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The space of multipliers and convolutors of Orlicz spaces on a locally compact group

by

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Abstract. Let G be a locally compact group, let (φ, ψ) be a complementary pair of Young functions, and let $L^{\varphi}(G)$ and $L^{\psi}(G)$ be the corresponding Orlicz spaces. Under some conditions on φ, we will show that for a Banach $L^{\varphi}(G)$-submodule X of $L^{\psi}(G)$, the multiplier space $\mathrm{Hom}_{L^{\varphi}(G)}(L^{\varphi}(G), X^*)$ is a dual Banach space with predual $L^{\varphi}(G) \bullet X := \overline{\mathrm{span}}\{ux : u \in L^{\varphi}(G), x \in X\}$, where the closure is taken in the dual space of $\mathrm{Hom}_{L^{\varphi}(G)}(L^{\varphi}(G), X^*)$. We also prove that if φ is a Δ_2-regular N-function, then $\mathrm{Cv}_{\varphi}(G)$, the space of convolutors of $M^{\varphi}(G)$, is identified with the dual of a Banach algebra of functions on G under pointwise multiplication.

1. Introduction. Let A be a Banach algebra and X, Y be right Banach A-modules. A right A-module homomorphism from X into Y is a linear operator $T : X \to Y$ such that $T(xa) = T(x)a$ for all $a \in A$, $x \in X$. The Banach space of all bounded right A-module homomorphisms from X into Y with the operator norm is denoted by $\mathrm{Hom}_A(X, Y)$. Characterizing the space $\mathrm{Hom}_A(X, Y)$ for various classes of Banach algebras A and right Banach A-modules X and Y is a longstanding problem that many mathematicians have paid special attention to it; for the example see [Gr, L, M, Ri1, Ri2]. Also, for a recent study, see for example [Da, HNR1, HNR2, K].

Let G be a locally compact group with a fixed left Haar measure λ. Let also (φ, ψ) be a complementary pair of Young functions, and let $L^{\varphi}(G)$ and $L^{\psi}(G)$ be the corresponding Orlicz spaces. Orlicz spaces are genuine generalizations of the usual L^p-spaces. They have been thoroughly investigated from the functional analysis point of view. For analysis of some aspects of Orlicz spaces see [CHL, JPU, R1, R2, R3].

In this paper, we will study the problem of characterizing $\mathrm{Hom}_A(X, Y)$, when $A = X = L^{\varphi}(G)$ and Y is the dual of a closed $L^{\varphi}(G)$-submodule of

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$L^\psi(G)$, with convolution as the module action. Also, if φ is a finite Young function, we will consider the space $\mathrm{Cv}_\varphi(G)$ of convolutors of $M^\varphi(G)$, where $M^\varphi(G)$ is the closure of $C_c(G)$ in $L^\varphi(G)$. Then we show that $\mathrm{Cv}_\varphi(G)$ is a dual space when φ is a Δ_2-regular N-function, and obtain its predual. Our results extend some interesting results of [M] to Orlicz spaces.

2. Preliminaries. Throughout this paper let G be a locally compact group with a fixed left Haar measure λ. By $\int_G f(x)\,dx$ we denote the integral of a function f defined on G with respect to λ. Also, let $L^0(G)$ denote the set of all equivalence classes of λ-measurable complex-valued functions on G. By [RR, p. 6], a function $\varphi : \mathbb{R} \to [0,\infty]$ is called a *Young function* if φ is a convex, even, and left continuous function with $\varphi(0) = 0$ which is neither identically zero nor identically infinite. We call a Young function φ an *N-function* (a nice Young function) if it satisfies the limit conditions

$$\lim_{x\to 0} \frac{\varphi(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty.$$

For any Young function φ let

$$\psi(x) = \sup\{xy - \varphi(y) : y \in \mathbb{R}\} \quad (x \in \mathbb{R}).$$

It is easily verified that ψ is a Young function, called the *complementary Young function* to φ. It should be remarked that φ is also the complementary Young function to ψ. Then (φ, ψ) is called a *complementary pair* of Young functions.

A Young function φ is said to satisfy the Δ_2-*condition*, written $\varphi \in \Delta_2$, if there exist $k > 0$ and $x_0 \geq 0$ such that

$$\varphi(2x) \leq k\varphi(x) \quad \text{for } x \geq x_0.$$

Let φ be a Young function. For $f \in L^0(G)$ define

$$\rho_\varphi(f) = \int_G \varphi(|f(x)|)\,dx.$$

Then the *Orlicz space* $L^\varphi(G)$ is defined by

$$L^\varphi(G) = \{f \in L^0(G) : \rho_\varphi(af) < \infty \text{ for some } a > 0\}.$$

We also set

$$M^\varphi(G) = \{f \in L^0(G) : \rho_\varphi(af) < \infty \text{ for all } a > 0\}.$$

Then $L^\varphi(G)$ and $M^\varphi(G)$ are both Banach spaces under the norm $N_\varphi(\cdot)$, called the *Luxemburg–Nakano norm*, defined for $f \in L^\varphi(G)$ by

$$N_\varphi(f) = \inf\{k > 0 : \rho_\varphi(f/k) \leq 1\}.$$

It is well known that $N_\varphi(f) \leq 1$ if and only if $\rho_\varphi(f) \leq 1$. Furthermore, if the Young function φ vanishes only at the origin and is finite, then using the complementary Young function ψ, another norm $\|\cdot\|_\varphi$, called the *Orlicz norm*, is defined on $L^\varphi(G)$ by

$$\|f\|_\varphi = \sup\Big\{\int_G |fg|\,d\lambda : \rho_\psi(g) \le 1\Big\}.$$

Let us remark that $\|\cdot\|_\varphi$ is equivalent to $N_\varphi(\cdot)$; in fact, $N_\varphi(f) \le \|f\|_\varphi \le 2N_\varphi(f)$ for every $f \in L^\varphi(G)$. For $1 \le p \le \infty$, the classical Lebesgue spaces on G with respect to the left Haar measure λ will be denoted by $L^p(G)$ with the norm $\|\cdot\|_p$ as defined in [F]. It is clear that $L^p(G)$ is an elementary example of an Orlicz space.

We say that a Young function $\varphi \in \Delta_2$ is Δ_2-*regular* and write $\varphi \in \Delta_2$-regular if φ satisfies the Δ_2-condition, with $x_0 = 0$ in the case when G is not compact. It is well known that if $\varphi \in \Delta_2$-regular then $L^\varphi(G) = M^\varphi(G)$ and $L^\varphi(G)$ is equal to the closure of $C_c(G)$ in the norm $N_\varphi(\cdot)$. Here $C_c(G)$ stands for the space of continuous functions on G with compact support.

