikodým compact lives in a separable dual, it is metrizable and so it is Eberlein compact. In the non-separable case, we know that it lives in a dual of an Asplund space where we can define a projectional resolution of the identity [6]. These projections are not, in general, weak-* continuous, but if they were, we could construct a weak-*-to-weak continuous injection into a $c_0(\Gamma)$ space, and so the compact space would be Eberlein compact [4, 18, 23].

We shall formulate here a "linking condition" that relates the separable pieces of a given Radon–Nikodým compact space with the separable pieces of the dual norm of the space where it lives, which will be necessary and sufficient for the Radon–Nikodým compactum to be Eberlein compact. This condition goes back to the transfer techniques developed in [11] for renormings, and studied in the non-metric case in [18].

To formulate our main results we need the following:

Definition 0.1. (1) Let X be a set and τ_1, τ_2 be two topologies on it. We say that X has *property* $\mathcal{L}(\tau_1, \tau_2)$ if for any $x \in X$ there exists a countable set $S(x)$ containing x so that if $A \subset X$ then $\overline{A}^{\tau_2} \subset \overline{\bigcup\{S(x) : x \in A\}}^{\tau_1}$.

(2) Let (X, τ) be a topological space. We say that X has the *Linking Separability Property* (LSP, for short) if there exists a metric d on X, with the metric topology finer than τ, such that X has $\mathcal{L}(d, \tau)$.

In [18] we studied LSP topological spaces and we shall point out some of their properties when needed.

Our main results are the following.

Theorem A. *Let (K, τ) be a compact Hausdorff space. The following are equivalent*:

(i) *K is Eberlein compact.*

(ii) *There exists a lower semicontinuous metric ϱ on K such that K has $\mathcal{L}(\varrho, \tau)$.*

Theorem B. *Let K be a Radon–Nikodým compact space. Then K is Eberlein compact if*, *and only if*, *K has LSP.*

As a corollary we obtain the following [7, 19, 22]:

Theorem C. *Let X be an Asplund generated Banach space*, *i.e.*, *there exists an Asplund space E and a map $T : E \to X$ with $\overline{T(E)}^{\|\cdot\|} = X$. Then X is WCG if*, *and only if*, *(B_{X^*}, w^*) has LSP.*

For further references on this topic we refer the reader to [5], Chapter 8.

1. Characterizing Eberlein compact spaces. In this section we give the proof of Theorem A. The first step is to prove that every K satisfying condition (ii) of Theorem A is Corson compact (Th. 1.6). To prove that, we need some lemmas. Let us begin by setting some notation.

In this paper we study compact Hausdorff spaces (K,τ) that admit a lower semicontinuous metric ϱ such that K has $\mathcal{L}(\varrho,\tau)$. We notice that, by a result of Jayne, Namioka and Rogers [8], the metric topology must then be finer than τ, which we denote by $\tau \preceq \varrho$. In the same paper they state the following result which improves a result by Ghoussoub and Maurey.

Let K be a compact Hausdorff space and let ϱ be a bounded lower semicontinuous metric on K. Then there is a dual Banach space E^* and a homeomorphism $\varphi : K \to E^*$ (where E^* is taken with its weak topology) with*

$$\|\varphi(x)-\varphi(y)\|_{E^*} = \varrho(x,y) \quad \textit{for all } x,y \in K.$$

The space E is the space of all continuous real-valued functions f on K that satisfy a uniform Lipschitz condition of order 1 with respect to ϱ. Then $\|f\|_{\mathrm{Lip}}$, defined to be the least constant $M>0$ such that

$$|f(z_1)-f(z_2)| \le M\varrho(z_1,z_2) \quad \text{for all } z_1,z_2 \in K,$$

is a norm on E. Another norm $\|\cdot\|$ on E is defined by

$$\|f\| = \max\{\|f\|_{\mathrm{Lip}}, \|f\|_\infty\}.$$

The map $\varphi : K \to E^*$ is defined as follows. Given $z \in K$, let $\varphi(z)$ be the linear map $\varphi(z) : E \to \mathbb{R}$ in E^* defined by $\varphi(z)(f) = f(z)$. (So φ sends a point in the compact space to its associated Dirac measure in $E^* \supset C(K)^*$.) We then have $\|\varphi(x)-\varphi(y)\|_{E^*} = \varrho(x,y)$.

If ϱ is not bounded, we could take a homeomorphism $\psi : \mathbb{R} \to (0,1)$ and consider $d = \psi\circ\varrho$, which would be a bounded lower semicontinuous metric on K.

