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Spectra originating from semi-B-Fredholm theory and commuting perturbations

by

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Abstract. Burgos, Kaidi, Mbekhta and Oudghiri [J. Operator Theory 56 (2006)] provided an affirmative answer to a question of Kaashoek and Lay and proved that an operator F is of power finite rank if and only if $\sigma_{\rm dsc}(T+F)=\sigma_{\rm dsc}(T)$ for every operator T commuting with F. Later, several authors extended this result to the essential descent spectrum, left Drazin spectrum and left essential Drazin spectrum. In this paper, using the theory of operators with eventual topological uniform descent and the technique used by Burgos et al., we generalize these results to various spectra originating from semi-B-Fredholm theory. As immediate consequences, we give affirmative answers to several questions posed by Berkani, Amouch and Zariouh. Moreover, we provide a general framework which allows us to derive in a unified way perturbation results for Weyl–Browder type theorems and properties (generalized or not). Our results improve many recent results by removing certain extra assumptions.

1. Introduction. In 1972, Kaashoek and Lay [32] showed that the descent spectrum is invariant under any commuting *power finite rank* perturbation F (that is, F^n is of finite rank for some $n \in \mathbb{N}$). Also they conjectured that this perturbation property characterizes such operators F. In 2006, Burgos, Kaidi, Mbekhta and Oudghiri [22] confirmed this conjecture: they proved that an operator F is of power finite rank if and only if $\sigma_{\rm dsc}(T+F)=\sigma_{\rm dsc}(T)$ for every operator T commuting with F. Later, Bel Hadj Fredj [8] generalized this result to the essential descent spectrum. Bel Hadj Fredj, Burgos and Oudghiri [9] extended this result to the left Drazin spectrum and left essential Drazin spectrum (in [9], they are called the ascent spectrum and essential ascent spectrum, respectively).

The present paper is concerned with commuting power finite rank perturbations of semi-B-Fredholm operators. As seen in Theorem 2.11 (our main result), we generalize the previous results to various spectra originat-

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ing from semi-B-Fredholm theory. The proof of our main result is mainly dependent upon the theory of operators with eventual topological uniform descent and the technique used in [22].

Spectra originating from semi-B-Fredholm theory include, in particular, the upper semi-B-Weyl spectrum σ_{USBW} (resp. the B-Weyl spectrum σ_{BW}) which is closely related to generalized a-Weyl's theorem, generalized a-Browder's theorem, property (gw) and property (gb) (resp. generalized Weyl's theorem, generalized Browder's theorem, property (gaw) and property (gab)). Concerning the upper semi-B-Weyl spectrum σ_{USBW}, Berkani and Amouch [13] posed the following question:

Question 1.1. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Do we always have*

$$\sigma_{\mathrm{USBW}}(T+N) = \sigma_{\mathrm{USBW}}(T) \ ?$$

Similarly, for the B-Weyl spectrum σ_{BW}, Berkani and Zariouh [19] posed the following question:

Question 1.2. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Do we always have*

$$\sigma_{\mathrm{BW}}(T+N) = \sigma_{\mathrm{BW}}(T) \ ?$$

Recently, Amouch, Zguitti, Berkani and Zariouh have given partial answers to Question 1.1 in [5, 7, 13, 16]. As immediate consequences of our main result (see Theorem 2.11), we provide positive answers to Questions 1.1 and 1.2 and some other questions posed by Berkani and Zariouh (see Corollaries 3.1, 3.3 and 3.8). Moreover, we provide a general framework which allows us to derive in a unified way commuting perturbations results for Weyl–Browder type theorems and properties (generalized or not). These results, in particular, improve many recent results of [13, 16, 19, 20, 39] by removing certain extra assumptions (see Corollary 3.9 and Remark 3.10).

Throughout this paper, $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X, and $\mathcal{F}(X)$ denotes the ideal of finite rank operators. For $T \in \mathcal{B}(X)$, let T^* denote its dual, $\mathcal{N}(T)$ its kernel, $\alpha(T)$ its nullity, $\mathcal{R}(T)$ its range, $\beta(T)$ its defect, $\sigma(T)$ its spectrum and $\sigma_{\mathrm{a}}(T)$ its approximate point spectrum. If $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp. *lower semi-Fredholm*). If $T \in \mathcal{B}(X)$ is both upper and lower semi-Fredholm, then it is *Fredholm*. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then it is *semi-Fredholm*, and its index is defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$.

For each $n \in \mathbb{N}$, we set $c_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ and $c'_n(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n)$. It follows from [31, Lemmas 3.1 and 3.2] that, for every $n \in \mathbb{N}$,

$$c_n(T) = \dim X/(\mathcal{R}(T) + \mathcal{N}(T^n)), \qquad c'_n(T) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^n).$$

Hence, it is easy to see that the sequences $\{c_n(T)\}_{n=0}^{\infty}$ and $\{c'_n(T)\}_{n=0}^{\infty}$ are decreasing. Recall that the *descent* and *ascent* of $T \in \mathcal{B}(X)$ are $\mathrm{dsc}(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and $\mathrm{asc}(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be ∞). That is,

$$\begin{aligned} \mathrm{dsc}(T) &= \inf\{n \in \mathbb{N} : c_n(T) = 0\}, \\ \mathrm{asc}(T) &= \inf\{n \in \mathbb{N} : c'_n(T) = 0\}. \end{aligned}$$

Similarly, the *essential descent* and *essential ascent* of $T \in \mathcal{B}(X)$ are

$$\begin{aligned} \mathrm{dsc_e}(T) &= \inf\{n \in \mathbb{N} : c_n(T) < \infty\}, \\ \mathrm{asc_e}(T) &= \inf\{n \in \mathbb{N} : c'_n(T) < \infty\}. \end{aligned}$$

If $\mathrm{asc}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{asc}(T)+1})$ is closed, then T is said to be *left Drazin invertible*. If $\mathrm{dsc}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{dsc}(T)})$ is closed, then T is *right Drazin invertible*. If $\mathrm{asc}(T) = \mathrm{dsc}(T) < \infty$, then T is *Drazin invertible*. Clearly, $T \in \mathcal{B}(X)$ is both left and right Drazin invertible if and only if it is Drazin invertible. If $\mathrm{asc_e}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{asc_e}(T)+1})$ is closed, then T is said to be *left essentially Drazin invertible*. If $\mathrm{dsc_e}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{dsc_e}(T)})$ is closed, then T is *right essentially Drazin invertible*.

For $T \in \mathcal{B}(X)$, let us define the *left Drazin spectrum*, *right Drazin spectrum*, *Drazin spectrum*, *left essential Drazin spectrum*, and *right essential Drazin spectrum* of T respectively as follows:

$$\begin{aligned} \sigma_{\mathrm{LD}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}, \\ \sigma_{\mathrm{RD}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Drazin invertible}\}, \\ \sigma_{\mathrm{D}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}, \\ \sigma^{\mathrm{e}}_{\mathrm{LD}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left essentially Drazin invertible}\}, \\ \sigma^{\mathrm{e}}_{\mathrm{RD}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right essentially Drazin invertible}\}. \end{aligned}$$

These spectra have been extensively studied (see e.g. [2, 7, 10, 11, 25, 9, 34]).