If φ is a finite Young function, then the dual space of $M^\varphi(G)$ is the Banach space $L^\psi(G)$ under the usual duality

$$\langle f, g\rangle = \int_G f(x)g(x)\,dx \qquad (f \in M^\varphi(G),\ g \in L^\psi(G)),$$

where φ and ψ are complementary Young functions.

The above concepts relating to Young functions are quite standard and can be found in any standard textbook on Orlicz spaces. Here we refer to the excellent monographs [KR, RR].

For measurable functions f and g on a locally compact group G, the *convolution* product

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy$$

is defined at each point $x \in G$ for which this makes sense. For any function $f : G \to \mathbb{C}$ we denote by $\check{f}$ the function defined by $\check{f}(x) = f(x^{-1})$ for all $x \in G$.

Let φ be a finite Young function whose right derivative φ' is strictly positive at the origin. For such φ by [R2, Proposition 4.1], $L^\varphi(G)$ is a Banach algebra with convolution multiplication, and is contained in $L^1(G)$ with $\|f\|_1 \le \frac{1}{\varphi'(0)} N_\varphi(f)$ for any $f \in L^\varphi(G)$. Also we have the following easy lemma.

Lemma 2.1. *Let G be a locally compact group and φ a finite Young function with $\varphi'(0) > 0$. Then $L^\varphi(G)$ is a left Banach $L^1(G)$-module.*

Proof. Take arbitrary positive elements $f \in L^1(G)$, $g \in L^\varphi(G)$ and $h \in L^\psi(G)$, where ψ is the complementary Young function to φ. Then we have

$$\langle f*g,h\rangle = \int\limits_G\int\limits_G f(t)g(t^{-1}s)h(s)\,dt\,ds = \int\limits_G\int\limits_G f(t)g(s)h(ts)\,ds\,dt$$
$$\le 2\|f\|_1 N_\varphi(g)N_\psi(h) < \infty.$$

Since $f*g$ has σ-compact support, $f*g\in L^\varphi(G)$, by [RR, Proposition IV.4.1]. ∎

For two Banach spaces X and Y, we denote by $X\mathbin{\widehat{\otimes}} Y$ their projective tensor product, and by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X into Y. We write $\mathcal{L}(X)$ in place of $\mathcal{L}(X,X)$. The projective tensor norm on $X\mathbin{\widehat{\otimes}} Y$ will be denoted by $\|\cdot\|_\wedge$.

Let A be a Banach algebra, let X be a Banach A-bimodule, and let Y be a left Banach A-module. Then $X\mathbin{\widehat{\otimes}} Y$ becomes a left A-module with the following action:

$$a\cdot(x\otimes y) = ax\otimes y \qquad (a\in A,\ x\in X,\ y\in Y).$$

Then clearly the closed linear span of the set

$$\{xa\otimes y - x\otimes ay : a\in A,\ x\in X,\ y\in Y\},$$

denoted by E, in $X\mathbin{\widehat{\otimes}} Y$ is a closed submodule of $X\mathbin{\widehat{\otimes}} Y$. Now $X\mathbin{\widehat{\otimes}}_A Y := (X\mathbin{\widehat{\otimes}} Y)/E$ is a Banach left A-module; for more details see [D, Section 2.6].

For two Banach spaces X and Y, the mapping $\Phi:\mathcal{L}(X,Y^*)\to(X\mathbin{\widehat{\otimes}} Y)^*$ defined by

$$\langle x\otimes y,\Phi(T)\rangle = \langle y,T(x)\rangle \qquad (x\in X,\ y\in Y,\ T\in\mathcal{L}(X,Y^*)),$$

is an isometric isomorphism. In particular, if X is a reflexive Banach space, then $(X\mathbin{\widehat{\otimes}} X^*)^*\cong\mathcal{L}(X)$ [D, Proposition A.3.70]. Here X^* denotes the topological dual space of X equipped with its dual Banach norm.

Finally, let us recall that if X is a Banach left A-module, then X^* is a right Banach A-module under the dual module action defined by $\langle x^*\cdot a,x\rangle = \langle x^*,ax\rangle$ for $x^*\in X^*$, $x\in X$, and $a\in A$.

3. The multiplier space $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G),X^*)$. In this section, among other things, we characterize the multiplier space $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G),X^*)$ as the dual of a natural space, namely, the closed linear span of $L^\varphi(G)X$.

Throughout this section, φ will denote a finite Young function with $\varphi'(0)>0$.

Let X be a left $L^\varphi(G)$-submodule of $L^\psi(G)$ which is a Banach space with the norm $\|\cdot\|_X$ satisfying $N_\psi(\cdot)\le\|\cdot\|_X$. We will show in this section that $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G),X^*)$ is a dual Banach space and characterize its predual in terms of elements in $L^\varphi(G)$ and X. To see this we note that for every $u\in L^\varphi(G)$ and $x\in X$, ux is a bounded linear functional on

$\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ defined by

$$\langle ux, T\rangle = \langle x, T(u)\rangle \qquad \text{for } T \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$$

with $\|ux\| \le N_\varphi(u)\|x\|_X$. We denote by $L^\varphi(G) \bullet X$ the norm closed linear span of $L^\varphi(G)X := \{ux : u \in L^\varphi(G) \text{ and } x \in X\}$ in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)^*$. So each element of $L^\varphi(G) \bullet X$ becomes a bounded linear functional on $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$.

The following two results are parallel to Theorems 2.2 and 2.3 of [M].

Theorem 3.1. *Let G be a locally compact group and φ a finite Young function with $\varphi'(0) > 0$. If $(X, \|\cdot\|_X)$ is a left Banach $L^\varphi(G)$-submodule of $L^\psi(G)$, then* $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*) = (L^\varphi(G) \bullet X)^*$.