LEMMA 1.1 (Main construction). *Let (K,τ) be a compact Hausdorff space and ϱ be a lower semicontinuous metric on K such that K has $\mathcal{L}(\varrho,\tau)$. Let $A_0 \subset C(K)$ and $M_0 \subset K$, with $|A_0| = |M_0|$. Then there are sets A and M with the following properties*:

(i) *$A_0 \subset A \subset C(K)$, A is a $\mathbb{Q}$-linear algebra with $\mathbf{1} \in A$ and $|A| = |A_0|$.*

(ii) *$M_0 \subset M \subset K$ and $|M_0| = |M|$.*

(iii) *$A \cap B_E$ is a norming set for* $\operatorname{span}\varphi(S(M)) \subset E^*$ *and a norming set for* $\operatorname{span}\overline{\varphi(M)}^{\sigma(E^*,E)} \subset E^*$. *(Here $S(M)$ is the set associated with M by property $\mathcal{L}$.)*

(iv) *If x and y are in $\overline{M}$, $x \neq y$, then there is $f \in A$ with $f(x) \neq f(y)$ and for every $f \in A$ there is $\xi(f) \in M$ with $|f(\xi(f))| = \sup\{|f(x)| : x \in K\}$.*

Proof. We shall construct M and A by an "exhaustion argument" of countable type thanks to $\mathcal{L}(\varrho,\tau)$ we have on K.

For $x \in K$, $S(x)$ is the countable set given by $\mathcal{L}(\varrho,\tau)$ and $S(N) = \bigcup\{S(x) : x \in N\}$. For any $f \in C(K)$, let $\xi(f) \in K$ be such that $|f(\xi(f))| = \max\{|f(x)| : x \in K\}$. For any subset $N \subset K$, define a subset of E^* by

$$\Phi(N) = \mathbb{Q}\text{-linear span}\{\varphi(S(N))\}.$$

For $y \in \Phi(N)$ consider a countable subset $\{f_y^n : n \in \mathbb{N}\}$ of B_E such that

$$\|y\|_{E^*} = \sup\{|f_y^n(y)| : n \in \mathbb{N}\}\}.$$

Finally set

$$\Psi(N) = \bigcup\{f_y^n : n \in \mathbb{N},\ y \in \Phi(N)\}.$$

Consider $M_0 \subset K$, $A_0 \subset C(K)$ and define

$$A_1 = \mathbb{Q}\text{-linear algebra generated by } \{\mathbf{1}, \Psi(M_0), A_0)\} \subset C(K),$$

and $M_1 = M_0 \cup \{\xi(f) : f \in A_1\}$. It is clear that $|A_1| = |A_0|$, $|M_1| = |M_0|$ and $A_1 \cap B_E$ is a norming set for the $\mathbb{Q}$-linear span $\varphi(S(M_0))$.

Assume we have defined sequences of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ and $M_0 \subset M_1 \subset \ldots \subset M_n$ as A_1 and M_1 above. Define $A_{n+1} = \mathbb{Q}$-linear algebra generated by $\{1, \Psi(M_n), A_n\}$ and $M_{n+1} = M_n \cup \{\xi(f) : f \in A_{n+1}\}$. Take $A = \bigcup\{A_n : n \in \mathbb{N}\}$ and $M = \bigcup\{M_n : n \in \mathbb{N}\}$. Let us show that M and A are the sets we are looking for.

(i) and (ii) are quite clear by construction and since for any point x the set $S(x)$ is at most countable.

By construction, $A \cap B_E$ is norming for $\operatorname{span}\varphi(S(M)) \subset E^*$. Thus, $A \cap B_E$ norms

$$\operatorname{span}\overline{\varphi(S(M))}^{\|\cdot\|^*} \subset \overline{\operatorname{span}\varphi(S(M))}^{\|\cdot\|^*}.$$

Now by $\mathcal{L}(\varrho,\tau)$, $\overline{\varphi(M)}^{w^*} \subset \overline{\varphi(S(M))}^{\|\cdot\|^*}$ and that implies that $A \cap B_E$ norms $\operatorname{span}\overline{\varphi(M)}^{w^*}$.

Let us check (iv). Take $x, y \in \overline{M}$, $x \neq y$, and assume that $f(x) = f(y)$ for all $f \in A$. Since φ injects K homeomorphically in E^*, we have $\varphi(x) \neq \varphi(y)$. Now since x, y belong to $K = \overline{M}^{\tau}$, there must be $(x_n) \in S(M)$ and $(y_n) \in S(M)$ converging to x and y in the ϱ-distance by $\mathcal{L}(\varrho,\tau)$.

