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The evolution and Poisson kernels on nilpotent meta-abelian groups

by

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Abstract. Let S be a semidirect product $S = N \rtimes A$ where N is a connected and simply connected, non-abelian, nilpotent meta-abelian Lie group and A is isomorphic to $\mathbb{R}^k$, $k > 1$. We consider a class of second order left-invariant differential operators on S of the form $\mathcal{L}_\alpha = L^a + \Delta_\alpha$, where $\alpha \in \mathbb{R}^k$, and for each $a \in \mathbb{R}^k$, L^a is left-invariant second order differential operator on N and $\Delta_\alpha = \Delta - \langle \alpha, \nabla \rangle$, where Δ is the usual Laplacian on $\mathbb{R}^k$. Using some probabilistic techniques (e.g., skew-product formulas for diffusions on S and N respectively) we obtain an upper estimate for the transition probabilities of the evolution on N generated by $L^{\sigma(t)}$, where σ is a continuous function from $[0, \infty)$ to $\mathbb{R}^k$. We also give an upper bound for the Poisson kernel for $\mathcal{L}_\alpha$.

1. Introduction

1.1. The evolution kernel on NA groups. We say that a solvable Lie group S is an NA *group* if it is a semidirect product $S = N \rtimes A$ where N is a connected and simply connected nilpotent Lie group and A is isomorohic to $\mathbb{R}^k$. There is a remarkable probabilistic formula (formula (1.9) below) for the heat semigroup defined by a fairly general second order elliptic, or even degenerate elliptic, left-invariant, differential operator on an NA group that has long played a central role in their analysis. (See [4, 6, 9, 8, 17, 18, 19] for example.) The idea behind formula (1.9) goes back to [14, 15, 22].

To describe this formula in our context, let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of A and N respectively. In general, we identify connected, simply connected nilpotent Lie groups with their Lie algebras using the exponential map so that in particular, A and N are identified with $\mathfrak{a}$ and $\mathfrak{n}$. We assume that there is a basis $\mathcal{B} = \{X_1, \ldots, X_d\}$ of $\mathfrak{n}$ that diagonalizes the A-action. We

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typically think of the X_i as left-invariant differential operators on N. When thought of as left-invariant operators on S, they are denoted $\tilde{X}_i$. Thus, for $f \in C^\infty(S)$,

$$\tilde{X}_i f(n,a) = e^{\lambda_i(a)} X_i f(n,a) \tag{1.1}$$

where $\lambda_i \in \mathfrak{a}^*$ is the root functional corresponding to X_i, i.e., $[H, X_i] = \lambda_i(H) X_i$ for all $H \in \mathfrak{a}$. We also choose a basis $\{A_1, \ldots, A_k\}$ for $\mathfrak{a}$, which we use to identify $\mathfrak{a}$ with $\mathbb{R}^k$.

The Euclidean space $\mathbb{R}^k$ is endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding ℓ^2 norm $\| \cdot \|$. For the vector $x \in \mathbb{R}^k$ we write $x^2 = x \cdot x = \langle x, x \rangle = \sum_{i=1}^k x_i^2$. By $\| \cdot \|_\infty$, we denote the ℓ^∞ norm $\|x\|_\infty = \max_{1 \le i \le k} |x_i|$.

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, let

$$\mathcal{L}_\alpha = \sum_{j=1}^d \tilde{X}_j^2 + \sum_{j=1}^k (A_j^2 - 2\alpha_j A_j) = \sum_{j=1}^d e^{2\lambda_j(a)} X_j^2 + \Delta_\alpha, \tag{1.2}$$

where

$$\Delta_\alpha = \sum_{j=1}^k (\partial_{a_j}^2 - 2\alpha_j \partial_{a_j}). \tag{1.3}$$

For $a \in \mathbb{R}^k$ we let

$$\mathcal{L}_N^a = \sum_{j=1}^d e^{2\lambda_j(a)} X_j^2.$$

For $\sigma \in C^\infty([0,\infty), \mathbb{R}^k)$ and $s \le t < \infty$, let $P_{t,s}^\sigma(x)$, $x \in N$, be the fundamental solution for the operator

$$L = \partial_s + \mathcal{L}_N^{\sigma(s)}. \tag{1.4}$$

Thus $P_{t,s}^\sigma$ is a non-negative function on N such that

$$\int_N P_{t,s}^\sigma(x)\, dx = 1 \tag{1.5}$$

and, for $s \le u \le t$,

$$P_{t,u}^\sigma * P_{u,s}^\sigma = P_{t,s}^\sigma. \tag{1.6}$$

Moreover, if $\phi \in C_c^\infty(N)$ then

$$\phi * P_{t,s}^\sigma \equiv U_{s,t}^\sigma(\phi) \tag{1.7}$$

is the solution to the Dirichlet problem on $N \times (s, \infty)$ with boundary data ϕ, i.e.,

$$LU_{s,t}^\sigma(\phi) = 0 \quad \text{on } N \times (0,t), \qquad \lim_{t \to s^+} U_{s,t}^\sigma(\phi)(x) = \phi(x). \tag{1.8}$$

(For the existence of $P^{\sigma}_{t,s}$ see [6, 21].) In probabilistic terms, $P^{\sigma}_{t,s}$ is the kernel for the evolution defined by the time dependent family of operators $\mathcal{L}_N^{\sigma_s}$.

Of course, $U^{\sigma}_{s,t}$ can also be thought of as an integral operator. With obvious abuse of notation, we denote the corresponding kernel by $P^{\sigma}_{t,s}(x;y)$. Thus

$$P^{\sigma}_{t,s}(x;y) = P^{\sigma}_{t,s}(y^{-1}x).$$

For $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$, we put

$$T_t f(x,a) = \mathbf{E}_a U^{\sigma}_{0,t} f(x,\sigma_t) = \mathbf{E}_a (f *_N P^{\sigma}_{t,0})(x,\sigma(t)), \tag{1.9}$$

where the expectation is taken with respect to the distribution of the process $\sigma(t)$ (Brownian motion with drift) in $\mathbb{R}^k$ with generator Δ_α. The operator $U^{\sigma}(0,t)$ acts on the first variable of the function f (as a convolution operator).