Proof. Define $\Psi : L^\varphi(G) \mathbin{\widehat{\otimes}} X \to L^\varphi(G) \bullet X$ by $\Psi(\sum_{i=1}^\infty u_i \otimes x_i) = \sum_{i=1}^\infty u_i x_i$, where $u_i \in L^\varphi(G)$, $x_i \in X$ and $\sum_{i=1}^\infty N_\varphi(u_i)\|x_i\|_X < \infty$. Then Ψ is well defined: in fact, if $\sum_{i=1}^\infty u_i \otimes x_i = 0$ in $L^\varphi(G) \mathbin{\widehat{\otimes}} X$, then $\langle T, \sum_{i=1}^\infty u_i \otimes x_i\rangle = 0$ for all $T \in \mathcal{L}(L^\varphi(G), X^*)$. Therefore $\sum_{i=1}^\infty u_i x_i = 0$ in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)^*$. It also follows that

$$\Big\|\Psi\Big(\sum_{i=1}^\infty u_i \otimes x_i\Big)\Big\| \le \sum_{i=1}^\infty N_\varphi(u_i)\|x_i\|_X.$$

So $\Psi(\sum_{i=1}^\infty u_i \otimes x_i) \in L^\varphi(G) \bullet X$ and $\|\Psi\| \le 1$. Hence, we have the adjoint operator $\Psi^* : (L^\varphi(G) \bullet X)^* \to (L^\varphi(G) \mathbin{\widehat{\otimes}} X)^*$ with $\|\Psi^*\| \le 1$. As $(L^\varphi(G) \mathbin{\widehat{\otimes}} X)^* = \mathcal{L}(L^\varphi(G), X^*)$, for each $T \in (L^\varphi(G) \bullet X)^*$, $\Psi^*(T) : L^\varphi(G) \to X^*$ is a bounded linear operator.

We will show that $\Psi^*(T) \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$. Let $u, v \in L^\varphi(G)$ and $x \in X$. Then

$$\begin{aligned}\langle \Psi^*(T)(uv), x\rangle &= \langle \Psi^*(T), uv \otimes x\rangle = \langle T, \Psi(uv \otimes x)\rangle = \langle T, (uv)x\rangle \\ &= \langle T, u(vx)\rangle = \langle T, \Psi(u \otimes vx)\rangle = \langle \Psi^*(T), u \otimes vx\rangle \\ &= \langle \Psi^*(T)(u), vx\rangle = \langle \Psi^*(T)(u) \cdot v, x\rangle.\end{aligned}$$

Hence $\Psi^*(T)(uv) = \Psi^*(T)(u) \cdot v$ for all $u, v \in L^\varphi(G)$. Thus $\Psi^*(T) \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$.

We know that the restriction of any $T \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ to $L^\varphi(G) \bullet X$ is in $(L^\varphi(G) \bullet X)^*$. Also for every $T \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ we have

$$\begin{aligned}\Big\langle \Psi^*(T), \sum_{i=1}^\infty u_i \otimes x_i\Big\rangle &= \Big\langle T, \sum_{i=1}^\infty u_i x_i\Big\rangle = \sum_{i=1}^\infty \langle T, u_i x_i\rangle \\ &= \sum_{i=1}^\infty \langle T(u_i), x_i\rangle = \Big\langle T, \sum_{i=1}^\infty u_i \otimes x_i\Big\rangle,\end{aligned}$$

and so $\Psi^*(T) = T$. Since the image of Ψ contains $L^\varphi(G)X$, by [Ru, p. 99, Corollary] or [S, Proposition 26.20], Ψ^* is one-to-one. An application of the Hahn–Banach theorem shows $\Psi^* : (L^\varphi(G) \bullet X)^* \to \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ is a surjective isometry. ∎

PROPOSITION 3.2. *Let G be a locally compact group and φ a finite Young function with $\varphi'(0) > 0$. Then $\xi \in L^\varphi(G) \bullet X$ if and only if there exist sequences $(u_i) \subseteq L^\varphi(G)$ and $(x_i) \subseteq X$ such that $\sum_{i=1}^\infty N_\varphi(u_i)\|x_i\|_X < \infty$ with $\xi = \sum_{i=1}^\infty u_i x_i$ and*

$$\|\xi\| = \inf\Big\{\sum_{i=1}^\infty N_\varphi(u_i)\|x_i\|_X : \xi = \sum_{i=1}^\infty u_i x_i,\ \sum_{i=1}^\infty N_\varphi(u_i)\|x_i\|_X < \infty\Big\}.$$

Proof. By definition, each element of the form $\sum_{i=1}^\infty u_i x_i$ as in (the proof of) Theorem 3.1 lies in $L^\varphi(G) \bullet X$.

For the converse, let $\mathfrak{A}$ be the closed subspace of $L^\varphi(G)\widehat{\otimes} X$ generated by $uv\otimes x - u\otimes vx$ for $u, v \in L^\varphi(G)$, $x \in X$. Then an element $T \in \mathcal{L}(L^\varphi(G), X^*)$ is in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ if and only if $T = 0$ on $\mathfrak{A}$.

Let $B : (L^\varphi(G) \widehat{\otimes} X)/\mathfrak{A} \to L^\varphi(G) \bullet X$ be defined by

$$B\Big(\sum_{i=1}^\infty u_i \otimes x_i + \mathfrak{A}\Big) = \sum_{i=1}^\infty u_i x_i.$$

It is clear that B is well defined and $\|B\| \le 1$. Also $((L^\varphi(G) \widehat{\otimes} X)/\mathfrak{A})^* = \mathfrak{A}^\perp = \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ and $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*) = (L^\varphi(G) \bullet X)^*$ imply that $B^* : (L^\varphi(G) \bullet X)^* \to ((L^\varphi(G) \widehat{\otimes} X)/\mathfrak{A})^*$ is one-to-one and onto. So, B is surjective by [Ru, Theorem 4.15] and one-to-one by [Ru, p. 99, Corollary]. This proves the first part of the proposition.

For the second part, let $\xi \in L^\varphi(G) \bullet X$ and $\epsilon > 0$ be given. Then there are sequences $(u_i) \subseteq L^\varphi(G)$ and $(x_i) \subseteq X$ such that $\sum_{i=1}^\infty N_\varphi(x_i)\|x_i\|_X < \infty$ and $\xi = \sum_{i=1}^\infty u_i x_i$. Let $\eta = \sum_{i=1}^\infty u_i \otimes x_i + \mathfrak{A}$ be in $(L^\varphi(G) \widehat{\otimes} X)/\mathfrak{A}$. Since $\langle T, \eta\rangle = \langle T, \xi\rangle$ for all $T \in \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$, we have $\|\eta\| = \|\xi\|$. Thus there exist $p_i \in L^\varphi(G)$ and $q_i \in X$, for any $i \ge 1$, such that $\sum_{i=1}^\infty N_\varphi(p_i)\|q_i\|_X < \|\xi\| + \epsilon$ and $\eta = \sum_{i=1}^\infty p_i \otimes q_i + \mathfrak{A}$ by the definition of the quotient norm. Thus $\xi = \sum_{i=1}^\infty p_i q_i$ on $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$, which was to be shown. ∎

It is natural to consider relations between $L^\varphi(G) \bullet X$ and X. Since X^* is a Banach right $L^\varphi(G)$-module, we can consider the mapping

$$\iota : X^* \to \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*), \quad \iota(f) = \mathfrak{L}_f,$$

where $\mathfrak{L}_f : L^\varphi(G) \to X^*$ is left multiplication by f, i.e., $\mathfrak{L}_f(u) = f \cdot u$ for all $u \in L^\varphi(G)$. Then it is easily seen that ι is an embedding with $\|\iota(f)\| \le \|f\|$, and so we can assume that $X^* \subseteq \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$.