Fix $n \in \mathbb{N}$. There must be $p \in \mathbb{N}$ such that $x_n, y_n \in S(M_p)$ (since $S(M_j)$ is an increasing sequence), therefore

$$\varphi(x_n) - \varphi(y_n) \in \mathbb{Q}\text{-linear span}\{\varphi(S(M_p))\} \subset E^*$$

whose members are normed in $\Psi(M_p) \subset A_{p+1} \subset A$. The same argument holds for any $n \in \mathbb{N}$ and so $\varphi(x_n) - \varphi(y_n)$ is normed in $A \cap B_E$ for any $n \in \mathbb{N}$. Finally we have

$$\begin{aligned}\varrho(x_n,y_n) &= \|\varphi(x_n)-\varphi(y_n)\|_{E^*} = \sup\{|f(\varphi(x_n)-\varphi(y_n))| : f\in A\cap B_E\}\\ &\le \sup\{|f(\varphi(x_n)-\varphi(x)|+|f(\varphi(x)-\varphi(y))|\\ &\qquad + |f(\varphi(y)-\varphi(y_n))| : f\in A\cap B_E\}\\ &\le \|\varphi(x_n)-\varphi(x)\|_{E^*}+\|\varphi(y)-\varphi(y_n)\|_{E^*} = \varrho(x_n,x)+\varrho(y_n,y)\end{aligned}$$

and that implies that $\lim_{n\to\infty}\varrho(x_n,y_n)=0$, hence $x=y$, which contradicts the hypothesis.

The second part of (iv) is clear by construction. ■

LEMMA 1.2. *For sets A and M as in Lemma* 1.1, *there exists a norm-one projection $P: C(K)\to C(K)$ with*:

(i) $P(C(K)) = \overline{A}^{\|\cdot\|_\infty}$.

(ii) *P is a homomorphism of algebras with $P(\mathbf{1})=\mathbf{1}$.*

(iii) *There is a continuous retraction $r: K\to \overline{M}$ such that $P(f)=f\circ r$ for all $f\in C(K)$.*

(iv) *$\varrho(r(x),r(y))\le\varrho(x,y)$ for all $x,y\in K$.*

Proof. Consider $C(\overline{M})$ with its supremum norm $|||\cdot|||$ and let R be the restriction map $R: C(K)\to C(\overline{M})$, $R(f)=f_{|\overline{M}}$.

Given $\varepsilon>0$ and $f\in\overline{A}^{\|\cdot\|_\infty}$, there exists $g\in A$ with $\|g-f\|_\infty<\varepsilon$. Let $\xi(g)\in M$ with $|g(\xi(g))|=\|g\|_\infty$. Then

$$\begin{aligned}\|f\|_\infty &\le \|f-g\|_\infty+\|g\|_\infty\le\varepsilon+|g(\xi(g))| = |||Rg|||+\varepsilon\\ &\le |||Rg-Rf|||+|||Rf|||+\varepsilon\le |||Rf|||+2\varepsilon.\end{aligned}$$

Since the reasoning is valid for every $\varepsilon>0$ we have $\|f_\infty\|\le|||Rf|||$ for all $f\in\overline{A}^{\|\cdot\|_\infty}$ and R is an isometry and algebraic homomorphism between $\overline{A}^{\|\cdot\|_\infty}$ and $(C(\overline{M}),|||\cdot|||)$. Since A separates the points of $\overline{M}$ and contains $\mathbf{1}$, $R(\overline{A}^{\|\cdot\|_\infty})$ must coincide with $C(\overline{M})$ by the Stone–Weierstrass theorem. Then

$$R^{-1}: C(\overline{M})\to\overline{A}^{\|\cdot\|_\infty}\hookrightarrow C(K)$$

is a linear extension operator and the projection P defined by $P=R^{-1}\circ R$ obviously satisfies (i) and (ii).

(iii) follows from a very special case of variants of the theorems of Banach–Stone and Gelfand–Naimark. Indeed, every measure δ_x for $x\in K$ gives us a character for the algebra $C(K)$, i.e., a multiplicative linear functional sending $\mathbf{1}$ to $\mathbf{1}$, and every character is a Dirac measure. Any algebraic homomorphism and linear isometry between algebras puts in one-to-one correspondence the characters of the algebras by the transpose isomorphism. Consequently, if we consider A with the weak* topology, then for every $x\in K$, δ_x provides a character for the algebra A which corresponds to a

Dirac measure $\delta_{r(x)} \in \overline{M}$. See [21]. This provides us with a continuous retraction $r : K \to \overline{M}$ and $P(f) = f \circ r$ since $f \circ r$ is continuous on K and $f \circ r_{|\overline{M}} = f_{|\overline{M}}$.