Recall that $T \in \mathcal{B}(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is Fredholm and $\mathrm{asc}(T) = \mathrm{dsc}(T) < \infty$ (resp. T is upper semi-Fredholm and $\mathrm{asc}(T) < \infty$, T is lower semi-Fredholm and $\mathrm{dsc}(T) < \infty$).

For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm, Browder, upper semi-Browder, lower semi-Browder), then T is called *B-Fredholm* (resp. *upper semi-B-Fredholm*, *lower semi-B-Fredholm*, *B-Browder*, *upper semi-B-Browder*, *lower semi-B-Browder*). If $T \in \mathcal{B}(X)$ is upper or lower semi-B-Browder, then it is called *semi-B-Browder*. If $T \in \mathcal{B}(X)$ is upper or lower semi-B-Fredholm, then it is *semi-B-Fredholm*. It follows from [15, Proposition 2.1] that if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is semi-Fredholm, then $\mathcal{R}(T^m)$ is closed, T_m is semi-Fredholm and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for all $m \geq n$. This enables us to define the index of a semi-B-Fredholm operator T as the index of the semi-Fredholm operator T_n, where n is an integer such that $\mathcal{R}(T^n)$ is closed and T_n is semi-Fredholm. An operator $T \in \mathcal{B}(X)$ is called *B-Weyl* (resp. *upper semi-B-Weyl*, *lower semi-B-Weyl*) if T is B-Fredholm and $\operatorname{ind}(T) = 0$ (resp. T is upper semi-B-Fredholm and $\operatorname{ind}(T) \leq 0$, T is lower semi-B-Fredholm and $\operatorname{ind}(T) \geq 0$). If $T \in \mathcal{B}(X)$ is upper or lower semi-B-Weyl, then it is *semi-B-Weyl*.

For $T \in \mathcal{B}(X)$, let us define the *upper semi-B-Fredholm spectrum*, *lower semi-B-Fredholm spectrum*, *semi-B-Fredholm spectrum*, *B-Fredholm spectrum*, *upper semi-B-Weyl spectrum*, *lower semi-B-Weyl spectrum*, *semi-B-Weyl spectrum*, *B-Weyl spectrum*, *upper semi-B-Browder spectrum*, *lower semi-B-Browder spectrum*, *semi-B-Browder spectrum*, and *B-Browder spectrum* of T respectively as follows:

$$\begin{aligned}
\sigma_{\mathrm{USBF}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Fredholm}\},\\
\sigma_{\mathrm{LSBF}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Fredholm}\},\\
\sigma_{\mathrm{SBF}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-B-Fredholm}\},\\
\sigma_{\mathrm{BF}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm}\},\\
\sigma_{\mathrm{USBW}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Weyl}\},\\
\sigma_{LSBW}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Weyl}\},\\
\sigma_{\mathrm{SBW}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-B-Weyl}\},\\
\sigma_{\mathrm{BW}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl}\},\\
\sigma_{\mathrm{USBB}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Browder}\},\\
\sigma_{\mathrm{LSBB}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Browder}\},\\
\sigma_{\mathrm{SBB}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-B-Browder}\},\\
\sigma_{\mathrm{BB}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Browder}\}.
\end{aligned}$$

These spectra originating from semi-B-Fredholm theory have also been extensively studied (see e.g. [2, 7, 10, 12, 15, 19, 25]).

For any $T \in \mathcal{B}(X)$, Berkani have found in [10, Theorem 3.6] the following elegant equalities:

$$\begin{gathered}
\sigma_{\mathrm{LD}}(T) = \sigma_{\mathrm{USBB}}(T), \quad \sigma_{\mathrm{RD}}(T) = \sigma_{\mathrm{LSBB}}(T),\\
\sigma_{\mathrm{LD}}^{\mathrm{e}}(T) = \sigma_{\mathrm{USBF}}(T), \quad \sigma_{\mathrm{RD}}^{\mathrm{e}}(T) = \sigma_{\mathrm{LSBF}}(T),\\
\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BB}}(T).
\end{gathered}$$

This paper is organized as follows. In Section 2, by using the theory of operators with eventual topological uniform descent and the technique used in [22], we characterize power finite rank operators via various spectra originating from semi-B-Fredholm theory. In Section 3, as applications, we provide affirmative answers to some questions of Berkani, Amouch and Zariouh. Moreover, we derive in a unified way several perturbation results for Weyl–Browder type theorems and properties (generalized or not).

2. Main result. We begin with some lemmas.

LEMMA 2.1. *Let $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$. If $T \in \mathcal{B}(X)$ is upper semi-B-Fredholm and commutes with F, then $T + F$ is also upper semi-B-Fredholm.*

Proof. Since T is upper semi-B-Fredholm, by [10, Theorem 3.6], T is left essentially Drazin invertible. Hence by [9, Proposition 3.1], $T + F$ is left essentially Drazin invertible. By [10, Theorem 3.6] again, T is upper semi-B-Fredholm. ∎

LEMMA 2.2. *Let $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$. If $T \in \mathcal{B}(X)$ is lower semi-B-Fredholm and commutes with F, then $T + F$ is also lower semi-B-Fredholm.*

Proof. Since $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$, $\mathcal{R}(F^n)$ is a closed and finite-dimensional subspace, and hence $\dim \mathcal{R}(F^{*n}) = \dim \mathcal{N}(F^n)^\perp = \dim \mathcal{R}(F^n)$, thus $\mathcal{R}(F^{*n})$ is finite-dimensional, which implies that $F^{*n} \in \mathcal{F}(X^*)$. It is obvious that T^* commutes with F^*. Since T is lower semi-B-Fredholm, by [10, Theorem 3.6], T is right essentially Drazin invertible. Then from the observations preceding Section IV of [34], T^* is left essentially Drazin invertible. Hence by [9, Proposition 3.1], $(T+F)^* = T^* + F^*$ is left essentially Drazin invertible. From the observations in [34] again, $T + F$ is right essentially Drazin invertible. Consequently, by [10, Theorem 3.6], $T + F$ is lower semi-B-Fredholm. ∎

It follows from [10, Corollary 3.7 and Theorem 3.6] that T is B-Fredholm if and only if T is both upper and lower semi-B-Fredholm.

COROLLARY 2.3. *Let $T \in \mathcal{B}(X)$, and let $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$. If T commutes with F, then*

(1) $\sigma_{\mathrm{USBF}}(T + F) = \sigma_{\mathrm{USBF}}(T)$;
(2) $\sigma_{\mathrm{LSBF}}(T + F) = \sigma_{\mathrm{LSBF}}(T)$;
(3) $\sigma_{\mathrm{SBF}}(T + F) = \sigma_{\mathrm{SBF}}(T)$;
(4) $\sigma_{\mathrm{BF}}(T + F) = \sigma_{\mathrm{BF}}(T)$.