We have the following

THEOREM 1.1. *The family T_t defined in* (1.9) *is the semigroup of operators generated by $\mathcal{L}_\alpha$. That is,*

$$\partial_t T_t f = \mathcal{L}_\alpha T_t f \quad \textit{and} \quad \lim_{t\to 0} T_t f = f.$$

Of course, the Brownian motion with a drift is an extremely well understood object. Clearly, then, a good understanding of $P^{\sigma}_{t,s}$ is key to understanding the heat semigroup as well as objects derived from it, such as the Poisson kernel.

It is not difficult to give an explicit formula for $P^{\sigma}_{t,s}$ when N is abelian. (See Proposition 2.9 below.) Our first main result is a skew-product formula for $P^{\sigma}_{t,s}$ (Theorem 1.2) similar to formula (1.9) that describes $P^{\sigma}_{t,s}$ on a meta-abelian group. Specifically, we assume that

$$N = M \rtimes V,$$

where M and V are abelian Lie groups with the corresponding Lie algebras $\mathfrak{m}$ and $\mathfrak{v}$. Let $\mathcal{B}_1 = \{Y_1,\dots,Y_m\}$ and $\mathcal{B}_2 = \{X_1,\dots,X_n\}$ be ordered bases of $\mathfrak{m}$ and $\mathfrak{v}$ respectively such that $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2$ forms an ordered Jordan–Hölder basis for the Lie algebra $\mathfrak{n}$ of N, ordered so that the matrix of ad_X in this basis is strictly lower triangular for all $X \in \mathfrak{n}$. We use $\mathcal{B}'$ in place of the basis $\mathcal{B}$ mentioned above (1.1). Hence, in this case,

$$\mathcal{L}_\alpha = \Delta_\alpha + \sum_{j=1}^{m} e^{2\xi_j(a)} Y_j^2 + \sum_{j=1}^{n} e^{2\vartheta_j(a)} X_j^2 = \Delta_\alpha + \mathcal{L}_N^a, \tag{1.10}$$

where $\xi_1,\dots,\xi_m$ and $\vartheta_1,\dots,\vartheta_n$ are the root functionals in $\mathfrak{a}^*$ corresponding to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively.

The time dependent family of operators

$$\mathcal{L}_V^{\sigma,t} = \sum_{j=1}^{n} e^{2\vartheta_j(\sigma(t))} X_j^2 \tag{1.11}$$

gives rise to an evolution on $V = \mathbb{R}^n$ that is described by a kernel $P_{t,s}^{V,\sigma}$ which may be explicitly computed, since V is abelian. In fact, it turns out that the process $\eta(t)$ generated by $\mathcal{L}_V^{\sigma,t}$ has coordinates $\eta_j(t)$ which are independent Brownian motions with time scaled by

$$A_{V,i}^{\sigma}(s,t) = \int_s^t e^{2\vartheta_j(\sigma(u))}\,du. \tag{1.12}$$

For $\eta \in C^\infty([0,\infty), V)$ let

$$\mathcal{L}_M^{\sigma,\eta,t} = \sum_{j=1}^{m} e^{2\xi_j(\sigma(t))}\big(\mathrm{Ad}(\eta(t))Y_j\big)^2.$$

This family of operators gives rise to an evolution on $M = \mathbb{R}^m$ that is described by a kernel $P_{t,s}^{M,\sigma,\eta}$ which may also be explicitly computed. Specifically, for $a \in A$, let $S(a)$ be the $m \times m$ matrix

$$S(a) = \mathrm{diag}[e^{\xi_1(a)}, \ldots, e^{\xi_m(a)}].$$

For $v \in V$, we identify $\mathrm{Ad}(v)|_{\mathfrak{m}}$ with the $m \times m$ matrix of this linear transformation with respect to the basis $\mathcal{B}_1$. Let

$$[a_M^{\sigma,\eta}(t)] = 2[\mathrm{Ad}(\eta(t))|_{\mathfrak{m}} S^\sigma(t)][\mathrm{Ad}(\eta(t))|_{\mathfrak{m}} S^\sigma(t)]^*,$$

where

$$S^\sigma(t) := S(\sigma(t)),$$

and

$$A_M^{\sigma,\eta}(s,t) = \int_s^t a_M^{\sigma,\eta}(u)\,du.$$

Finally, for a $d \times d$ invertible matrix A we set

$$\mathcal{B}(A)(x) = \tfrac{1}{2}A^{-1}x \cdot x \quad \text{and} \quad \mathcal{D}(A) = (2\pi)^{-d/2}(\det A)^{-1/2}. \tag{1.13}$$

We prove in §3.1 that for $m^1, m^2 \in M = \mathbb{R}^m$,

$$P_{t,s}^{M,\sigma,\eta}(m^1, m^2) = \mathcal{D}(A_M^{\sigma,\eta}(t,s))e^{-\mathcal{B}(A_M^{\sigma,\eta}(t,s))(m^1-m^2)}. \tag{1.14}$$

Our main tool is the following theorem. To the best of our knowledge, this result represents the first known formula for the evolution defined by $P_{t,s}^\sigma$ for a non-abelian N other than the similar result for the Heisenberg group from our work [19].