Let us finish proving (iv). For x and y in K we have $r(x) \in \overline{M}$ and $r(y) \in \overline{M}$, so

$$\varphi(r(x)) - \varphi(r(y)) \in \operatorname{span} \overline{\varphi(M)}^{\sigma(E^*,E)}$$

and by in Lemma 1.1(iii) we have

$$\begin{aligned}
\varrho(r(x), r(y)) = \|\varphi(r(x)) - \varphi(r(y))\|_{E^*} &= \sup_{f \in A \cap B_E} \{|\langle \varphi(r(x)) - \varphi(r(y)), f\rangle|\} \\
&= \sup_{f \in A \cap B_E} \{|f(r(x)) - f(r(y))|\} = \sup_{f \in A \cap B_E} \{|\langle f \circ r, \delta_x - \delta_y\rangle|\} \\
&= \sup_{f \in A \cap B_E} \{|\langle P(f), \delta_x - \delta_y\rangle|\} \le \sup_{f \in B_E} \{|\langle f, \delta_x - \delta_y\rangle|\} \\
&= \sup_{f \in B_E} \{|\langle f, \varphi(x) - \varphi(y)\rangle|\} = \|\varphi(x) - \varphi(y)\|_{E^*} = \varrho(x, y). \ \blacksquare
\end{aligned}$$

Proposition 1.3. *Let (K, τ) be a compact Hausdorff space and ϱ be a lower semicontinuous metric on it such that K has $\mathcal{L}(\varrho, \tau)$. Then*

$$\operatorname{dens}(K, \tau) = \operatorname{dens}(K, \varrho) = \operatorname{dens}(C(K), \|\cdot\|_\infty).$$

Proof. The definition of $\mathcal{L}(\varrho, \tau)$ clearly implies $\operatorname{dens}(K, \tau) = \operatorname{dens}(K, \varrho)$. Since always $\operatorname{dens}(K, \tau) \le \operatorname{dens}(C(K), \|\cdot\|_\infty)$, we only have to show that $\operatorname{dens}(C(K), \|\cdot\|_\infty) \le \operatorname{dens}(K, \tau)$.

Let $M_0 = \{x_\alpha : 0 \le \alpha < \mu\}$ be a dense subset of K, where μ is the first ordinal number whose cardinality $|\mu|$ is $\operatorname{dens}(K, \tau)$. And let A_0 be any subset of $C(K)$ of the same cardinality as M_0.

Applying Lemmas 1.1 and 1.2 to A_0 and M_0, we obtain $A \supset A_0$ and $M \supset M_0$ with the properties stated in both results. But $\overline{M} = K$ and therefore the restriction R is the identity. So $\overline{A} = C(K)$ and the density character of $\overline{A}$, and hence of $C(K)$, is at most the cardinality of M_0. ∎

The previous lemmas can be applied to obtain the following:

Theorem 1.4. *Let (K, τ) be a compact Hausdorff space and ϱ be a lower semicontinuous metric on it with $\mathcal{L}(\varrho, \tau)$. Then there exists a PRI $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ on $C(K)$, and a family of continuous retractions $r_\alpha : K \to K$ with $P_\alpha(f) = f \circ r_\alpha$, $\operatorname{dens}(r_\alpha(K)) \le |\alpha|$ and $\varrho(r_\alpha(x), r_\alpha(y)) \le \varrho(x, y)$ for all $x, y \in K$ and for all $\alpha \in [\alpha, \mu]$. Moreover, $r_\alpha \to r_\beta$ as $\alpha \to \beta$ pointwise on K in the ϱ-topology. The latter implies that given $x \in K$ and $\varepsilon > 0$, the set*

$$\{\alpha : \omega_0 \le \alpha \le \mu,\ \varrho(r_{\alpha+1}(x), r_\alpha(x)) > \varepsilon\}$$

is finite. Thus, the set $\{\alpha : \omega_0 \le \alpha \le \mu,\ r_{\alpha+1}(x) \ne r_\alpha(x)\}$ is at most countable.

Proof. Let $|\mu|$ be the first ordinal such that $|\mu| = \mathrm{dens}(C(K))$ and let $\{x_\alpha : 0 \le \alpha < \mu\}$ and $\{f_\alpha : 0 \le \alpha < \mu\}$ be dense subsets of K and $C(K)$ respectively.

Let us begin by applying Lemmas 1.1 and 1.2 to the sets $A_0 = \{f_\alpha : 0 \le \alpha \le \omega_0\}$ and $M_0 = \{x_\alpha : 0 \le \alpha \le \omega_0\}$. We obtain $A_{\omega_0} = A$, M_{ω_0} and P_{ω_0} with the properties stated in both lemmas.

Now let $\beta \le \mu$ be any ordinal and assume that for any $\alpha < \beta$, we have constructed families $A_{\omega_0} \subset \ldots \subset A_\alpha$ of $\mathbb{Q}$-algebras and $M_{\omega_0} \subset \ldots \subset M_\alpha \subset K$ with $S(M_\alpha) \subset M_{\alpha+1}$, as well as the corresponding linear projections $\{P_\alpha : \omega_0 \le \alpha < \beta\}$ satisfying the conditions in both lemmas and $|\alpha| = |M_\alpha| = |A_\alpha|$.