Now consider the conjugate map $\iota^* : (L^\varphi(G) \bullet X)^{**} \to X^{**}$ which is the restriction map with $\|\iota^*\| \le 1$. Also for each $u \in L^\varphi(G)$, $x \in X$ and $f \in X^*$,

$$\langle \iota^*(ux), f\rangle = \langle ux, \iota(f)\rangle = \langle ux, \mathfrak{L}_f\rangle = \langle x, \mathfrak{L}_f(u)\rangle = \langle x, f \cdot u\rangle = \langle ux, f\rangle.$$

Therefore, $\iota^*(L^\varphi(G) \bullet X) \subseteq X$.

The following result is a direct consequence of well known results about adjoints of linear maps; see for example Theorems 4.12, 4.14 and 4.15 in [Ru].

Proposition 3.3. *Let G be a locally compact group and φ a finite Young function with $\varphi'(0) > 0$. Then the restriction map $\iota^* : L^\varphi(G) \bullet X \to X$ is a bijection if and only if X^* is homeomorphic to* $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$.

We denote by $\mathrm{LUC}(G)$ the space of all bounded left uniformly continuous functions on G. Then $\mathrm{LUC}(G)$ is a Banach $L^1(G)$-bimodule for which the left and right module actions are given by

$$\varphi f = f * \check{\varphi}, \qquad f\varphi = \frac{1}{\Delta}\check{\varphi} * f$$

for all $f \in \mathrm{LUC}(G)$ and $\varphi \in L^1(G)$, where Δ denotes the modular function of G. Thus, $\mathrm{LUC}(G)^*$ is a Banach $L^1(G)$-bimodule. Let us remark that $L^1(G)\,\mathrm{LUC}(G) = \mathrm{LUC}(G)$; for more details see Section 32.45 in [HR]. The following corollary is a direct consequence of [Ri1, Theorem 4.4] and Proposition 3.3. We point out that this result has been proved before in [La, Theorem 1].

Corollary 3.4. *For any locally compact group G,*

$$\mathrm{Hom}_{L^1(G)}(L^1(G), \mathrm{LUC}(G)^*) = \mathrm{LUC}(G)^*.$$

Let φ be a Δ_2-regular Young function and $f \in L^\varphi(G)$. Then, using Hölder's inequality [RR, Proposition III.3.1], f can be viewed as an element of X^*. Now define the linear map $L_f : L^\varphi(G) \to L^\varphi(G)$ by $L_f(g) = f * g$ for each $g \in L^\varphi(G)$. If X is a Banach left $L^\varphi(G)$-module with module action $f \diamond g = g * \check{f}$, then by this definition, $L^\varphi(G) \subseteq \mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$. We are interested in when $L^\varphi(G)$ is w^*-dense in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$. For this reason, let $f \in L^\varphi(G)$ and $g \in X$. Then the function $g * \check{f}$ belongs to $L^\psi(G)$, with $N_\psi(g * \check{f}) \le 2N_\varphi(f)\|g\|_X$. Since the mapping $(f, g) \mapsto g * \check{f}$ from $L^\varphi(G) \times X$ into $L^\psi(G)$ is bilinear and continuous, there is a unique continuous linear mapping $\Phi : L^\varphi(G)\widehat{\otimes}X \to L^\psi(G)$ satisfying $\Phi(f \otimes g) = g * \check{f}$ for all $f \in L^\varphi(G)$ and $g \in X$.

Definition 3.5. Let $A_\varphi(X)$ denote the range of the mapping

$$\Phi : L^\varphi(G) \mathbin{\widehat{\otimes}} X \to L^\psi(G), \quad \Phi(f \otimes g) = g * \check{f}.$$

We endow $A_\varphi(X)$ with the quotient norm from $L^\varphi(G)\widehat{\otimes}X$. Then $A_\varphi(X)$ becomes a Banach space, and $\xi \in A_\varphi(X)$ if and only if there are sequences

$(f_i) \subseteq L^\varphi(G)$ and $(g_i) \subseteq X$ such that $\xi = \sum_{i=1}^\infty g_i * \check{f}_i$ with $\sum_{i=1}^\infty N_\varphi(g_i)\|g_i\|_X < \infty$.

We conclude this section with the following result that characterizes the w^*-denseness of $L^\varphi(G)$ in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$.

Theorem 3.6. *Let G be a locally compact group and let φ a finite Young function with $\varphi'(0) > 0$. Then $L^\varphi(G)$ is w^*-dense in* $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$ *if and only if $A_\varphi(X)$ is isometrically isomorphic to $L^\varphi(G) \bullet X$.*

Proof. Let $L^\varphi(G)$ be w^*-dense in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$. Define the mapping $\Theta : L^\varphi(G) \bullet X \to A_\varphi(X)$ by

$$\Theta\Big(\sum_{i=1}^\infty f_i g_i\Big) = \sum_{i=1}^\infty g_i * \check{f}_i.$$

Since $L^\varphi(G)$ separates the points of $L^\varphi(G) \bullet X$ [H, Corollary 3, p. 68], and for any $h \in L^\varphi(G)$, $(f_i) \subseteq L^\varphi(G)$ and $(g_i) \subseteq X$,

$$\begin{aligned}\Big\langle L_h, \sum_{i=1}^\infty f_i g_i\Big\rangle &= \sum_{i=1}^\infty \langle L_h(f_i), g_i\rangle = \sum_{i=1}^\infty \langle h * f_i, g_i\rangle \\ &= \sum_{i=1}^\infty \langle h, g_i * \check{f}_i\rangle = \Big\langle h, \sum_{i=1}^\infty g_i * \check{f}_i\Big\rangle,\end{aligned}$$

Θ is a linear isomorphism. Also, by Proposition 3.2 it is an isometry.