THEOREM 1.2. *Let $N = M \rtimes V$. For every $m \in M$ and $v \in V$ and a.e.* (*with respect to the corresponding Wiener measure*) *trajectory σ of the* *process generated by* Δ_α,

$$\begin{aligned}\int_N & P^\sigma_{t,0}(m,v;m',v')f(m',v')\,dm'\,dv' \\ &= \int\int_M P^{M,\sigma,\eta}_{t,0}(m,m')f(m,\eta(t))\,dm'\,d\mathbf{W}^{V,\sigma}_y(\eta) \\ &= \int\int_M \mathcal{D}(A^{\sigma,\eta}_M(0,t))e^{-\mathcal{B}(A^{\sigma,\eta}_M(0,t))(m-m')}f(m',\eta(t))\,dm'\,d\mathbf{W}^{V,\sigma}_y(\eta),\end{aligned}$$

where $\mathbf{W}^{V,\sigma}_y$ *is the product of* n *one-dimensional Wiener measures transformed according to* (1.12), *i.e., for the trajectory* $\eta(t) = (\eta_1(t),\dots,\eta_n(t))$ *its coordinates* $\eta_i(t)$ *are the one-dimensional Brownian motions* $b_i(t)$ *starting from* v_i *with their time changed to* $A^\sigma_{V,i}(0,t)$, *i.e.*,

$$\eta_i(t) = b_i(A^\sigma_{V,i}(0,t)).$$

Theorem 1.2 yields a new estimate on $P^\sigma_{t,0}$ which is our second main result. To state this result let, for a continuous function $\sigma : [0,\infty) \to A = \mathbb{R}^k$,

$$\begin{aligned} A^\sigma_{M,i}(s,t) &= \int_s^t e^{2\xi_i(\sigma(u))}\,du, \qquad i = 1,\dots,m, \\ A^\sigma_{V,j}(s,t) &= \int_s^t e^{2\vartheta_j(\sigma(u))}\,du, \qquad j = 1,\dots,n, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} A^\sigma_{M,\Sigma}(s,t) &= \sum_{i=1}^m A^\sigma_{M,j}(s,t), \qquad A^\sigma_{V,\Sigma}(s,t) = \sum_{j=1}^n A^\sigma_{V,j}(s,t), \\ A^\sigma_{M,\Pi}(s,t) &= \prod_{i=1}^m A^\sigma_{M,j}(s,t), \qquad A^\sigma_{V,\Pi}(s,t) = \prod_{j=1}^n A^\sigma_{V,j}(s,t). \end{aligned}$$

We also set

$$\begin{aligned} A^\sigma_{N,\Pi}(0,t) &= A^\sigma_{M,\Pi}(0,t)A^\sigma_{V,\Pi}(0,t), \\ A^\sigma_{N,\Sigma}(0,t) &= A^\sigma_{M,\Sigma}(0,t) + A^\sigma_{V,\Sigma}(0,t). \end{aligned}$$

We also let k_o be the smallest non-negative integer such that

$$(\mathrm{ad}_X)^{k_o+1}|_{\mathfrak{m}} = 0, \qquad \forall X \in \mathfrak{v}. \tag{1.16}$$

Note that if $k_o = 0$, then $\mathfrak{v}$ centralizes $\mathfrak{m}$; hence N is abelian. Thus our hypotheses imply that $k_o > 0$.

The following theorem is a simplified version of Theorem 4.1 which is one of our main results.

For, $a, b \in \mathbb{R}$ we write $a \wedge b = \min\{a, b\}$.

THEOREM 1.3. *There are positive constants C, D such that for all $(m, v) \in N = M \rtimes V$,*

$$P^{\sigma}_{t,0}(m,v) \le C(A^{\sigma}_{N,\Pi}(0,t))^{-1/2}\big(\|m\|^{1/(2k_0)} + 1 + A_{V,\Sigma}(0,t)^{1/2}\big) \times \exp\bigg(-D\frac{\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)} - D\frac{\|m\|^{1/k_0} \wedge \|m\|^2}{A^{\sigma}_{N,\Sigma}(0,t)}\bigg).$$

It is interesting to compare this result with what is known in the general case. The best general result that we are aware of in the literature is, when specialized to our current context, Theorem 1.4 below. (See [6, 8] and [17].) Theorem 1.3 is an improvement in two respects. First, it applies to all (m, v), not just points in a compact set not containing e. Secondly, it does not contain a term such as $\tau(x)/4$ in the exponent which is large when $\tau(x)$ is large. Of course, this term will be eventually dominated by the $\tau(x)^2$, but the point at which this domination takes place depends on the sizes of both $\tau(x)$ and $A^{\sigma}_{N,\Sigma}(0,t)$ which are very hard to control. We conjecture that a result such as Theorem 1.3 holds in general.

THEOREM 1.4. *Let $K \subset N$ be closed and $e \notin K$. Then there exist positive constants C_1, C_2, and c such that for every $x \in K$ and every t,*

$$P^{\sigma}_{t,0}(x) \le C_1\Big(\int_0^t (A^{\sigma}_{N,\Pi}(0,t))^{2/c}\,du\Big)^{-c/2} \exp\bigg(\frac{\tau(x)}{4} - \frac{\tau(x)^2}{C_2 A^{\sigma}_{N,\Sigma}(0,t)}\bigg),$$

where τ is a subadditive norm which is smooth on $N \setminus \{e\}$.

1.2. Poisson kernel for $\mathcal{L}_\alpha$. As mentioned above, we expect improved estimates for $P^{\sigma}_{t,0}$ to yield better estimates for objects derived from the heat semigroup such as the Poisson kernel. As an illustration of this we use Theorem 1.3 to prove Theorem 1.5 below that, in the current context, improves the estimates from [17] and [18]. (See §6.) This result is our final "main result." To state it we again require some notation.