If β is not a limit ordinal, i.e., $\beta = \alpha + 1$, set

$$A_0 = A_\alpha \cup \{f_{\alpha+1}\} \quad \text{and} \quad M_0 = S(M_\alpha \cup \{x_{\alpha+1}\}).$$

Apply the lemmas to these sets to obtain $A_{\alpha+1}$ and $M_{\alpha+1}$ satisfying all the conditions required.

If β is a limit ordinal define

$$A_\beta = \bigcup\{A_{\alpha+1} : \omega_0 \le \alpha < \beta\}, \qquad M_\beta = \bigcup\{M_{\alpha+1} : \omega_0 \le \alpha < \beta\}.$$

We shall now see that A_β and M_β satisfy the conditions of Lemma 1.1.

First let us show that $A_\beta \cap B_E$ norms $\operatorname{span}\varphi(S_{M_\beta}) \subset E^*$.

Take $x \in \operatorname{span}\varphi(S(M_\beta))$. Then x is a finite linear combination of points in $\bigcup\{\varphi(S(M_{\alpha+1})) : \omega_0 \le \alpha < \mu\}$. Hence, by construction, there must be α such that $x \in \operatorname{span}\varphi(S(M_\alpha))$, which is normed, by induction hypothesis, by $A_\alpha \cap B_E$, which is contained in $A_\beta \cap B_E$.

Consequently, as in Lemma 1.1, $A_\beta \cap B_E$ norms $\operatorname{span}\overline{\varphi(S(M_\beta))}^{w^*} \subset E^*$. It also norms $\operatorname{span}\overline{\varphi(S(M_\beta))}^{\|\cdot\|^*}$ since

$$\operatorname{span}\overline{\varphi(S(M_\beta))}^{\|\cdot\|^*} \subset \overline{\operatorname{span}\varphi(S(M_\beta))}^{\|\cdot\|^*}.$$

Now by $\mathcal{L}(\varrho,\tau)$, $\overline{\varphi(M_\beta)}^{w^*} \subset \overline{\varphi(S(M_\beta))}^{\|\cdot\|^*}$ and that implies that $A_\beta \cap B_E$ norms $\operatorname{span}\overline{\varphi(M_\beta)}^{w^*}$.

To prove (iv) we essentially follow the proof of Lemma 1.1. Take $x, y \in \overline{M_\beta}$, $x \ne y$, and assume that $f(x) = f(y)$ for all $f \in A$. Then $\varphi(x) \ne \varphi(y)$. Since x, y belong to $K_\beta = \overline{M_\beta}^\tau$, there must be $(x_n) \in S(M_\beta)$ and $(y_n) \in S(M_\beta)$ converging to x and y in the ϱ-distance by $\mathcal{L}(\varrho,\tau)$.

Fix $n \in \mathbb{N}$. There must be $\alpha(n) < \beta$ such that $x_n, y_n \in S(M_{\alpha(n)})$ (since $S(M_\alpha)$ is an increasing sequence), therefore

$$\varphi(x_n) - \varphi(y_n) \in \mathbb{Q}\text{-linear } \operatorname{span}\varphi(S(M_{\alpha(n)})) \subset E^*$$

whose members are normed in $A_{\alpha(n)} \subset A_\beta$. The same argument holds for any $n \in \mathbb{N}$ and so $\varphi(x_n) - \varphi(y_n)$ are normed in $A_\beta \cap B_E$ for any $n \in \mathbb{N}$. And, as in Lemma 1.1, we would get $x = y$. The second part of (iv) in Lemma 1.1 is clear.

Consequently, by Lemma 1.2, we have a projection P_β with range $\overline{A}^{\|\cdot\|_\infty}$, and a continuous retraction r_β of K onto $\overline{M_\beta}$ and $\mathrm{dens}(r_\beta(K)) \le |\beta|$.

To finish let us show that for each $x \in K$, $r_\alpha(x) \to r_\beta(x)$ in the ϱ-topology.

Since $S(M_\alpha) \subset M_{\alpha+1}$ for any α, for any limit ordinal β we have

$$\overline{M_\beta}^\tau \subset \overline{\bigcup_{\alpha<\beta} M_{\alpha+1}}^\varrho \subset \overline{M_\beta}^\beta,$$

therefore $\overline{M_\beta}^\tau = \overline{M_\beta}^\varrho$.

Trivially, $r_\alpha(x) \to r_\beta(x)$ for any $x \in M_\beta$. Since $\{r_\alpha\}$ are ϱ-uniformly equicontinuous, and $\overline{M_\beta}^\tau = \overline{M_\beta}^\varrho$, we have $r_\alpha(x) \to r_\beta(x)$ for all $x \in \overline{M_\beta}^\tau$. ∎

The following result is in [18].