Conversely, let $A_\varphi(X)$ be homeomorphic to $L^\varphi(G) \bullet X$. Since $L^\varphi(G)$ separates the points of $A_\varphi(X)$, again by [H, Corollary 3, p. 68], $L^\varphi(G)$ is w^*-dense in $\mathrm{Hom}_{L^\varphi(G)}(L^\varphi(G), X^*)$. ■

4. $\mathrm{Cv}_\varphi(G)$, the space of convolutors of $M^\varphi(G)$. In this section we deal with operators on $M^\varphi(G)$ which commute with certain functions, and show that the dual of $\overline{A}_\varphi(G)$, defined below, can be identified with the space of such operators.

Throughout this section we will assume that (φ, ψ) is a complementary pair of N-functions.

We commence with a definition.

Definition 4.1. An operator $T \in \mathcal{L}(M^\varphi(G))$ is termed a *convolutor* if

$$T(f * g) = T(f) * g \quad \text{whenever} \quad f, g \in C_c(G).$$

The space of all convolutors is denoted by $\mathrm{Cv}_\varphi(G)$, and is a closed subspace of $\mathcal{L}(M^\varphi(G))$.

Let $\mathcal{K}(G)$ be the set of all compact subsets of G with nonvoid interiors which contain the identity element e of G. Given K in $\mathcal{K}(G)$, we define

$$\check{A}_{\varphi,K}(G)=\Big\{u\in C_c(G):u=\sum_{n=1}^{\infty}g_n*\check{f}_n,\ (f_n)\subseteq M^{\varphi}(K),\\ (g_n)\subseteq M^{\psi}(K),\ \sum_{n=1}^{\infty}N_{\varphi}(f_n)N_{\psi}(g_n)<\infty\Big\}.$$

The norm of u in $\check{A}_{\varphi,K}(G)$ is defined by

$$\|u\|_{\check{A}_{\varphi,K}}=\inf\Big\{\sum_{n=1}^{\infty}N_{\varphi}(f_n)N_{\psi}(g_n):u=\sum_{n=1}^{\infty}g_n*\check{f}_n\Big\}.$$

We now define

$$\check{A}_{\varphi}(G)=\bigcup_{K\in\mathcal{K}(G)}\check{A}_{\varphi,K}(G),$$

and endow $u\in\check{A}_{\varphi}(G)$ with the norm

$$\|u\|_{\check{A}_{\varphi}}=\inf\{\|u\|_{\check{A}_{\varphi,K}}:u\in\check{A}_{\varphi,K}(G),\ K\in\mathcal{K}(G)\}.$$

The following two lemmas are needed to prove our main theorem in this section.

LEMMA 4.2. *Let G be a locally compact group and φ be an N-function. If $T\in\mathrm{Cv}_{\varphi}(G)$, then there exists a net $(e_\alpha)\subseteq C_c(G)$ with $\|e_\alpha\|_1=1$ such that if we set $T_\alpha(f)=T(e_\alpha*f)$ for every $f\in M^{\varphi}(G)$, then*

(i) *$\|T_\alpha\|\le\|T\|$ for each α,*
(ii) *$\lim_\alpha N_{\varphi}(T_\alpha(f)-T(f))=0$ for each $f\in C_c(G)$.*

Proof. By [R2, Proposition 1] we may choose a net $(e_\alpha)\subseteq C_c(G)$ with $\|e_\alpha\|_1=1$ for any α such that $\lim_\alpha N_{\varphi}(e_\alpha*f-f)=0$ whenever $f\in M^{\varphi}(G)$. Since $T_\alpha\in\mathrm{Cv}_{\varphi}(G)$, we have $N_{\varphi}(T_\alpha(f))\le\|T\|N_{\varphi}(f)$ for any $f\in M^{\varphi}(G)$, and so $\|T_\alpha\|\le\|T\|$. Moreover,

$$\begin{aligned}\lim_\alpha N_{\varphi}(T_\alpha(f)-T(f))&=\lim_\alpha N_{\varphi}(T(e_\alpha*f-f))\\&\le\|T\|\lim_\alpha N_{\varphi}(e_\alpha*f-f)=0.\ \blacksquare\end{aligned}$$

For $K\in\mathcal{K}(G)$ we write $C(K)$ for the space of complex-valued continuous functions on K.

LEMMA 4.3. *Let G be a locally compact group and let (φ,ψ) be a complementary pair of N-functions. Then $\check{A}_{\varphi}(G)$ is a normed algebra under pointwise multiplication.*

Proof. Let $u,v\in\check{A}_{\varphi}(G)$. Then there exist subsets $K,F\in\mathcal{K}(G)$ and sequences $(f_n),(g_n)\subseteq C(K)$ and $(h_n),(l_n)\subseteq C(F)$ such that $u\in\check{A}_{\varphi,K}(G)$

and $v\in \check{A}_{\varphi,F}(G)$ with representations $u=\sum_{n=1}^{\infty} g_n*\check{f}_n$ and $v=\sum_{n=1}^{\infty} h_n*\check{l}_n$. For $y\in G$ and $n,m\in\mathbb{N}$ define functions $F_{(n,m)y}$ and $G_{(n,m)y}$ on G by

$$F_{(n,m)y}(x)=f_n(x)l_m(xy) \quad \text{and} \quad G_{(n,m)y}(x)=g_n(x)h_m(xy).$$

Then $F_{(n,m)y}, G_{(n,m)y}\in C_c(G)$ and the map $y\mapsto G_{(n,m)y}*\check{F}_{(n,m)y}$ from G into $\check{A}_{\varphi,K\cup F}(G)$ is continuous and vanishes outside the compact subset

$$C_{(n,m)}=(\operatorname{supp} f_n)^{-1}\operatorname{supp} l_m\cap(\operatorname{supp} g_n)^{-1}\operatorname{supp} h_m$$

of G. Indeed, for $y_0,y\in G$ such that $y_0^{-1}y\in U$, where U is a symmetric neighborhood of identity,

$$\begin{aligned}
&\|G_{(n,m)y}*\check{F}_{(n,m)y}-G_{(n,m)y_0}*\check{F}_{(n,m)y_0}\|_{\check{A}_{\varphi,K\cup F}(G)}\\
&\qquad\le N_\varphi(F_{(n,m)y}-F_{(n,m)y_0})\big(N_\psi(G_{(n,m)y}-G_{(n,m)y_0})+N_\psi(G_{(n,m)y_0})\big)\\
&\qquad\quad+N_\varphi(F_{(n,m)y_0})N_\psi(G_{(n,m)y}-G_{(n,m)y_0}),
\end{aligned}$$