Proof. The first equation follows easily from Lemma 2.1, and the second from Lemma 2.2. The third equation is true because $\sigma_{\mathrm{SBF}}(T) = \sigma_{\mathrm{USBF}}(T) \cap$

$\sigma_{\mathrm{LSBF}}(T)$, and the fourth because $\sigma_{\mathrm{BF}}(T) = \sigma_{\mathrm{USBF}}(T) \cup \sigma_{\mathrm{LSBF}}(T)$, for every $T \in \mathcal{B}(X)$. ∎

We now recall some classical definitions. Using the isomorphism $X/\mathcal{N}(T^d) \approx \mathcal{R}(T^d)$ and following [28], a topology on $\mathcal{R}(T^d)$ is defined as follows.

Definition 2.4. Let $T \in \mathcal{B}(X)$. For every $d \in \mathbb{N}$, the *operator range topology* on $\mathcal{R}(T^d)$ is defined by the norm $\|\cdot\|_{\mathcal{R}(T^d)}$ such that for all $y \in \mathcal{R}(T^d)$,

$$\|y\|_{\mathcal{R}(T^d)} = \inf\{\|x\| : x \in X,\, y = T^d x\}.$$

For a detailed discussion of operator ranges and their topologies, we refer the reader to [26] and [27]. If $T \in \mathcal{B}(X)$, for each $n \in \mathbb{N}$, T induces a linear transformation from $\mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ to $\mathcal{R}(T^{n+1})/\mathcal{R}(T^{n+2})$. We let $k_n(T)$ be the dimension of its null space. From [28, Lemma 2.3] it follows that, for every $n \in \mathbb{N}$,

$$\begin{aligned} k_n(T) &= \dim\,(\mathcal{N}(T) \cap \mathcal{R}(T^n))/(\mathcal{N}(T) \cap \mathcal{R}(T^{n+1})) \\ &= \dim\,(\mathcal{R}(T) + \mathcal{N}(T^{n+1}))/(\mathcal{R}(T) + \mathcal{N}(T^n)). \end{aligned}$$

Definition 2.5. Let $T \in \mathcal{B}(X)$ and $d \in \mathbb{N}$. Then T has *uniform descent* for $n \geq d$ if $k_n(T) = 0$ for all $n \geq d$. If in addition $\mathcal{R}(T^n)$ is closed in the operator range topology of $\mathcal{R}(T^d)$ for all $n \geq d$, then we say that T has *eventual topological uniform descent*, and, more precisely, that T has *topological uniform descent* for $n \geq d$.

Operators with eventual topological uniform descent were introduced by Grabiner [28]. This class includes all classes of operators introduced in the Introduction. It also includes many other classes, such as operators of Kato type, quasi-Fredholm operators, operators with finite descent, operators with finite essential descent, and so on. A very detailed and far-reaching account of these notions can be seen in [1, 10, 34]. Especially, the operators with topological uniform descent for $n \geq 0$ are precisely the *semi-regular* operators studied by Mbekhta [33]. Discussions of operators with eventual topological uniform descent may be found in [14, 23, 28, 29, 30, 41].

An operator $T \in \mathcal{B}(X)$ is said to be *essentially semi-regular* if $\mathcal{R}(T)$ is closed and $k(T) := \sum_{n=0}^{\infty} k_n(T) < \infty$. The *hyperrange* and *hyperkernel* of $T \in \mathcal{B}(X)$ are the subspaces of X defined by $\mathcal{R}(T^\infty) = \bigcap_{n=1}^{\infty} \mathcal{R}(T^n)$ and $\mathcal{N}(T^\infty) = \bigcup_{n=1}^{\infty} \mathcal{N}(T^n)$, respectively. From [28, Theorem 3.7] it follows that

$$k(T) = \dim \mathcal{N}(T)/(\mathcal{N}(T) \cap \mathcal{R}(T^\infty)) = \dim\,(\mathcal{R}(T) + \mathcal{N}(T^\infty))/\mathcal{R}(T).$$

Hence, being an essentially semi-regular operator can be characterized by: $\mathcal{R}(T)$ is closed and there exists a finite-dimensional subspace $F \subseteq X$ such that $\mathcal{N}(T) \subseteq \mathcal{R}(T^\infty) + F$. In addition, if T is essentially semi-regular, then T^n is essentially semi-regular, and hence $\mathcal{R}(T^n)$ is closed for all $n \in \mathbb{N}$ (see [1, Theorem 1.51]). Hence it is easy to verify that if $T \in \mathcal{B}(X)$ is essentially semi-regular, then there exists $p \in \mathbb{N}$ such that T has topological uniform descent for $n \geq p$.

Also, an operator $T \in \mathcal{B}(X)$ is called *Riesz* if its essential spectrum $\sigma_{\mathrm{e}}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$ is $\{0\}$.

LEMMA 2.6. *Suppose that $T \in \mathcal{B}(X)$ has topological uniform descent for $m \geq d$. If $S \in \mathcal{B}(X)$ is a Riesz operator commuting with T, and $V = S + T$ has topological uniform descent for $n \geq l$, then:*

(a) $\dim(\mathcal{R}(T^\infty) + \mathcal{R}(V^\infty))/(\mathcal{R}(T^\infty) \cap \mathcal{R}(V^\infty)) < \infty$;
(b) $\dim(\overline{\mathcal{N}(T^\infty)} + \overline{\mathcal{N}(V^\infty)})/(\overline{\mathcal{N}(T^\infty)} \cap \overline{\mathcal{N}(V^\infty)}) < \infty$;
(c) $\dim \mathcal{R}(V^n)/\mathcal{R}(V^{n+1}) = \dim \mathcal{R}(T^m)/\mathcal{R}(T^{m+1})$ *for sufficiently large m and n*;
(d) $\dim \mathcal{N}(V^{n+1})/\mathcal{N}(V^n) = \dim \mathcal{N}(T^{m+1})/\mathcal{N}(T^m)$ *for sufficiently large m and n.*

Proof. Parts (c) and (d) follow directly from [41, Theorems 3.8 and 3.12 and Remark 4.5]; so do (a) and (b) when $d \neq 0$ (that is, T is not semi-regular).

When $d = 0$ (that is, T is semi-regular), then by [41, Theorem 3.8] we find that $V = T + S$ is essentially semi-regular. So, there exists $p \in \mathbb{N}$ such that V has topological uniform descent for $n \geq p$. If $p \neq 0$ (that is, V is not semi-regular), then (a) and (b) follow directly from [41, Theorem 3.12 and Remark 4.5]. If $p = 0$ (that is, V is semi-regular), noting that $(M+N)/N \approx M/(M \cap N)$ for any subspaces M and N of X (see [31, Lemma 2.2]), then (a) and (b) follow from [41, Theorem 3.8 and Remark 4.5]. ∎

THEOREM 2.7. *Let $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$.*

(1) *If $T \in \mathcal{B}(X)$ is semi-B-Fredholm and commutes with F, then*
  (a) $\dim(\mathcal{R}(T^\infty) + \mathcal{R}((T+F)^\infty))/(\mathcal{R}(T^\infty) \cap \mathcal{R}((T+F)^\infty)) < \infty$;
  (b) $\dim(\overline{\mathcal{N}(T^\infty)} + \overline{\mathcal{N}((T+F)^\infty)})/(\overline{\mathcal{N}(T^\infty)} \cap \overline{\mathcal{N}((T+F)^\infty)}) < \infty$.
(2) *If $T \in \mathcal{B}(X)$ is upper (resp. lower) semi-B-Fredholm and commutes with F, then $T + F$ is also upper (resp. lower) semi-B-Fredholm and* $\mathrm{ind}(T+F) = \mathrm{ind}(T)$.