Define

$$\rho_0 = \sum_{j=1}^{d} \lambda_j \tag{1.17}$$

and set

$$\chi(g) = \det(\mathrm{Ad}(g)) = e^{\rho_0(a)}, \tag{1.18}$$

where

$$\mathrm{Ad}(g)s = gsg^{-1}, \qquad s \in S.$$

Let ds be left-invariant Haar measure on S. We have

$$\int_S f(sg)\,ds = \chi(g)^{-1}\int_S f(s)\,ds.$$

Let

$$A^+ = \mathrm{Int}\{a \in \mathbb{R}^k : \lambda_j(a) \geq 0 \text{ for } 1 \leq j \leq r\}.$$

If $\alpha \in A^+$ then there exists a *Poisson kernel* ν for $\mathcal{L}_\alpha$ [4]. That is, there is a C^∞ function ν on N such that every bounded $\mathcal{L}_\alpha$-harmonic function F on S may be written as a *Poisson integral* against a bounded function f on $S/A = N$,

$$F(g) = \int_{S/A} f(gy)\nu(y)\,dy = \int_N f(y)\check{\nu}^a(y^{-1}x)\,dy, \quad \text{where} \quad g = (x, a),$$

and

$$\check{\nu}^a(z) = \check{\nu}(a^{-1}za)\chi(a)^{-1}, \quad \text{where} \quad \check{\nu}(z) = \nu(z^{-1}).$$

Conversely, the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}_\alpha$-harmonic function.

For $t \in \mathbb{R}^+$ and $\wp \in A^+$, let

$$\delta_t^\wp = \mathrm{Ad}((\log t)\wp)|_N.$$

Then $t \mapsto \delta_t^\wp$ is a one-parameter group of automorphisms of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [12] that then N has a $\delta_t^\wp$-homogeneous norm, a non-negative and subadditive continuous function $|\cdot|_\wp$ on N such that $|n|_\wp = 0$ if and only if $n = e$ and

$$|\delta_t^\rho x|_\wp = t|x|_\wp.$$

For many years the best pointwise estimate in higher rank available in the literature was

$$\nu(x) \leq C_\wp(1 + |x|_\wp)^{-\varepsilon}$$

for some $\varepsilon > 0$, where $\wp \in A^+$ [4, 5]. These results, however, provide no way of determining ε. (For estimates for the Poisson kernel and its derivatives on rank-one NA groups, i.e., with $\dim A = 1$, see [9, 8, 23, 3, 7].)

A formula for determining an appropriate value of ε was provided by the authors in [17, 18], although it is clear that the value of ε produced is far from best possible. Assume that the rank (dimension of A) is $k > 1$. Let ν be the Poisson kernel for the operator $\mathcal{L}_\alpha$ with $\alpha \in A^+$.

To simplify our notation we write Λ to denote the set of roots

$$\Lambda = \Xi \cup \Theta,$$

where

$$\Xi = \{\xi_1, \ldots, \xi_m\}, \qquad \Theta = \{\vartheta_1, \ldots, \vartheta_n\}.$$

For $\Lambda_o \subset \Lambda$ and $a \in A^+$ we set

$$\gamma_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \lambda(a), \qquad \overline{\gamma}_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \frac{\lambda(a)}{\lambda^2}. \tag{1.19}$$

In this setting our final main result is the following.

THEOREM 1.5. *Let ν be the Poisson kernel for the operator $\mathcal{L}_\alpha$, defined in* (1.10), *with $\alpha \in A^+$. Under the above assumptions on N, for every $\wp \in A^+$ and $\varepsilon > 0$ there exists a constant $C = C_{\wp,\varepsilon} > 0$ such that*

$$\nu(m, v) \le C(1 + |(m, v)|_\wp)^{-\gamma},$$

where

$$\gamma = \begin{cases} \gamma_\Theta(\rho)\overline{\gamma}_\Theta(\alpha) =: \gamma_1 & \text{for } \|m\| < \varepsilon,\ \|v\| \ge \varepsilon, \\ \gamma_\Lambda(\rho)\overline{\gamma}_\Lambda(\alpha) =: \gamma_2 & \text{for } \|m\| \ge \varepsilon,\ \|v\| < \varepsilon, \\ \max\{\gamma_1, \gamma_2\} & \text{for } \|m\| \ge \varepsilon,\ \|v\| \ge \varepsilon. \end{cases}$$

We provide an example in §6 demonstrating that this result does in fact provide a sharper estimate that the estimates found in [17] and [18].

1.3. Structure of the paper. The outline of the rest of the paper is as follows:

In §2.1 and §2.2 we recall some basic facts about exponential functionals of Brownian motion and some estimates for the joint distribution of the maximum of the absolute value of the Brownain motion on the time interval $[0, t]$ and its position at time t. In §2.3 we give a formula for the evolution kernels in the special case that the nilpotent group is $\mathbb{R}^n$. In §5.1 we recall the construction of the Poisson kernel ν on N and its extension $\nu^a(x)$ to $N \times \mathbb{R}^k$. In §3.1 and §3.2 we consider diffusions on M and V respectively. Theorem 1.2 is proved in §3. Our main results are proved in §4 (Theorem 1.3) and §5 (Theorem 1.5). In §6 we compare the estimate from Theorem 1.5 with our previous results from [17, 18].

2. Preliminaries

2.1. Exponential functionals of Brownian motion. Let $b(s)$, $s \ge 0$, be the Brownian motion on $\mathbb{R}$ starting from $a \in \mathbb{R}$ and normalized so that

$$\mathbf{E}_a f(b(s)) = \int_{\mathbb{R}} f(x + a) \frac{1}{\sqrt{4\pi s}} e^{-x^2/4s}\, dx. \tag{2.1}$$

Hence $\mathbf{E} b(s) = a$ and $\operatorname{Var} b(s) = 2s$.

REMARK. Our normalization of the Brownian motion $b(s)$ is different than that typically used by probabilists who tend to assume that $\operatorname{Var} b(s) = s$.

For $d > 0$ and $\mu > 0$ we define the following exponential functional

$$I_{d,\mu} = \int_0^\infty e^{d(b(s) - \mu s)}\, ds. \tag{2.2}$$

Such functionals are called *perpetual functionals* in financial mathematics and they play an important role there (see e.g. [11, 25]).