and

$$N_\varphi(F_{(n,m)y}-F_{(n,m)y_0})\le\|R_y l_m-R_{y_0}l_m\|_\infty N_\varphi(f_n),$$

where R_y denotes right translation, i.e., $R_y(f)(x)=f(xy)$ for $x,y\in G$. Thus by [F, Theorem A3.1], the vector valued integral

$$H=\int_{C_{(n,m)}} G_{(n,m)y}*\check{F}_{(n,m)y}\,dy=\int_G G_{(n,m)y}*\check{F}_{(n,m)y}\,dy$$

exists and defines an element of $\check{A}_{\varphi,K\cup F}(G)$. Moreover,

$$\begin{aligned}
u(s)v(s)&=\sum_{n=1}^{\infty} g_n*\check{f}_n(s)\sum_{m=1}^{\infty} h_m*\check{l}_m(s)\\
&=\sum_{n=1}^{\infty}\int_G g_n(x)\check{f}_n(x^{-1}s)\,dx\sum_{m=1}^{\infty}\int_G h_m(y)\check{l}_m(y^{-1}s)\,dy\\
&=\sum_{n,m=1}^{\infty}\int_G\int_G g_n(x)\check{f}_n(x^{-1}s)h_m(xy)\check{l}_m(y^{-1}x^{-1}s)\,dy\,dx\\
&=\sum_{n,m=1}^{\infty}\int_G\int_G g_n(x)h_m(xy)\check{f}_n(x^{-1}s)\check{l}_m(y^{-1}x^{-1}s)\,dx\,dy\\
&=\sum_{n,m=1}^{\infty}\int_G G_{(n,m)y}*\check{F}_{(n,m)y}(s)\,dy=\sum_{n,m=1}^{\infty}\Big(\int_G G_{(n,m)y}*\check{F}_{(n,m)y}\,dy\Big)(s).
\end{aligned}$$

It follows that $uv\in\check{A}_{\varphi,K\cup F}(G)$. Moreover, if we put $F_{(n,m)}(y)=N_\varphi(F_{(n,m)y})$ and $G_{(n,m)}(y)=N_\psi(G_{(n,m)y})$, then

$$\begin{aligned}\|uv\|_{\check{A}_{\varphi,K\cup F}(G)} &\le \sum_{n,m=1}^{\infty}\int_G N_\varphi(F_{(n,m)y})N_\psi(G_{(n,m)y})\,dy\\ &\le 2\sum_{n,m=1}^{\infty} N_\varphi(F_{(n,m)})N_\psi(G_{(n,m)}) \le 2\sum_{n,m=1}^{\infty}\|F_{(n,m)}\|_\varphi\|G_{(n,m)}\|_\psi\\ &\le 32\sum_{n=1}^{\infty}N_\varphi(f_n)N_\psi(g_n)\sum_{m=1}^{\infty}N_\varphi(l_m)N_\psi(h_m).\end{aligned}$$

Therefore,

$$\|uv\|_{\check{A}_{\varphi,K\cup F}(G)} \le 32\|u\|_{\check{A}_{\varphi,K}(G)}\|v\|_{\check{A}_{\varphi,F}(G)}. \ \blacksquare$$

The following is the main theorem of this section, which extends [C, Theorem 2]. For the proof we use some ideas from [C].

THEOREM 4.4. *If G is a locally compact group and φ is a Δ_2-regular N-function, then the dual of $\check{A}_\varphi(G)$ can be identified with* $\mathrm{Cv}_\varphi(G)$.

Proof. Let $T \in \mathrm{Cv}_\varphi(G)$. If $h \in \check{A}_\varphi(G)$, then there is a set K in $\mathcal{K}(G)$ with $h = \sum_{n=1}^{\infty} g_n * \check{f}_n$ such that all f_n and g_n are supported inside K. Set

$$\Phi_T(h) = \sum_{n=1}^{\infty}\langle Tf_n, g_n\rangle.$$

Then

$$\begin{aligned}|\Phi_T(h)| = \Big|\sum_{n=1}^{\infty}\langle Tf_n, g_n\rangle\Big| &\le 2\sum_{n=1}^{\infty}N_\varphi(Tf_n)N_\psi(g_n)\\ &\le 2\|T\|\sum_{n=1}^{\infty}N_\varphi(f_n)N_\psi(g_n) < \infty.\end{aligned}$$

It is apparent that Φ_T is linear. To show that $\Phi_T(h)$ is independent of the representation of h, it suffices to show that $\Phi_T(h) = 0$ whenever $h = 0$. Suppose that $K \in \mathcal{K}(G)$ and $h \in \check{A}_{\varphi,K}(G)$ with $h = \sum_{n=1}^{\infty} g_n * \check{f}_n = 0$. By Lemma 4.2, there exists a net $(e_\alpha) \subseteq C_c(G)$ such that

$$T(e_\alpha * f) = (Te_\alpha) * f = T_\alpha f \quad (f \in M^\varphi(G)).$$

For each α we have

$$\sum_{n=1}^{\infty}|\langle T_\alpha f_n, g_n\rangle| \le 2\sum_{n=1}^{\infty}N_\varphi(T_\alpha f_n)N_\psi(g_n) \le 2\|T\|\sum_{n=1}^{\infty}N_\varphi(f_n)N_\psi(g_n) < \infty.$$

Hence the series $\sum_{n=1}^{\infty}\langle T_\alpha f_n, g_n\rangle$ converges in the supremum norm, uniformly with respect to each α, and thus

$$\lim_\alpha \sum_{n=1}^{\infty}\langle T_\alpha f_n, g_n\rangle = \sum_{n=1}^{\infty}\lim_\alpha \langle T_\alpha f_n, g_n\rangle = \sum_{n=1}^{\infty}\langle Tf_n, g_n\rangle.$$

We also have

$$\sum_{n=1}^{\infty}\langle T_\alpha f_n, g_n\rangle = \sum_{n=1}^{\infty}\langle (Te_\alpha) * f_n, g_n\rangle = \sum_{n=1}^{\infty}\langle \chi_{K_1}\cdot Te_\alpha, g_n * \check{f}_n\rangle$$
$$= \Big\langle \chi_{K_1}\cdot Te_\alpha, \sum_{n=1}^{\infty} g_n * \check{f}_n\Big\rangle = 0,$$

where $K_1 = KK^{-1}$. Thus Φ_T is a well defined linear form on $\check{A}_\varphi(G)$ and $\|\Phi_T\| \le 2\|T\|$. Furthermore, we have

$$\begin{aligned}\|T\| &= \sup\{N_\varphi(Tf) : f \in C_c(G),\ N_\varphi(f) \le 1\}\\ &\le \sup\{|\langle Tf, g\rangle| : f, g \in C_c(G),\ N_\varphi(f) \le 1,\ N_\psi(g) \le 1\}\\ &\le \sup\{|\Phi_T(h)| : h = g * \check{f},\ \|h\|_{\check{A}_\varphi} \le 1\} \le \|\Phi_T\|.\end{aligned}$$

Hence, $\|T\| \le \|\Phi_T\| \le 2\|T\|$.