Proof. Since $F^n \in \mathcal{F}(X)$, it follows that F^n is Riesz, that is, $\sigma_{\mathrm{e}}(F^n) = \{0\}$. By the spectral mapping theorem for the essential spectrum, we get $\sigma_{\mathrm{e}}(F) = \{0\}$, so F is Riesz.

(1) Since T is semi-B-Fredholm and commutes with F, by Lemmas 2.1 and 2.2, $T + F$ is also semi-B-Fredholm. Since every semi-B-Fredholm operator has eventual topological uniform descent, by Lemma 2.6(a) & (b), parts (a) and (b) follow immediately.

(2) By Lemmas 2.1 and 2.2, it remains to prove that $\mathrm{ind}(T+F)=\mathrm{ind}(T)$. Since every semi-B-Fredholm operator has eventual topological uniform descent, 2.6(c) & (d) and [15, Proposition 2.1] imply that $\mathrm{ind}(T+F)=\mathrm{ind}(T)$. ∎

THEOREM 2.8. *Let $T\in\mathcal{B}(X)$, and let $F\in\mathcal{B}(X)$ with $F^n\in\mathcal{F}(X)$ for some $n\in\mathbb{N}$. If T commutes with F, then*

(1) $\sigma_{\mathrm{USBW}}(T+F)=\sigma_{\mathrm{USBW}}(T)$;
(2) $\sigma_{\mathrm{LSBW}}(T+F)=\sigma_{\mathrm{LSBW}}(T)$;
(3) $\sigma_{\mathrm{SBW}}(T+F)=\sigma_{\mathrm{SBW}}(T)$;
(4) $\sigma_{\mathrm{BW}}(T+F)=\sigma_{\mathrm{BW}}(T)$.

Proof. This follows directly from Theorem 2.7(2). ∎

Next, we turn to characterizations of power finite rank operators via various spectra originating from semi-B-Fredholm theory. For that, some notations are needed.

For $T\in\mathcal{B}(X)$, let us define the *descent spectrum*, *essential descent spectrum* and *eventual topological uniform descent spectrum* of T respectively as follows:

$$\sigma_{\mathrm{dsc}}(T)=\{\lambda\in\mathbb{C}:\mathrm{dsc}(T-\lambda I)=\infty\},$$
$$\sigma_{\mathrm{dsc}}^{\mathrm{e}}(T)=\{\lambda\in\mathbb{C}:\mathrm{dsc}_{\mathrm{e}}(T-\lambda I)=\infty\},$$
$$\sigma_{\mathrm{ud}}(T)=\{\lambda\in\mathbb{C}:T-\lambda I\text{ does not have eventual topological uniform descent}\}.$$

Several authors discussed the emptiness of various spectra, for example, the descent spectrum [22], essential descent spectrum [8], left Drazin spectrum and left essential Drazin spectrum [9], Drazin spectrum [21]. In [30], Jiang, Zhong and Zhang obtained a classification of the components of *eventual topological uniform descent resolvent set* $\rho_{\mathrm{ud}}(T):=\mathbb{C}\backslash\sigma_{\mathrm{ud}}(T)$. As an application, they generalized the corresponding results of [21, 22, 8, 9].

LEMMA 2.9 ([30, Corollary 4.5]). *Let $T\in\mathcal{B}(X)$ and let $\sigma_*\in\{\sigma_{\mathrm{ud}},\sigma_{\mathrm{dsc}},\sigma_{\mathrm{dsc}}^{\mathrm{e}},\sigma_{\mathrm{USBF}}=\sigma_{\mathrm{LD}}^{\mathrm{e}},\sigma_{\mathrm{USBB}}=\sigma_{\mathrm{LD}},\sigma_{\mathrm{BB}}=\sigma_{\mathrm{D}}\}$. Then the following statements are equivalent:*

(1) *$\sigma_*(T)=\emptyset$;*
(2) *T is algebraic (that is, there exists a non-zero complex polynomial p for which $p(T)=0$).*

COROLLARY 2.10. *Let $T\in\mathcal{B}(X)$ and let $\sigma_*\in\{\sigma_{\mathrm{ud}},\sigma_{\mathrm{dsc}},\sigma_{\mathrm{dsc}}^{\mathrm{e}},\sigma_{\mathrm{USBF}}=\sigma_{\mathrm{LD}}^{\mathrm{e}},\sigma_{\mathrm{LSBF}}=\sigma_{\mathrm{RD}}^{\mathrm{e}},\sigma_{\mathrm{SBF}},\sigma_{\mathrm{BF}},\sigma_{\mathrm{USBW}},\sigma_{\mathrm{LSBW}},\sigma_{\mathrm{SBW}},\sigma_{\mathrm{BW}},\sigma_{\mathrm{USBB}}=\sigma_{\mathrm{LD}},\sigma_{\mathrm{LSBB}}=\sigma_{\mathrm{RD}},\sigma_{\mathrm{SBB}},\sigma_{\mathrm{BB}}=\sigma_{\mathrm{D}}\}$. Then the following statements are equivalent:*

(1) *$\sigma_*(T)=\emptyset$;*

(2) *T is algebraic.*

Proof. If $\sigma_* \in \{\sigma_{\mathrm{ud}}, \sigma_{\mathrm{dsc}}, \sigma_{\mathrm{dsc}}^{\mathrm{e}}, \sigma_{\mathrm{USBF}} = \sigma_{\mathrm{LD}}^{\mathrm{e}}, \sigma_{\mathrm{USBB}} = \sigma_{\mathrm{LD}}, \sigma_{\mathrm{BB}} = \sigma_{\mathrm{D}}\}$, the conclusion is given by Lemma 2.9. Note that

$$\begin{aligned}\sigma_{\mathrm{ud}}(\cdot) \subseteq \sigma_{\mathrm{SBF}}(\cdot) &\subseteq \begin{cases}\sigma_{\mathrm{SBW}}(\cdot)\\ \sigma_{\mathrm{USBF}}(\cdot) = \sigma_{\mathrm{LD}}^{\mathrm{e}}(\cdot)\end{cases}\\ &\subseteq \sigma_{\mathrm{USBW}}(\cdot) \subseteq \begin{cases}\sigma_{\mathrm{BW}}(\cdot)\\ \sigma_{\mathrm{USBB}}(\cdot) = \sigma_{\mathrm{LD}}(\cdot)\end{cases}\\ &\subseteq \sigma_{\mathrm{BB}}(\cdot) = \sigma_{\mathrm{D}}(\cdot)\end{aligned}$$