THEOREM 2.1 (Dufresne, [11]). *Let $b(0) = 0$. Then the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\gamma_{\mu/2}$ has a density $(1/\Gamma(\mu/2))x^{\mu/2-1}e^{-x}1_{[0,\infty)}(x)$.*

The proof of Dufresne's theorem can be found in many places. See for example [10, 9, 24] or the survey paper [16] and the references therein.

The *inverse gamma density* on $(0,\infty)$, with respect to dx, is defined by

$$h_{\mu,\gamma}(x) = C_{\mu,\gamma}x^{-\mu-1}e^{-\gamma/x}1_{(0,\infty)}(x),$$

where $C_{\mu,\gamma}$ is the normalizing constant such that $\int_0^\infty h_{\mu,\gamma}(x)\,dx = 1$.

As a corollary of Theorem 2.1, by scaling the Brownian motion and changing the variable, we get the following theorem.

THEOREM 2.2. *Let $b(0) = a$. Then*

$$\mathbf{E}_a f(I_{d,\mu}) = c_{d,\mu}e^{\mu a}\int_0^\infty f(x)x^{-\mu/d}\exp\left(-\frac{e^{da}}{d^2x}\right)\frac{dx}{x}.$$

In particular, $I_{2,\mu}$ has the inverse gamma density $h_{\mu/2,1/4}$.

We will also need the following lemma.

LEMMA 2.3. *Let $\sigma(u) = b(u) - 2\alpha u$ be the k-dimensional Brownian motion with a drift, $d > 0$, and let $\ell \in (\mathbb{R}^k)^*$ be such that $\ell(\alpha) > 0$. Then*

$$\mathbf{E}_a f\left(\int_0^\infty e^{d\ell(\sigma(u))}\,du\right) = c_{d,\ell,\alpha}e^{\gamma\ell(a)}\int_0^\infty f(u)u^{-\gamma/d}\exp\left(-\frac{e^{d\ell(a)}}{2d^2\ell^2u}\right)\frac{du}{u},$$

where $\gamma = 2\ell(\alpha)/\ell^2$. In particular, the functional $\int_0^\infty e^{d\ell(b(u)-2\alpha u)}\,du$ has the inverse gamma density $h_{2\ell(\alpha)/(d\ell^2),1/(d^2\ell^2)}$.

Proof. This follows from Theorem 2.2. See [17, Lemma 5.4] for details. ∎

2.2. Some probabilistic lemmas. If $b(t)$ is the Brownian motion starting from $x \in \mathbb{R}$ then the corresponding Wiener measure on the space $C([0,\infty),\mathbb{R})$ is denoted by $\mathbf{W}_x$. The following lemma follows from [1, formula 1.1.4, p. 125].

LEMMA 2.4. *There exists a constant $c > 0$ such that for all $x \le y$,*

$$\mathbf{W}_x\Big(\sup_{0<s<t}|b(s)| \ge y\Big) \le ce^{-(y-x)^2/4t}.$$

The following two equalities follow easily from the reflection principle for the Brownian motion [13].

LEMMA 2.5. *If $x > a > 0$, then*

$$\mathbf{W}_0\Big(\sup_{u\in[0,t]} b(u) \ge a \text{ and } b(t) \le x\Big) = 2\mathbf{W}_0(b(t) > a) - \mathbf{W}_0(b(t) > x),$$

whereas if $x < a$ with $a > 0$, then

$$\mathbf{W}_0\Big(\sup_{u\in[0,t]} b(u) \ge a \text{ and } b(t) \le x\Big) = \mathbf{W}_0\big(b(t) > 2a - x\big).$$

Let

$$\Phi(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{x} e^{-u^2/4}\, du.$$

LEMMA 2.6. *Let $t > 0$. For $a \ge 0$, $x, y \in \mathbb{R}$ with $x < y$, let*

$$R_1 = \{-a \le x < y \le a\}, \qquad R_2 = \{x < y < -a\},$$
$$R_3 = \{a < x < y\}, \qquad R_4 = \{0 < x < a < y\}.$$

Then

$$\mathbf{W}_0\Big(\sup_{u\in[0,t]} |b(u)| \ge a \text{ and } b(t) \in [x,y]\Big) \le \tag{2.3}$$
$$\begin{cases} 2\Phi\big(\frac{2a-x}{\sqrt{t}}\big) - 2\Phi\big(\frac{2a-y}{\sqrt{t}}\big) + 2\Phi\big(\frac{2a+y}{\sqrt{t}}\big) - 2\Phi\big(\frac{2a+x}{\sqrt{t}}\big) & \text{on } R_1,\\ 2\Phi\big(\frac{2a-x}{\sqrt{t}}\big) - 2\Phi\big(\frac{2a-y}{\sqrt{t}}\big) + \Phi\big(\frac{-x}{\sqrt{t}}\big) - \Phi\big(\frac{-y}{\sqrt{t}}\big) & \text{on } R_2,\\ \Phi\big(\frac{y}{\sqrt{t}}\big) - \Phi\big(\frac{x}{\sqrt{t}}\big) + 2\Phi\big(\frac{2a+y}{\sqrt{t}}\big) - 2\Phi\big(\frac{2a+x}{\sqrt{t}}\big) & \text{on } R_3,\\ 2\big(1 - \Phi\big(\frac{a}{\sqrt{t}}\big)\big) - \Phi\big(\frac{y}{\sqrt{t}}\big) - \Phi\big(\frac{2a-x}{\sqrt{t}}\big) + \Phi\big(\frac{2a+x}{\sqrt{t}}\big) - \Phi\big(\frac{2a+y}{\sqrt{t}}\big) & \text{on } R_4. \end{cases}$$

Proof. We use

$$\begin{aligned}\mathbf{W}_0\Big(\sup_{u\in[0,t]} |b(u)| \ge a \text{ and } b(t) \in [x,y]\Big) &\le \mathbf{W}_0\Big(\sup_{u\in[0,t]} b(u) \ge a \text{ and } b(t) \in [x,y]\Big)\\ &\quad + \mathbf{W}_0\Big(\sup_{u\in[0,t]} -b(u) \ge a \text{ and } -b(t) \in [-y,-x]\Big).\end{aligned}$$