To complete the proof we must show that Φ is onto. Let $F \in \check{A}_\varphi(G)^*$ and $f \in C_c(G)$. For each $g \in C_c(G)$ define $F_f(g) = F(g * \check{f})$. Then

$$|F_f(g)| = |F(g * \check{f})| \le \|F\| N_\varphi(f) N_\psi(g).$$

Hence, F_f defines a continuous linear form on $C_c(G)$ considered as a subspace of $M^\psi(G)$. Since $C_c(G)$ is dense in $M^\psi(G)$ and $M^\psi(G)^* = L^\varphi(G) = M^\varphi(G)$, there exists a unique function $T(f) \in M^\varphi(G)$ such that

$$F_f(g) = F(g * \check{f}) = \langle Tf, g\rangle \quad \text{for each } g \in C_c(G),$$

and we have

$$N_\varphi(Tf) \le \|F\| N_\varphi(f) \quad \text{for each } f \in C_c(G).$$

Since $C_c(G)$ is dense in $M^\varphi(G)$, T can be extended to a continuous linear mapping on $M^\varphi(G)$ with $\|T\| \le \|F\|$. Furthermore, for each $f, g \in C_c(G)$ and $h \in L^\psi(G)$ we have

$$\langle (Tf) * g, h\rangle = \langle Tf, h * \check{g}\rangle = F_f(h * \check{g}) = F(h * \check{g} * \check{f}) = \langle T(f * g), h\rangle.$$

Since $h \in L^\psi(G)$ is arbitrary, we get $T \in \mathrm{Cv}_\varphi(G)$. ∎

By Lemma 4.3 and Theorem 4.4 we have:

Corollary 4.5. *Let $\overline{A}_\varphi(G)$ be the norm completion of $\check{A}_\varphi(G)$. Then $\mathrm{Cv}_\varphi(G)$ is the dual of $\overline{A}_\varphi(G)$, and $\overline{A}_\varphi(G)$ is a Banach algebra under pointwise multiplication.*

We denote by $C_0(G)$ the Banach space of all continuous functions on G vanishing at infinity. Since $C_c(G)$ is dense in both $M^\psi(G)$ and $M^\varphi(G)$, for any functions $f \in M^\varphi(G)$ and $g \in M^\psi(G)$, the function $g * \check{f}$ belongs to $C_0(G)$ with $\|g * \check{f}\|_\infty \le 2N_\varphi(f)N_\psi(g)$. Let $A_\varphi(G)$ be the range of the mapping

$$M^\varphi(G) \mathbin{\widehat{\otimes}} M^\psi(G) \to C_0(G), \quad f \otimes g \mapsto g * \check{f},$$

equipped with the quotient norm. Then $A_\varphi(G)$ becomes a Banach space and similar to Proposition 3.1.6 of [De], one can see that

$$A_\varphi(G) = \Big\{u \in C_0(G) : u = \sum_{n=1}^{\infty} g_n * \check{f}_n,\ (f_n), (g_n) \subseteq C_c(G),\\ \sum_{n=1}^{\infty} N_\varphi(f_n) N_\psi(g_n) < \infty\Big\},$$

and

$$\|u\|_{A_\varphi} = \inf\Big\{\sum_{n=1}^{\infty} N_\varphi(f_n)N_\psi(g_n) : u = \sum_{n=1}^{\infty} g_n * \check{f}_n\Big\}.$$

Since $C_c(G)$ is dense in $M^\varphi(G)$ and $M^\psi(G)$, the algebraic tensor product $C_c(G) \otimes C_c(G)$ is dense in $M^\varphi(G) \mathbin{\widehat{\otimes}} M^\psi(G)$. It follows that $A_\varphi(G) \cap C_c(G)$ is dense in $A_\varphi(G)$.

PROPOSITION 4.6. *Let G be a locally compact group and φ a Young function. Then $\check{A}_\varphi(G)$ is dense in $A_\varphi(G)$.*

Proof. Let Λ be a base for the neighborhoods of e, the identity element of G, consisting of compact sets. For each $V \in \Lambda$, set $e_V = \frac{1}{\lambda(V)}\chi_V$. Then similar to [R2, Proposition 1] one can show that $\{e_V : V \in \Lambda\}$ is an approximate identity for $M^\psi(G)$, when Λ is partially ordered downwards by set inclusion. Now let $u \in A_\varphi(G)$ with representation $u = \sum_{n=1}^{\infty} g_n * \check{f}_n$, where $(f_n), (g_n) \subseteq C_c(G)$. Given $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} N_\varphi(f_n)N_\psi(g_n) < \epsilon/2.$$

Also if we put $h = \sum_{n=1}^{n_0} g_n * \check{f}_n$, then there is a $V \in \Lambda$ such that

$$\|e_V * h - h\|_{A_\varphi} \le \sum_{n=1}^{n_0} N_\varphi(f_n)N_\psi(e_V * g_n - g_n) < \epsilon/2.$$

Consequently,

$$\|e_V * h - u\|_{A_\varphi} \le \|e_V * h - h\|_{A_\varphi} + \|u - h\|_{A_\varphi} < \epsilon. \ \blacksquare$$

For the notion of *amenability* of a locally compact group we refer the reader to [P]. Let us remark that a locally compact group G is amenable if and only if it satisfies *Leptin's condition*: for each $\varepsilon > 0$ and any compact subset $K \subseteq G$, there exists a measurable subset $U \subseteq G$ such that $0 < \lambda(U) < \infty$ and $\lambda(KU) < (1+\varepsilon)\lambda(U)$; see [P, Theorem 7.3].