Remark 1.5. *Let (X,τ) be an LSP topological space. Then any subspace of X is also LSP. In fact if d is a metric on X such that X has $\mathcal{L}(d,\tau)$ and $H \subset X$ then H has $\mathcal{L}(d,\tau)$.*

Theorem 1.6. *Let (K,τ) be a compact Hausdorff space and ϱ be a lower semicontinuous metric on it with $\mathcal{L}(\varrho,\tau)$. Then K is Corson compact.*

Proof. We argue by induction on the density character of the compactum.

If (K,τ) is separable, then by Proposition 1.7 below, it is metrizable, hence Corson compact.

Now let μ be the first ordinal with cardinality $\mathrm{dens}(K,\tau)$, and assume that any compact space of density character less than $|\mu|$ and having LSP for a lower semicontinuous metric is Corson.

Let $\{r_\alpha : \omega_0 \le \alpha < \mu\}$ be the family of retractions on K given by Theorem 1.4.

Let $K_\alpha = r_\alpha(K) \subset K$. By construction $\mathrm{dens}(K_\alpha) \le |\alpha|$. Since property $\mathcal{L}$ is hereditary (Remark 1.5), by the induction hypothesis each K_α is Corson compact. Hence, for any α with $\omega_0 \le \alpha < \mu$ there exists a set Γ_α and a homeomorphism $\Psi_\alpha : K_\alpha \to \Sigma(\Gamma_\alpha) \subset \mathbb{R}^{\Gamma_\alpha}$.

Let Γ be the disjoint union of the sets $\{\Gamma_\alpha\}_{\omega_0<\alpha<\mu}$ and $\mathbb{N}$, and define $T : K \to \mathbb{R}^\Gamma$ by

$$T(x)(n) = \Psi_{\omega_0}(r_{\omega_0}(x))(n), \qquad n \in \mathbb{N},$$
$$T(x)(\gamma) = \Psi_{\alpha+1}(r_{\alpha+1}(x))(\gamma) - \Psi_{\alpha+1}(r_\alpha(x))(\gamma) \qquad \text{for } \gamma \in \Gamma_{\alpha+1}.$$

Given $x \in K$, since the set $\{\alpha : r_{\alpha+1}(x) \ne r_\alpha(x)\}$ is at most countable and $\Psi_\alpha(r_\alpha(x))$ lives in $\Sigma(\Gamma_\alpha)$ for any α, it clearly follows that $T(x)$ lives in $\Sigma(\Gamma)$.

T is clearly continuous. To see that it is an injection, take $x, y \in K$ and suppose $T(x) = T(y)$. Let us show that $r_\alpha(x) = r_\alpha(y)$ for all α, which implies $x = y$.

In particular, $\Psi_{\omega_0}(r_{\omega_0}(x)) = \Psi_{\omega_0}(r_{\omega_0}(y))$, and since Ψ_{ω_0} is one-to-one, $r_{\omega_0}(x) = r_{\omega_0}(y)$. So assume $r_\alpha(x) = r_\alpha(y)$ for all $\alpha < \beta$. Since $r_\alpha(x) \to r_\beta(x)$ we obtain $r_\beta(x) = r_\beta(y)$. Moreover, $x = r_\mu(x) = r_\mu(y) = y$. For non-limit ordinals the reasoning is also trivial.

Hence T injects (K, τ) homeomorphically into a sigma product. Thus, K is Corson compact. ∎

The conditions of the following two propositions are clearly satisfied if K has $\mathcal{L}(\varrho, \tau)$.

PROPOSITION 1.7. *Let (K, τ) be a compact Hausdorff space and ϱ a lower semicontinuous metric on K. If every separable subset of K is also ϱ-separable, then separable subsets of K are metrizable.*

Proof. Since the ϱ-topology is finer than τ, the result follows from the fact that any compact image of a separable metrizable space is metrizable ([3], Theorem 3.1.20). ∎

It is known after Namioka [14] that a compact space is Radon–Nikodým compact if and only if it is fragmented by a lower semicontinuous metric. Recall that a topological space is said to be *fragmented* by a metric if for any $\varepsilon > 0$, and any non-empty subset A of the space, there exists a relatively open subset of A with diameter less than ε.

PROPOSITION 1.8. *Let (K, τ) be a compact Hausdorff space and ϱ a lower semicontinuous metric on K. If every separable subset of K is also ϱ-separable, then ϱ is a fragmenting metric. Hence, K is RN compact.*

Proof. The result follows immediately from Theorem 4.1(c)⇒(j) of [8], where one should consider the irreducible map p. ∎

We can now *prove Theorem A* of the introduction:

(i)⇒(ii). K is Radon–Nikodým (Proposition 1.8) and Corson (Theorem 1.6), so by the already mentioned result of [19, 22], we conclude that K is Eberlein.