and

$$\begin{aligned}\sigma_{\mathrm{ud}}(\cdot) \subseteq \sigma_{\mathrm{SBF}}(\cdot) &\subseteq \begin{cases}\sigma_{\mathrm{SBW}}(\cdot)\\ \sigma_{\mathrm{LSBF}}(\cdot) = \sigma_{\mathrm{RD}}^{\mathrm{e}}(\cdot)\end{cases}\\ &\subseteq \sigma_{\mathrm{LSBW}}(\cdot) \subseteq \begin{cases}\sigma_{\mathrm{BW}}(\cdot)\\ \sigma_{\mathrm{LSBB}}(\cdot) = \sigma_{\mathrm{RD}}(\cdot)\end{cases}\\ &\subseteq \sigma_{\mathrm{BB}}(\cdot) = \sigma_{\mathrm{D}}(\cdot).\end{aligned}$$

By Lemma 2.9, if $\sigma_* \in \{\sigma_{\mathrm{LSBF}} = \sigma_{\mathrm{RD}}^{\mathrm{e}}, \sigma_{\mathrm{SBF}}, \sigma_{\mathrm{USBW}}, \sigma_{\mathrm{LSBW}}, \sigma_{\mathrm{SBW}}, \sigma_{\mathrm{BW}}, \sigma_{\mathrm{LSBB}} = \sigma_{\mathrm{RD}}\}$, the conclusion follows easily. Note that

$$\sigma_{\mathrm{ud}}(\cdot) \subseteq \sigma_{\mathrm{SBB}}(\cdot) \subseteq \sigma_{\mathrm{BB}}(\cdot) = \sigma_{\mathrm{D}}(\cdot)$$

and

$$\sigma_{\mathrm{ud}}(\cdot) \subseteq \sigma_{\mathrm{BF}}(\cdot) \subseteq \sigma_{\mathrm{BB}}(\cdot) = \sigma_{\mathrm{D}}(\cdot).$$

Again by Lemma 2.9, if $\sigma_* \in \{\sigma_{\mathrm{SBB}}, \sigma_{\mathrm{BF}}\}$, the conclusion follows easily. ∎

In [9, Theorem 3.2], O. Bel Hadj Fredj et al. proved that $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ if and only if $\sigma_{\mathrm{LD}}^{\mathrm{e}}(T+F) = \sigma_{\mathrm{LD}}^{\mathrm{e}}(T)$ (equivalently, $\sigma_{\mathrm{LD}}(T+F) = \sigma_{\mathrm{LD}}(T)$) for every operator T in the commutant of F.

We are now in a position to give the proof of the following main result.

THEOREM 2.11. *Let $F \in \mathcal{B}(X)$ and $\sigma_* \in \{\sigma_{\mathrm{dsc}}, \sigma_{\mathrm{dsc}}^{\mathrm{e}}, \sigma_{\mathrm{USBF}} = \sigma_{\mathrm{LD}}^{\mathrm{e}}, \sigma_{\mathrm{LSBF}} = \sigma_{\mathrm{RD}}^{\mathrm{e}}, \sigma_{\mathrm{SBF}}, \sigma_{\mathrm{BF}}, \sigma_{\mathrm{USBW}}, \sigma_{\mathrm{LSBW}}, \sigma_{\mathrm{SBW}}, \sigma_{\mathrm{BW}}, \sigma_{\mathrm{USBB}} = \sigma_{\mathrm{LD}}, \sigma_{\mathrm{LSBB}} = \sigma_{\mathrm{RD}}, \sigma_{\mathrm{SBB}}, \sigma_{\mathrm{BB}} = \sigma_{\mathrm{D}}\}$. Then the following statements are equivalent:*

(1) *$F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$;*
(2) *$\sigma_*(T+F) = \sigma_*(T)$ for all $T \in \mathcal{B}(X)$ commuting with F.*

Proof. For $\sigma_* \in \{\sigma_{\mathrm{dsc}}, \sigma_{\mathrm{dsc}}^{\mathrm{e}}, \sigma_{\mathrm{USBB}} = \sigma_{\mathrm{LD}}, \sigma_{\mathrm{USBF}} = \sigma_{\mathrm{LD}}^{\mathrm{e}}\}$, the conclusion can be found in [22, Theorem 3.1], [8, Theorem 3.1] and [9, Theorem 3.2]. In the following, we prove the conclusion for the other spectra.

(1)⇒(2). For $\sigma_* \in \{\sigma_{\mathrm{LSBF}} = \sigma_{\mathrm{RD}}^{\mathrm{e}}, \sigma_{\mathrm{SBF}}, \sigma_{\mathrm{BF}}, \sigma_{\mathrm{USBW}}, \sigma_{\mathrm{LSBW}}, \sigma_{\mathrm{SBW}}, \sigma_{\mathrm{BW}}\}$, the conclusion follows directly from Corollary 2.3 and Theorem 2.8.

For $\sigma_* \in \{\sigma_{\mathrm{LSBB}} = \sigma_{\mathrm{RD}}\}$, suppose that $F \in \mathcal{B}(X)$ with $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F. It is clear that $F^* \in \mathcal{B}(X^*)$ with $F^{*n} \in \mathcal{F}(X^*)$ and that $T^* \in \mathcal{B}(X^*)$ commutes with F^*.

From the observation before this theorem, we get $\sigma_{\mathrm{LD}}(T^*+F^*)=\sigma_{\mathrm{LD}}(T^*)$, hence dually, $\sigma_{\mathrm{RD}}(T+F)=\sigma_{\mathrm{RD}}(T)$.

For $\sigma_* \in \{\sigma_{\mathrm{SBB}}, \sigma_{\mathrm{BB}}=\sigma_{\mathrm{D}}\}$, noting that $\sigma_{\mathrm{SBB}}(\cdot)=\sigma_{\mathrm{USBB}}(\cdot)\cap\sigma_{\mathrm{LSBB}}(\cdot)$ and $\sigma_{\mathrm{BB}}(\cdot)=\sigma_{\mathrm{USBB}}(\cdot)\cup\sigma_{\mathrm{LSBB}}(\cdot)$, the conclusion follows.