LEMMA 4.7. *Let G be an amenable locally compact group, K a compact subset of G and φ an N-function. Then for every $\epsilon > 0$ there exist $0 \le f \in M^\varphi(G)$ and $0 \le g \in M^\psi(G)$ such that $u = g * \check{f} \in A_\varphi(G) \cap C_c(G)$,*

$\|u\|_{A_\varphi} < 1+\epsilon$ and $u = 1$ on K, where ψ is the complementary N-function to φ.

Proof. Take a compact neighborhood V of e, and let

$$u(x) = \frac{1}{\lambda(V)}(\chi_{KV} * \check{\chi}_V)(x) = \frac{\lambda(xV \cap KV)}{\lambda(V)}.$$

Then $u \in A_\varphi(G)$ and $0 \le u \le 1$. If $x \in K$, then $\lambda(xV \cap KV) = \lambda(xV) = \lambda(V)$, so that $u(x) = 1$, whereas if $x \notin KVV^{-1}$, then $xV \cap KV = \emptyset$ and hence $u(x) = 0$. Thus $\operatorname{supp} u \subseteq KVV^{-1}$ which is compact. Now, since G is amenable, by Leptin's condition we may choose V in such a way that $\lambda(KV) < (1+\epsilon)\lambda(V)$. Let $f = \frac{1}{\lambda(V)}\chi_V$, $g = \chi_{KV}$ and $u = g * \check{f}$. Then

$$\begin{aligned}
\|u\|_{A_\varphi} &\le \frac{1}{\lambda(V)} N_\psi(\chi_{KV}) N_\varphi(\chi_V) \\
&= \frac{1}{\lambda(V)} \left[\psi^{-1}\left(\frac{1}{\lambda(KV)}\right)\right]^{-1} \left[\varphi^{-1}\left(\frac{1}{\lambda(V)}\right)\right]^{-1} \\
&\le \frac{1}{\lambda(V)} \left[\psi^{-1}\left(\frac{1}{(1+\epsilon)\lambda(V)}\right)\right]^{-1} \left[\varphi^{-1}\left(\frac{1}{(1+\epsilon)\lambda(V)}\right)\right]^{-1} \\
&< 1+\epsilon. \ \blacksquare
\end{aligned}$$

As an immediate consequence of Theorem 4.4 we have the following.

COROLLARY 4.8. *If G is an amenable locally compact group and φ a Δ_2-regular N-function, then $\mathrm{Cv}_\varphi(G)$ can be identified with the dual of $A_\varphi(G)$.*

Proof. It is sufficient to show that the norms on $\check{A}_\varphi(G)$ and $A_\varphi(G)$ are equivalent on the dense subspace $\check{A}_\varphi(G)$. It is clear that $\|v\|_{A_\varphi} \le \|v\|_{\check{A}_\varphi}$ for all $v \in \check{A}_\varphi(G)$.

On the other hand, let $v \in \check{A}_\varphi(G)$ with $K = \mathrm{supp}(v)$. Given $\varepsilon > 0$, by Lemma 4.7, there exists $u \in C_c(G)$ such that $u = 1$ on K and $\|u\|_{A_\varphi} < 1+\varepsilon$. Thus, $v = vu$ and

$$\|v\|_{\check{A}_\varphi} = \|uv\|_{\check{A}_\varphi} \le 32(1+\varepsilon)\|v\|_{A_\varphi},$$

as the proof of Lemma 4.3. Hence $\|v\|_{A_\varphi} \le \|v\|_{\check{A}_\varphi} \le 32\|v\|_{A_\varphi}$. $\blacksquare$

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References

[C] M. Cowling, *The predual of the space of convolutors on a locally compact group*, Bull. Austral. Math. Soc. 57 (1998), 409–414.

[CHL] Y. Cui, H. Hudzik, and J. Li, *Some fundamental properties for duals of Orlicz spaces*, Nonlinear Anal. 73 (2010), 2353–2360.

[D] H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.

[Da] M. Daws, *Multipliers, self-induced and dual Banach algebras*, Dissertationes Math. 470 (2010), 62 pp.

[De] A. Derighetti, *Convolution Operators on Groups*, Lecture Notes Un. Mat. Ital. 11, Springer, Heidelberg, 2011.

[F] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, 1995.

[Gr] M. Grosser, *Bidualräume und Vervollständigungen von Banachmoduln*, Lecture Notes in Math. 717, Springer, Berlin, 1979.

[H] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer, New York, 1975.

[He] C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) 23 (1973), no. 3, 91–123.

[HR] E. Hewitt and K. Ross, *Abstract Harmonic Analysis II*, Springer, New York, 1970.

[HNR1] Z. Hu, M. Neufang and Z.-J. Ruan, *Completely bounded multipliers over locally compact quantum groups*, Proc. London Math. Soc. 103 (2011), 1–39.

[HNR2] Z. Hu, M. Neufang and Z.-J. Ruan, *Multipliers on a new class of Banach algebras, locally compact quantum groups, and topological centres*, Proc. London Math. Soc. 100 (2010), 429–458.

[JPU] P. Jain, L. E. Persson, and P. Upreti, *Inequalities and properties of some generalized Orlicz classes and spaces*, Acta Math. Hungar. 117 (2007), 161–174.

[K] L. A. Khan, *Topological modules of continuous homomorphisms*, J. Math. Anal. Appl. 343 (2008), 141–150.

[KR] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[L] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer, Berlin, 1971.

[La] A. T.-M. Lau, *Operators which commute with convolution on subspaces of $L^\infty(G)$*, Colloq. Math. 39 (1978), 351–359.

[M] T. Miao, *Predual of the multiplier algebra of $A_p(G)$ and amenability*, Canad. J. Math. 56 (2004), 344–355.

[P] J. P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984.

[R1] M. M. Rao, *Convolutions of vector fields-I*, Math. Z. 174 (1980), 63–79.

[R2] M. M. Rao, *Convolutions of vector fields-II: Random walk models*, Nonlinear Anal. 47 (2001), 3599–3615.

[R3] M. M. Rao, *Convolutions of vector fields, III. Amenability and spectral properties*, in: Real and Stochastic Analysis, Trends Math., Birkhäuser, Boston, 2004, 375–401.

[RR] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

[Ri1] M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Funct. Anal. 1 (1967), 443–491.

[Ri2] M. A. Rieffel, *Multiplires and tensor products on L^p-spaces of locally compact groups*, Studia Math. 33 (1969), 71–82.

[Ru] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[S] C. Swartz, *An Introduction to Functional Analysis*, Monogr. Textbooks Pure Appl. Math. 157, Dekker, New York, 1992.

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