Then the bound (2.3) on each set R_i follows from Lemma 2.5 by an easy calculation. ∎

COROLLARY 2.7. *Assume that $a > |n| + \delta$, $\delta > 0$, $\varepsilon < 1$, and $0 < \varepsilon/2 < \delta$. Then*

$$\begin{aligned}\varepsilon^{-1}\mathbf{W}_0\Big(\sup_{u\in[0,t]} |b(u)| \ge a \text{ and } b(t) \in [n - \varepsilon/2, n + \varepsilon/2]\Big)&\\ \le \frac{1}{\sqrt{\pi t}}\big(e^{-(2a-n)^2/(4t)} + e^{-(2a+n-\varepsilon/2)^2/(4t)}\big)&\\ \le \frac{1}{\sqrt{\pi t}}\big(e^{-(2a-n)^2/(4t)} + e^{-(2a+n-1/2)^2/(4t)}\big).&\end{aligned}$$

Proof. Let $x = n - \varepsilon/2$ and $y = n + \varepsilon/2$. Our hypotheses imply that $-a < x < y < a$. In particular

$$0 < (2a - y)/\sqrt{t} < (2a - x)/\sqrt{t} \quad \text{and} \quad 0 < (2a + x)/\sqrt{t} < (2a + y)/\sqrt{t}.$$

Hence, from Lemma 2.6,

$$\varepsilon^{-1}\mathbf{W}_0\Big(\sup_{u\in[0,t]} |b(u)| \ge a \text{ and } b(t) \in [n - \varepsilon/2, n + \varepsilon/2]\Big) \le \frac{1}{\sqrt{\pi t}}(e^{-(2a-x)^2/(4t)} + e^{-(2a+y)^2/(4t)}),$$

proving the corollary. ∎

COROLLARY 2.8. *Assume that $a \ge 0$. Then*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbf{W}_0\Big(\sup_{u\in[0,t]} |b(u)| \ge a \textit{ and } b(t) \in [n - \varepsilon/2, n + \varepsilon/2]\Big) \le \begin{cases} \dfrac{2}{\sqrt{\pi t}} e^{-(2a-|n|)^2/(4t)}, & |n| < a, \\ \dfrac{1}{\sqrt{4\pi t}} e^{-n^2/(4t)}, & 0 \le a \le |n|. \end{cases}$$

Proof. The first statement is immediate from Corollary 2.7. For the second statement note that

$$\mathbf{W}_0\Big(\sup_{u\in[0,t]} b(t) \ge a \text{ and } b(t) \in [n - \varepsilon/2, n + \varepsilon/2]\Big) \le \mathbf{W}_0(b(t) \in [n - \varepsilon/2, n + \varepsilon/2]) = \frac{1}{\sqrt{4\pi t}} \int_{n-\varepsilon/2}^{n+\varepsilon/2} e^{-u^2/(4t)}\, du,$$

from which the lemma follows. ∎

2.3. Evolution equations in $\mathbb{R}^n$. Let

$$L^t = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t)\partial_i\partial_j + \sum_{j=1}^{n} b_j(t)\partial_j \tag{2.4}$$

be a differential operator on $C^\infty(\mathbb{R}^n)$, where $\partial_i = \partial_{x_i}$ and $a(t) = [a_{ij}(t)]$ is a symmetric, positive definite matrix and the a_{ij} and b_j belong to $C([0,\infty), \mathbb{R})$. For $s \le t$, let $P_{t,s}$ be the fundamental solution for $L = \partial_s + L^s$ which is defined by formulas (1.4)–(1.6) and (1.8) where $\mathcal{L}^{\sigma(s)}$ is replaced by L^s. Let

$$A_{ij}(s,t) = \int_s^t a_{ij}(u)\, du \equiv A_{i,j}, \qquad B_j(s,t) = \int_s^t b_j(u)\, du \equiv B_j.$$

PROPOSITION 2.9. *Let $A = [A_{ij}]$ and $B = (B_1, \ldots, B_n)^t$. Then*

$$P_{t,s}(x) = (2\pi)^{-n/2}(\det A)^{-1/2} e^{-\frac{1}{2}(A^{-1}(x-B))\cdot(x-B)}. \tag{2.5}$$

Proof. For $f_o \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we write

$$f(x,t) = U(s,t)f_o(x) = f_o * P_{t,s}(x).$$

We note that

$$\partial_t f(x,t) = L^t f(x,t), \quad t > s,$$
$$f(x,s) = f_o(x).$$

We solve the above equation using the Fourier transform. See [19, Proposition 2.10] for details. ∎

3. Meta-abelian groups. Let the notation be as in §1.1. We consider a family $\{\Phi(a)\}_{a\in\mathbb{R}^k}$ of automorphisms of $\mathfrak{n}$ that leave $\mathfrak{m}$ and $\mathfrak{v}$ invariant. We identify linear transformations on $\mathfrak{n}$ with $(m+n)\times(m+n)$ matrices, allowing us to write

$$\Phi(a) = \begin{bmatrix} S(a) & 0 \\ 0 & T(a) \end{bmatrix},$$

where

$$S(a) = \mathrm{diag}[e^{\xi_1(a)}, \ldots, e^{\xi_m(a)}], \quad T(a) = \mathrm{diag}[e^{\vartheta_1(a)}, \ldots, e^{\vartheta_n(a)}].$$

We denote the diagonal entries of $S(a)$ and $T(a)$ by

$$s_i(a) = e^{\xi_i(a)}, \quad i = 1,\ldots,m, \quad t_j(a) = e^{\vartheta_j(a)}, \quad j = 1,\ldots,n.$$