(2)⇒(1). Apply the proof of [22, Theorem 3.1(i)⇒(ii)] to the spectra under consideration, using in particular Corollary 2.10. ∎

Remark 2.12. By [13, Lemma 2.3], [34, pp. 135–136] and a similar argument of [12, Proposition 3.3], we know that $\sigma_*(T+F)=\sigma_*(T)$ for all finite rank operator F not necessarily commuting with T, where $\sigma_* \in \{\sigma^{\mathrm{e}}_{\mathrm{dsc}}, \sigma_{\mathrm{USBF}}=\sigma^{\mathrm{e}}_{\mathrm{LD}}, \sigma_{\mathrm{LSBF}}=\sigma^{\mathrm{e}}_{\mathrm{RD}}, \sigma_{\mathrm{SBF}}, \sigma_{\mathrm{BF}}, \sigma_{\mathrm{USBW}}, \sigma_{\mathrm{LSBW}}, \sigma_{\mathrm{SBW}}, \sigma_{\mathrm{BW}}\}$. By [34, Observation 5, p. 136], σ_* is not stable under non-commuting finite rank perturbations, where $\sigma_*\in\{\sigma_{\mathrm{dsc}}, \sigma_{\mathrm{USBB}}=\sigma_{\mathrm{LD}}, \sigma_{\mathrm{LSBB}}=\sigma_{\mathrm{RD}}, \sigma_{\mathrm{SBB}}, \sigma_{\mathrm{BB}}=\sigma_{\mathrm{D}}\}$.

3. Some applications.

3. Some applications. Rashid claimed in [38, Theorem 3.15] that if $T\in\mathcal{B}(X)$ and Q is a quasi-nilpotent operator that commutes with T, then (in [38], σ_{USBW} is denoted as $\sigma_{\mathrm{SBF}^-_+}$)

$$\sigma_{\mathrm{USBW}}(T+Q)=\sigma_{\mathrm{USBW}}(T).$$

In [42, Example 2.13], the authors showed that this equality does not hold in general.

As an immediate consequence of our main Theorem 2.11, we obtain the following corollary which, in particular, is a corrected version of [38, Theorem 3.15] and also provides positive answers to Questions 1.1 and 1.2.

Corollary 3.1. *Let $T\in\mathcal{B}(X)$, and let $N\in\mathcal{B}(X)$ be a nilpotent operator commuting with T. Then*

(1) $\sigma_{\mathrm{USBW}}(T+N)=\sigma_{\mathrm{USBW}}(T)$;
(2) $\sigma_{\mathrm{LSBW}}(T+N)=\sigma_{\mathrm{LSBW}}(T)$;
(3) $\sigma_{\mathrm{SBW}}(T+N)=\sigma_{\mathrm{SBW}}(T)$;
(4) $\sigma_{\mathrm{BW}}(T+N)=\sigma_{\mathrm{BW}}(T)$.

Besides Question 1.2, Berkani and Zariouh [19] also posed the following question:

Question 3.2. *Let $T\in\mathcal{B}(X)$, and let $N\in\mathcal{B}(X)$ be a nilpotent operator commuting with T. Under which conditions*

$$\sigma_{\mathrm{BF}}(T+N)=\sigma_{\mathrm{BF}}(T)\ ?$$

As an immediate consequence of Theorem 2.11, we obtain the following corollary which, in particular, provides an answer to Question 3.2.

Corollary 3.3. *Let $T\in\mathcal{B}(X)$, and let $N\in\mathcal{B}(X)$ be a nilpotent operator commuting with T. Then*

(1) $\sigma_{\mathrm{USBF}}(T+N) = \sigma_{\mathrm{USBF}}(T)$;
(2) $\sigma_{\mathrm{LSBF}}(T+N) = \sigma_{\mathrm{LSBF}}(T)$;
(3) $\sigma_{\mathrm{SBF}}(T+N) = \sigma_{\mathrm{SBF}}(T)$;
(4) $\sigma_{\mathrm{BF}}(T+N) = \sigma_{\mathrm{BF}}(T)$.

We say that $\lambda \in \sigma_{\mathrm{a}}(T)$ is a *left pole* of T if $T-\lambda I$ is left Drazin invertible. Let $\Pi_{\mathrm{a}}(T)$ denote the set of all left poles of T. An operator $T \in \mathcal{B}(X)$ is called *a-polaroid* if $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \Pi_{\mathrm{a}}(T)$. Henceforth, for $A \subseteq \mathbb{C}$, $\operatorname{iso} A$ is the set of isolated points of A. Besides Questions 1.2 and 3.2, Berkani and Zariouh [20] also posed the following three questions:

QUESTION 3.4. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Under which conditions*

$$\operatorname{asc}(T+N) < \infty \;\Leftrightarrow\; \operatorname{asc}(T) < \infty \text{ ?}$$

QUESTION 3.5. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Under which conditions, $\mathcal{R}((T+N)^m)$ is closed for m large enough if and only if $\mathcal{R}(T^m)$ is closed for m large enough?*

QUESTION 3.6. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Under which conditions*

$$\Pi_{\mathrm{a}}(T+N) = \Pi_{\mathrm{a}}(T) \text{ ?}$$

We mention that Question 3.4 is, in fact, an immediate consequence of an earlier result of Kaashoek and Lay [32, Theorem 2.2]. As for Question 3.5, suppose that $T \in \mathcal{B}(X)$ and that $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T. As a direct consequence of Theorem 2.11, if there exists $n \in \mathbb{N}$ such that $c_n(T) < \infty$ or $c'_n(T) < \infty$, then $\mathcal{R}((T+N)^m)$ is closed for m large enough if and only if $\mathcal{R}(T^m)$ is closed for m large enough.

As regards Question 3.6, we first recall a classical result.

LEMMA 3.7 (34). *If $T \in \mathcal{B}(X)$, and $Q \in \mathcal{B}(X)$ is a quasi-nilpotent operator commuting with T, then*

$$\sigma(T+Q) = \sigma(T) \quad \textit{and} \quad \sigma_{\mathrm{a}}(T+Q) = \sigma_{\mathrm{a}}(T). \tag{3.1}$$

As an immediate consequence of Theorem 2.11 and Lemma 3.7, we obtain the following corollary which provides an answer to Question 3.6.

COROLLARY 3.8. *Let $T \in \mathcal{B}(X)$, and let $N \in \mathcal{B}(X)$ be a nilpotent operator commuting with T. Then*

$$\Pi_{\mathrm{a}}(T+N) = \Pi_{\mathrm{a}}(T). \tag{3.2}$$

Let $\Pi(T)$ denote the set of all poles of T. It is proved in [16, Lemma 2.2] that if $T \in \mathcal{B}(X)$ and $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T, then

$$\Pi(T+N) = \Pi(T). \tag{3.3}$$

Let $E(T)$ and $E_{\mathrm{a}}(T)$ denote the set of all isolated eigenvalues of T and the set of all eigenvalues of T that are isolated in $\sigma_{\mathrm{a}}(T)$, respectively. That is,

$$E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda I)\},$$
$$E_{\mathrm{a}}(T) = \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T) : 0 < \alpha(T - \lambda I)\}.$$

An operator $T \in \mathcal{B}(X)$ is called *a-isoloid* if $\operatorname{iso}\sigma_{\mathrm{a}}(T) = E_{\mathrm{a}}(T)$.

We set $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$, $\Pi^0_{\mathrm{a}}(T) = \{\lambda \in \Pi_{\mathrm{a}}(T) : \alpha(T - \lambda I) < \infty\}$, $E^0(T) = \{\lambda \in E(T) : \alpha(T - \lambda I) < \infty\}$ and $E^0_{\mathrm{a}}(T) = \{\lambda \in E_{\mathrm{a}}(T) : \alpha(T - \lambda I) < \infty\}$.