(ii)⇒(i). We can view (K, τ) as a weakly compact subset of a WCG Banach space E. In [16] we showed that any WCG Banach space has $\mathcal{L}(\|\cdot\|, \text{weak})$, hence by Remark 1.5 so does K for τ and $\|\cdot\|$. ∎

2. Consequences in Banach spaces. In order to show Theorem B, we need the following definition by Jayne, Namioka and Rogers [9].

DEFINITION 2.1. Let (X, τ) be a topological space and d be a metric on X. We shall say that X is *σ-fragmented* by d if for every $\varepsilon > 0$, it is possible to represent X as $\bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ so that for each $n \in \mathbb{N}$ and any subset $A \subset X_n^{\varepsilon}$ there exists a relatively τ-open subset of A with d-diameter less than ε.

We also need the following result from [18].

Remark 2.2. *Let (X, τ) have LSP and ϱ be any metric on X finer than τ. If (X, τ) is σ-fragmented by ϱ, then X has $\mathcal{L}(\varrho, \tau)$.*

Now let us give the *proof of Theorem B.*

If K is Eberlein the reasoning in the proof of Theorem A applies. So let K have LSP, i.e., there exists a metric on K, say d, with the metric topology finer than τ and such that K has $\mathcal{L}(d, \tau)$.

Since K is Radon–Nikodým, there must be a lower semicontinuous metric ϱ fragmenting (K, τ). Remark 2.2 shows that K has $\mathcal{L}(\varrho, \tau)$; now Theorem A yields that K is Eberlein. ∎

We can also extend Theorem 8.3.4 of [5] giving the Banach space version of the former result, i.e., Theorem C of the introduction. The *proof of Theorem C* is as follows.

T^* is one-to-one and gives a homeomorphism between (B_{X^*}, w^*) and $(T^*(B_{X^*}), w^*)$.

If X is WCG we know that (B_{X^*}, w^*) is Eberlein compact and it has LSP.

Conversely, if (B_{X^*}, w^*) has LSP, then since it is Radon–Nikodým compact we deduce, by Theorem B, that it is Eberlein. Now Theorem 8.3.4 of [5] applies to show X is WCG. ∎

3. Final remarks. In [18] we studied the relationship between property $\mathcal{L}$, σ-fragmentability and property SLD of Jayne, Namioka and Rogers. The last property is defined as follows:

Definition 3.1. We say that X has a *countable cover by sets of small local diameter* (SLD) if for every $\varepsilon > 0$ it is possible to represent X as $\bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ so that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relative τ-neighbourhood of d-diameter less than ε.

It was shown that whenever (X, τ) is a metric space and ϱ is a metric on X finer than τ, the conditions: X has $\mathcal{L}(\varrho, \tau)$, (X, τ) is ϱ-σ-fragmented and (X, τ) has ϱ-SLD, are all equivalent.

Our aim now is to show that this is no longer true when τ is a nonmetrizable topology, i.e., we shall give examples of a space with property LSP but not SLD, and another with SLD but not LSP. First, one more property from [18] is needed:

Remark 3.2. *Let (X, τ) be σ-fragmented by a metric d finer than τ (resp. d-SLD). If ϱ is another metric such that X has $\mathcal{L}(\varrho, \tau)$, then (X, τ) is σ-fragmented by ϱ (resp. ϱ-SLD).*

Example 3.3. *Let (K, τ) be a separable non-metrizable RN compactum. Then K does not have property LSP.*

Proof. If there were a metric ϱ finer than the topology of K, with $\mathcal{L}(\varrho, \tau)$, then since K is RN, i.e., fragmented by a lower semicontinuous metric, by Remark 2.2, K would also have property $\mathcal{L}$ for this metric, and therefore by Proposition 1.7, K would be metrizable. ∎

The next example is due to A. Moltó, and can be found in [2].

Example 3.4. *There exists a compact Hausdorff space (K, τ) and a metric ϱ such that (K, τ) has the ϱ-SLD property and it fails to have $\mathcal{L}(\varrho, \tau)$. Moreover, (K, τ) does not have LSP.*

Proof. We denote by $\Delta = \{0, 1\}^{\mathbb{N}}$ the Cantor set, and by $\mathcal{D}$ the set of finite sequences of 0's and 1's. For $\sigma \in \mathcal{D}$, we denote by I_σ the clopen (i.e. closed and open) subset of Δ consisting of those sequences which start with σ. We consider the following set K_0 of functions on Δ: the set K_0 consists of the characteristic functions of the sets I_σ, $\sigma \in \mathcal{D}$ (denoted by χ_{I_σ}), and of the points of Δ, and the function identically equal to zero.