Let σ be a continuous function from $[0,\infty)$ to $A = \mathbb{R}^k$, and denote

$$\Phi^\sigma(t) = \Phi(\sigma(t)), \quad S^\sigma(t) = S(\sigma(t)), \quad T^\sigma(t) = T(\sigma(t)). \tag{3.1}$$

For $Z \in \mathfrak{n}$ let

$$Z(t) = \Phi^\sigma(t)Z.$$

For $v \in V$ let

$$\mathcal{L}_M^{\sigma,v,t} = \sum_{j=1}^m \big(\mathrm{Ad}(v)Y_j(t)\big)^2. \tag{3.2}$$

Then

$$\mathcal{L}_N^{\sigma(t)} f(m,v) = \mathcal{L}_V^{\sigma,t} f(m,\cdot)|_v + \mathcal{L}_M^{\sigma,v,t} f(\cdot,v)|_m, \quad t \in \mathbb{R}^+,$$

is a family of left-invariant operators on N depending on $t \in \mathbb{R}^+$. Our aim is to estimate the evolution kernel $P_{t,s}^\sigma$ for the time dependent operator $\mathcal{L}_N^{\sigma(t)}$.

3.1. Evolution on M. We choose coordinates y_i for M for which Y_i corresponds to $\partial_i = \partial_{y_i}$, $1 \le i \le m$. Let $\eta \in C([0,\infty), V)$ and consider the evolution on M generated by the time dependent operator

$$\mathcal{L}_M^{\sigma,\eta,t} = \sum_{j=1}^m \big(\mathrm{Ad}(\eta(t))Y_j(t)\big)^2,$$

where

$$Y_j(t) = \Phi^\sigma(t)Y_j.$$

Then

$$\mathrm{Ad}(\eta(t))Y_j(t) = \mathrm{Ad}(\eta(t))\Phi^\sigma(t)Y_j = \sum_{k=1}^{m} \psi_{j,k}(t)Y_k,$$

and consequently,

$$\sum_{j=1}^{m}\big(\mathrm{Ad}(\eta(t))Y_j(t)\big)^2 = \sum_{k,l=1}^{m}\sum_{j=1}^{m} \psi_{k,j}(t)\psi_{l,j}(t)Y_kY_l = \sum_{k,l=1}^{m}\big(\Psi(t)\Psi(t)^*\big)_{kl}Y_kY_l,$$

where $\Psi(t) = [\psi_{i,j}(t)]$ is the matrix of the operator $\mathrm{Ad}(\eta(t))\Phi^\sigma(t)|_M$. Thus the matrix $[a_{ij}(t)]$ from (2.4) for the operator $\mathcal{L}_M^{\sigma,\eta}$ is

$$[a_{ij}(t)] = 2\Psi(t)\Psi(t)^* = 2\,\mathrm{Ad}(\eta(t))\Phi^\sigma(t)|_M\big(\mathrm{Ad}(\eta(t))\Phi^\sigma(t)|_M M\big)^*.$$

It follows from Proposition 2.9 that the evolution kernel $P_{t,s}^{M,\sigma,\eta}$ for the operator $\mathcal{L}_M^{\sigma,\eta,t}$ is Gaussian, and in our notation, it is given by

$$P_{t,s}^{M,\sigma,\eta}(m,m') = \mathcal{D}(A_M^{\sigma,\eta}(t,s))e^{-\mathcal{B}(A_M^{\sigma,\eta}(t,s))(m-m')}, \tag{3.3}$$

where $m,m' \in M = \mathbb{R}^{\dim M}$, and $\mathcal{D},\mathcal{B}$ are defined in (1.13). We will need the following two lemmas:

Lemma 3.1. *Let A be a positive semidefinite symmetric matrix. Then*

$$\mathcal{B}(A)(x) \geq \frac{\|x\|^2}{2\|A\|},$$

where $\|A\|$ is the $\ell^2 \to \ell^2$ operator norm.

Proof. See e.g. [19, Lemma 4.1]. ∎

Lemma 3.2. *Let M and D be square matrices and let*

$$A = \begin{bmatrix} M & B \\ C & D \end{bmatrix}.$$

If $\det M \neq 0$ *then* $\det A = \det M \det(D - CM^{-1}B)$.

Proof. See e.g. [26]. ∎

Now we prove an upper bound on $\mathcal{D}(A_M^{\sigma,\eta}(s,t))$ that is independent of η. For simplicity of notation we identify M, V, and N with $\mathfrak{m}$, $\mathfrak{v}$, and $\mathfrak{n}$ using the exponential map.

Lemma 3.3. *There is a constant $C > 0$ such that*

$$\mathcal{D}(A_M^{\sigma,\eta}(s,t)) \leq C\Big(\prod_{i=1}^{m}\int_s^t s_i^\sigma(u)^2\,du\Big)^{-1/2} = CA_{M,\Pi}^\sigma(s,t)^{-1/2},$$

where $s_i^\sigma(t)$ are the entries of the diagonal matrix $S^\sigma(t)$ defined in (3.1).

Proof. We omit the t and σ dependence for the sake of simplicity. From the lower triangularity of the adjoint action of $\mathfrak{n}$, for $X \in \mathfrak{n} = N$,

$$\mathrm{ad}_X = \begin{bmatrix} X_o & 0 \\ v^{\mathrm{t}} & 0 \end{bmatrix}, \qquad \mathrm{Ad}_X = e^{\mathrm{ad}_X} = \begin{bmatrix} e^{X_o} & 0 \\ v(X)^{\mathrm{t}} & 1 \end{bmatrix},$$

where X_o is an $(m-1) \times (m-1)$-matrix and v is an $(m-1) \times 1$-column vector.