Suppose that $T \in \mathcal{B}(X)$ and that $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T. Then from the proof of [19, Theorem 3.5],

$$\alpha(T+N) > 0 \;\Leftrightarrow\; \alpha(T) > 0, \qquad \alpha(T+N) < \infty \;\Leftrightarrow\; \alpha(T) < \infty.$$

Hence by (3.1), we have

$$E(T+N) = E(T), \tag{3.4}$$
$$E_{\mathrm{a}}(T+N) = E_{\mathrm{a}}(T), \tag{3.5}$$
$$E^0(T+N) = E^0(T), \tag{3.6}$$
$$E^0_{\mathrm{a}}(T+N) = E^0_{\mathrm{a}}(T). \tag{3.7}$$

An operator $T \in \mathcal{B}(X)$ is said to be *upper semi-Weyl* if T is upper semi-Fredholm and $\operatorname{ind}(T) \le 0$, and *Weyl* if T is Fredholm and $\operatorname{ind}(T) = 0$. For $T \in \mathcal{B}(X)$, let us define the *upper semi-Browder spectrum*, *Browder spectrum*, *upper semi-Weyl spectrum* and *Weyl spectrum* of T respectively as follows:

$$\sigma_{\mathrm{USB}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Browder}\},$$
$$\sigma_{\mathrm{B}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\},$$
$$\sigma_{\mathrm{USW}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Weyl}\},$$
$$\sigma_{\mathrm{W}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.$$

Suppose that $T \in \mathcal{B}(X)$ and that $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T. Then it follows from [40, Proposition 5] and [37, Theorem 1] that

$$\sigma_{\mathrm{USW}}(T+R) = \sigma_{\mathrm{USW}}(T), \tag{3.8}$$
$$\sigma_{\mathrm{W}}(T+R) = \sigma_{\mathrm{W}}(T), \tag{3.9}$$
$$\sigma_{\mathrm{USB}}(T+R) = \sigma_{\mathrm{USB}}(T), \tag{3.10}$$
$$\sigma_{\mathrm{B}}(T+R) = \sigma_{\mathrm{B}}(T). \tag{3.11}$$

Suppose that $T \in \mathcal{B}(X)$ and that $Q \in \mathcal{B}(X)$ is a quasi-nilpotent operator commuting with T. Then, noting that $\Pi^0(T) = \sigma(T)\backslash\sigma_{\mathrm{B}}(T)$ and $\Pi^0_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USB}}(T)$ for any $T \in \mathcal{B}(X)$, it follows from (3.1), (3.10) and (3.11) that

$$\Pi^0(T+Q) = \Pi^0(T), \tag{3.12}$$

$$\Pi^0_{\mathrm{a}}(T+Q) = \Pi^0_{\mathrm{a}}(T). \tag{3.13}$$

In the following table, we use the abbreviations gaW, aW, gW, W, (gw), (w), (gaw) and (aw) to signify that an operator $T \in \mathcal{B}(X)$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (gaw) and property (aw). For example, an operator $T \in \mathcal{B}(X)$ is said to obey generalized a-Weyl's theorem (in symbols $T \in gaW$) if $\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USBW}}(T) = E_{\mathrm{a}}(T)$. Similarly, the abbreviations gaB, aB, gB, B, (gb), (b), (gab) and (ab) have analogous meanings with respect to Browder type theorems and properties.

gaW	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USBW}}(T) = E_{\mathrm{a}}(T)$	gaB	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USBW}}(T) = \Pi_{\mathrm{a}}(T)$
aW	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USW}}(T) = E^0_{\mathrm{a}}(T)$	aB	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USW}}(T) = \Pi^0_{\mathrm{a}}(T)$
gW	$\sigma(T)\backslash\sigma_{\mathrm{BW}}(T) = E(T)$	gB	$\sigma(T)\backslash\sigma_{\mathrm{BW}}(T) = \Pi(T)$
W	$\sigma(T)\backslash\sigma_{\mathrm{W}}(T) = E^0(T)$	B	$\sigma(T)\backslash\sigma_{\mathrm{W}}(T) = \Pi^0(T)$
(gw)	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USBW}}(T) = E(T)$	(gb)	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USBW}}(T) = \Pi(T)$
(w)	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USW}}(T) = E^0(T)$	(b)	$\sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{USW}}(T) = \Pi^0(T)$
(gaw)	$\sigma(T)\backslash\sigma_{\mathrm{BW}}(T) = E_{\mathrm{a}}(T)$	(gab)	$\sigma(T)\backslash\sigma_{\mathrm{BW}}(T) = \Pi_{\mathrm{a}}(T)$
(aw)	$\sigma(T)\backslash\sigma_{\mathrm{W}}(T) = E^0_{\mathrm{a}}(T)$	(ab)	$\sigma(T)\backslash\sigma_{\mathrm{W}}(T) = \Pi^0_{\mathrm{a}}(T)$

Weyl–Browder type theorems and properties, in their classical and more recent generalized form, have been studied by a large number of authors. Theorem 2.11 and equations (3.1)–(3.13) give us a unifying framework for establishing perturbation results for Weyl–Browder type theorems and properties (generalized or not).

Corollary 3.9.

(1) *If $T \in \mathcal{B}(X)$ obeys gaW (resp. aW, gW, W, (gw), (w), (gaw), (aw), (gb), (gab)) and $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T, then $T+N$ also obeys gaW (resp. aW, gW, W, (gw), (w), (gaw), (aw), (gb), (gab)).*

(2) *If $T \in \mathcal{B}(X)$ obeys gaB (resp. aB, gB, B) and $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T, then $T+R$ also obeys gaB (resp. aB, gB, B).*

(3) *If $T \in \mathcal{B}(X)$ obeys (b) (resp. (ab)) and $Q \in \mathcal{B}(X)$ is a quasi-nilpotent operator commuting with T, then $T+Q$ also obeys (b) (resp. (ab)).*

Proof. (1) follows directly from Theorem 2.11 and (3.1)–(3.9).

(2) By [6], we know that T obeys gB (resp. gaB) if and only if T obeys B (resp. aB) for any $T \in \mathcal{B}(X)$. Note that T obeys B (resp. aB) if and only

if $\sigma_{\mathrm{W}}(T) = \sigma_{\mathrm{B}}(T)$ (resp. $\sigma_{\mathrm{USW}}(T) = \sigma_{\mathrm{USB}}(T)$). Hence by (3.8)–(3.11), the conclusion follows immediately.

(3) follows directly from (3.1), (3.8), (3.9), (3.12) and (3.13). ∎

Corollary 3.9, in particular, improves many recent results of [13, 16, 19, 20, 39] by removing certain extra assumptions.