When equipped with the topology of pointwise convergence on Δ, K_0 becomes a compact set, which is separable, scattered, non-metrizable and $K_0^{(3)} = \emptyset$.

By a result of Deville $C(K)^*$ admits an equivalent dual LUR norm, which is equivalent ([20]) to $(C(K)^*, w^*)$ having the $\|\cdot\|^*$-SLD property.

So (K, τ) has ϱ-SLD for a τ-lower semicontinuous metric (ϱ is the restriction to K of the dual norm). Now, if K had $\mathcal{L}(\varrho, \tau)$, then by Proposition 1.7, (K, τ) would be metrizable (since it is separable), which is not true.

To prove the moreover part, we only have to apply Remark 2.2. ∎

So Example 3.4 shows that for a compact space (K, τ) that has the ϱ-SLD property we may not have LSP (not only $\mathcal{L}(\varrho, \tau)$).

Remark 3.5. In [16] we proved that under CH, ℓ^∞ has $\mathcal{L}(\|\cdot\|, \text{weak})$ and it does not have SLD [10].

The same arguments as in the example above work for the next result.

Proposition 3.6. *Let K be a scattered compact space with $K^{(\omega_1)} = \emptyset$, having separable subsets which are non-metrizable. Then K has the ϱ-SLD property for a lower semicontinuous metric and K does not have LSP.*

Example 3.7. *(B_{ℓ^∞}, w^*) is a metrizable compact space, and B_{ℓ^∞} does not have $\mathcal{L}(\|\cdot\|_\infty, w^*)$.*

Proof. This is clear since (ℓ^∞, w^*) is separable whereas $(\ell^\infty, \|\cdot\|_\infty)$ is not. (Of course (ℓ^∞, w^*) lacks the $\|\cdot\|_\infty$-SLD property [9].)

References

[1] W. J. Davies, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[2] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs Surveys Pure Appl. Math. 64, Longman, 1993.

[3] R. Engelking, *General Topology*, Monograf. Mat. 60, Polish Sci. Publ., Warszawa, 1977.

[4] M. Fabian, *Each weakly countably determined Asplund space admits a Fréchet differentiable norm*, Bull. Austral. Math. Soc. 35 (1987), 363–372.

[5] —, *Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces*, Canad. Math. Soc. Ser. Monographs Adv. Texts, Wiley, 1997.

[6] M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. 91 (1988), 141–151.

[7] M. Fabian and J. H. Whitfield, *On equivalent characterization of weakly compactly generated Banach spaces*, Rocky Mountain J. Math. 24 (1994), 1363–1378.

[8] J. E. Jayne, I. Namioka and C. A. Rogers, *Norm fragmented weak* compact sets*, Collect. Mat. 41 (1990), 133–163.

[9] —, —, —, *σ-fragmentable Banach spaces*, Mathematika 39 (1992), 161–188 and 197–215.

[10] —, —, —, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) 66 (1993), 651–672.

[11] A. Moltó, J. Orihuela and S. Troyanski, *Locally uniform rotund renorming and fragmentability*, Proc. London Math. Soc. (3) 75 (1997), 619–640.

[12] A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *On weakly locally uniformly rotund Banach spaces*, J. Funct. Anal. 163 (1999), 252–271.

[13] —, —, —, —, *Non-linear transfer technique*, preprint.

[14] I. Namioka, *Radon–Nikodym compact spaces and fragmentability*, Mathematika 34 (1989), 258–281.

[15] —, *Eberlein and Radon–Nikodym compact spaces*, lecture notes at University College London, Autumn 1985.

[16] L. Oncina, *Descriptive Banach spaces and Eberlein compacta*, doctoral thesis, Universidad de Murcia, 1999.

[17] —, *The JNR property and the Borel structure of a Banach space*, Serdica Math. J. 26 (2000), 13–32.

[18] —, *On WCG Asplund Banach spaces*, preprint.

[19] J. Orihuela, W. Schachermayer and M. Valdivia, *Every Radon–Nikodym Corson compact space is Eberlein compact*, Studia Math. 98 (1991), 157–174.

[20] M. Raja, *Borel measurability and renorming in Banach spaces*, doctoral thesis, Universidad de Murcia, 1999.

[21] C. Stegall, *A proof of the theorem of Amir and Lindenstrauss*, Israel J. Math. 68 (1989), 185–192.

[22] —, *More facts about conjugate Banach spaces with the Radon–Nikodým property II*, Acta Univ. Carolin. Math. Phys. 32 (1991), 47–54.

[23] M. Valdivia, *Resolution of the identity in certain Banach spaces*, Collect. Math. 39 (1988), 127–140.

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