REMARK 3.10. (1) For generalized a-Weyl's theorem, part (1) of Corollary 3.9 improves [20, Theorem 3.3] by removing the extra assumption that $E_{\mathrm{a}}(T) \subseteq \operatorname{iso} \sigma(T)$, and extends [20, Theorem 3.2]. For property (gw), on one hand, part (1) of Corollary 3.9 improves [39, Theorem 2.16] (resp. [13, Theorem 3.6]) by removing the extra assumption that T is a-isoloid (resp. T is a-polaroid) and extends [19, Theorem 3.8]; on the other hand, our proof corrects the proof of [39, Theorem 2.16]. For property (gab), part (1) of Corollary 3.9 improves [19, Theorem 3.2] by removing the extra assumption that T is a-polaroid, and extends [19, Theorem 3.4].

For generalized Weyl's theorem (resp. property (w), property (gaw)), part (1) of Corollary 3.9 has been proved in [13, Theorem 3.4] (resp. [3, Theorem 3.8] and [13, Theorem 3.1], [19, Theorem 3.6]) by using a different method.

For a-Weyl's theorem, some other perturbation theorems have been proved in [20, 24, 36].

For Weyl's theorem (resp. property (aw), property (gb)), part (1) of Corollary 3.9 has been proved in [35, Theorem 3] (resp. [19, Theorem 3.5], [42, Theorem 2.6]).

(2) It has been shown in [4] that Browder's theorem and a-Browder's theorem are stable under commuting Riesz perturbations.

(3) For property (b) (resp. (ab)), part (3) of Corollary 3.9 extends [16, Theorem 2.1] (resp. [19, Theorem 3.1]) to commuting quasi-nilpotent perturbations.

We conclude this paper by some examples to illustrate our perturbation results for Weyl–Browder type theorems and properties (generalized or not).

The following simple example shows that gaW, aW, gW, W, (gw), (w), (gaw) and (aw) are not stable under commuting quasi-nilpotent perturbations.

EXAMPLE 3.11. Let $Q : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be a quasi-nilpotent operator defined by

$$Q(x_1, x_2, \ldots) = (x_2/2, x_3/3, \ldots) \quad \text{for all } (x_n) \in l_2(\mathbb{N}).$$

Then Q is quasi-nilpotent, $\sigma(Q) = \sigma_{\mathrm{a}}(Q) = \sigma_{\mathrm{W}}(Q) = \sigma_{\mathrm{USW}}(Q) = \sigma_{\mathrm{BW}}(Q) = \sigma_{\mathrm{USBW}}(Q) = \{0\}$ and $E_{\mathrm{a}}(Q) = E_{\mathrm{a}}^0(Q) = E(Q) = E^0(Q) = \{0\}$. Take $T = 0$. Clearly, T satisfies gaW (resp. aW, gW, W, (gw), (w), (gaw), (aw)), but $T + Q = Q$ fails gaW (resp. aW, gW, W, (gw), (w), (gaw), (aw)).

The following example was given in [42, Example 2.14] to show that property (gb) is not stable under commuting quasi-nilpotent perturbations. Now, we use it to illustrate that property (gab) is also unstable under commuting quasi-nilpotent perturbations.

Example 3.12. Let $U : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$U(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \text{for } (x_n) \in l_2(\mathbb{N}).$$

Let $V : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ be the quasi-nilpotent operator defined by

$$V(x_1, x_2, \ldots) = (0, x_1, 0, x_3/3, x_4/4, \ldots) \quad \text{for } (x_n) \in l_2(\mathbb{N}).$$

Let $N : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the quasi-nilpotent operator defined by

$$N(x_1, x_2, \ldots) = (0, 0, 0, -x_3/3, -x_4/4, \ldots) \quad \text{for } (x_n) \in l_2(\mathbb{N}).$$

It is easy to verify that $VN = NV$. We consider the operators T and Q defined by $T = U \oplus V$ and $Q = 0 \oplus N$, respectively. Then Q is quasi-nilpotent and $TQ = QT$. Moreover,

$$\begin{aligned} \sigma(T) = \sigma(U) \cup \sigma(V) &= \{\lambda \in \mathbb{C} : 0 \le |\lambda| \le 1\}, \\ \sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(U) \cup \sigma_{\mathrm{a}}(V) &= \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}, \\ \sigma(T+Q) = \sigma(U) \cup \sigma(V+N) &= \{\lambda \in \mathbb{C} : 0 \le |\lambda| \le 1\}, \\ \sigma_{\mathrm{a}}(T+Q) = \sigma_{\mathrm{a}}(U) \cup \sigma_{\mathrm{a}}(V+N) &= \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}. \end{aligned}$$

It follows that $\Pi_{\mathrm{a}}(T) = \Pi(T) = \emptyset$ and $\{0\} = \Pi_{\mathrm{a}}(T+Q) \neq \Pi(T+Q) = \emptyset$. Hence by [18, Corollary 2.7], $T+Q$ fails property (gab). But since T has SVEP, it satisfies Browder's theorem or equivalently, by [6, Theorem 2.1], generalized Browder's theorem. Therefore by [18, Corollary 2.7] again, T has property (gab).

The following example was given in [42, Example 2.12] to show that property (gb) is not preserved under commuting finite rank perturbations. Now, we use it to illustrate that properties (b) and (ab) are also unstable under commuting finite rank (hence compact) perturbations.

Example 3.13. Let $U : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$U(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \text{for } (x_n) \in l_2(\mathbb{N}).$$

For fixed $0 < \varepsilon < 1$, let $F_\varepsilon : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the finite rank operator defined by

$$F_\varepsilon(x_1, x_2, \ldots) = (-\varepsilon x_1, 0, 0, \ldots) \quad \text{for } (x_n) \in l_2(\mathbb{N}).$$

We consider the operators T and F defined by $T = U \oplus I$ and $F = 0 \oplus F_\varepsilon$, respectively. Then F is a finite rank operator and $TF = FT$. Moreover,

$$\begin{aligned}
\sigma(T) &= \sigma(U) \cup \sigma(I) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\},\\
\sigma_{\mathrm{a}}(T) &= \sigma_{\mathrm{a}}(U) \cup \sigma_{\mathrm{a}}(I) = \{\lambda \in \mathbb{C} : |\lambda| = 1\},\\
\sigma(T+F) &= \sigma(U) \cup \sigma(I+F_\varepsilon) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\},\\
\sigma_{\mathrm{a}}(T+F) &= \sigma_{\mathrm{a}}(U) \cup \sigma_{\mathrm{a}}(I+F_\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{1-\varepsilon\}.
\end{aligned}$$

It follows that $\Pi^0_{\mathrm{a}}(T) = \Pi^0(T) = \emptyset$ and $\{1-\varepsilon\} = \Pi^0_{\mathrm{a}}(T+F) \neq \Pi^0(T+F) = \emptyset$. Hence by [17, Corollary 2.7] (resp. [18, Corollary 2.6]), $T+F$ fails property (b) (resp. (ab)). But since T has SVEP, T satisfies a-Browder's theorem (resp. Browder's theorem), therefore by [17, Corollary 2.7] (resp. [18, Corollary 2.6]) again, T has property (b) (resp. (ab)).

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