Then

$$\mathrm{Ad}_X\, S = e^{\mathrm{ad}_X} \begin{bmatrix} S_0 & 0 \\ 0 & s_m \end{bmatrix} = \begin{bmatrix} e^{X_o} S_o & 0 \\ v(X)^t S_o & s_m \end{bmatrix}. \tag{3.4}$$

Let

$$F^t = v(X)^t S_o.$$

Thus

$$\mathrm{Ad}_X\, S (\mathrm{Ad}_X\, S)^t = \begin{bmatrix} e^{X_o} S_o S_o^t e^{X_o^t} & G \\ G^t & s_m^2 + |F|^2 \end{bmatrix},$$

where

$$G = e^{X_o} S_o F = e^{X_o} S_o S_o^t v(X).$$

Hence,

$$A_M^{\sigma,\eta}(s,t) = 2 \begin{bmatrix} A_o & B \\ B^t & A + E \end{bmatrix},$$

where

$$A_o = \int_s^t e^{X_o(u)} S_o(u) S_o(u)^t e^{X_o(u)^t}\, du, \qquad B = \int_s^t G(u)\, du,$$

$$A = \int_s^t s_m^2(u)\, du, \qquad E = \int_s^t |F(u)|^2\, du.$$

From Lemma 3.2,

$$\begin{aligned} 2^{-m} \det A_M^{\sigma,\eta}(s,t) &= (\det A_o)(A + E - B^t A_o^{-1} B) \\ &= (\det A_o) A + (\det A_o)(E - B^t A_o^{-1} B) \\ &= (\det A_o) A + \det \begin{bmatrix} A_o & B \\ B^t & E \end{bmatrix}. \end{aligned}$$

The determinant on the right is non-negative since it is the $s_m = 0$ case of formula (3.4). Hence,

$$2^{-m} \det A_M^{\sigma,\eta}(s,t) \geq A \det A_o.$$

Our result follows by induction. ■

Now we estimate the operator norm of the matrix

$$A_M^{\sigma,\eta}(0,t) = 2\int_0^t [\mathrm{Ad}(\eta(u))S^\sigma(u)][\mathrm{Ad}(\eta(u))S^\sigma(u)]^* \, du. \tag{3.5}$$

LEMMA 3.4. *Let $\eta = \eta(u) = (\eta_1(u), \ldots, \eta_n(u))$ be a continuous function. Then there exists a constant $C > 0$ such that*

$$\|A_M^{\sigma,\eta}(0,t)\| \le C\big(1 + \Lambda^\eta(0,t)^{2k_o}\big) \sum_{j=1}^m \int_0^t s_j^\sigma(u)^2 \, du,$$

where

$$\Lambda^\eta(s,t) = \sup_{s\le u\le t} \|\eta(u)\|_\infty.$$

Proof. (We recall that $\|\cdot\|$ denotes the ℓ^2-norm.) We note first that for $X \in \mathfrak{n}$,

$$\begin{aligned} \mathrm{Ad}_X|_{\mathfrak{m}} &= \sum_{j=0}^{k_o} \frac{(\mathrm{ad}_X|_{\mathfrak{m}})^j}{j!}, \\ \|\mathrm{Ad}_X|_{\mathfrak{m}}\| &\le C(1 + \|\mathrm{ad}_X\|)^{k_o} \le C'(1 + \|X\|)^{k_o} \le C''(1 + \|X\|_\infty)^{k_o}. \end{aligned} \tag{3.6}$$

Our result follows by bringing the norm inside the integral in (3.5). ∎

3.2. Evolution on V. Recall that we identified V with $\mathbb{R}^n$. The matrix $T^\sigma(t) = \Phi^\sigma(t)|_V$ is of the form

$$T^\sigma(t) = \mathrm{diag}[e^{\vartheta_1(\sigma(t))}, \ldots, e^{\vartheta_n(\sigma(t))}],$$

where $\vartheta_1, \ldots, \vartheta_n \in (\mathbb{R}^n)^*$. Now we consider the evolution process $\eta(t)$ on V generated by

$$\mathcal{L}_V^{\sigma,t} = \sum_{j=1}^n X_j(t)^2 = \sum_{j=1}^n (T^\sigma(t)X_j)^2$$

(see the notation introduced in (1.11) on p. 72). Thus, since $X_j = \partial_{v_j}$,

$$\mathcal{L}_V^{\sigma,t} = \sum_{j=1}^n e^{2\vartheta_j(\sigma(t))}\partial_{v_j}^2.$$

The matrix $a(t) = [a_{ij}(t)]$, defined in (2.4), for $\mathcal{L}_V^{\sigma,t}$ is equal to

$$a_V^\sigma(t) = 2T^\sigma(t)T^\sigma(t)^* = 2\,\mathrm{diag}[e^{2\vartheta_1(\sigma(t))}, \ldots, e^{2\vartheta_n(\sigma(t))}].$$

Let $b(t)$ be the 1-dimensional Brownian motion normalized so that

$$\mathbf{W}_x(b(t) \in dy) = p_t(x, dy) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-y)^2/(4t)} \, dy.$$

Then, by (2.5),

$$P_{s,t}^{V,\sigma}(x, dz) = \prod_{1\le j\le n} p_{A_{V,i}^\sigma(s,t)}(x_j, dz_j). \tag{3.7}$$

Thus the process $\eta(t)$ generated by $\mathcal{L}_V^{\sigma,t}$ has coordinates $\eta_j(t)$ which are independent Brownian motions with time changed according to the clock governed by σ. Let

$$A_V^\sigma(0,t) = \int_0^t a_V^\sigma(u)\,du. \tag{3.8}$$

Since $A_V^\sigma(0,t)$ is diagonal we see

$$\begin{aligned}(\det A_V^\sigma(0,t))^{-1/2} &= \Big(\prod_{j=1}^n \int_0^t e^{2\vartheta_j(\sigma(u))}\,du\Big)^{-1/2},\\ \|A_V^\sigma(0,t)\| &\le \sum_{j=1}^n \int_0^t e^{2\vartheta_j(\sigma(u))}\,du = A_{V,\Sigma}^\sigma(0,t).\end{aligned} \tag{3.9}$$

3.3. Proof of Theorem 1.2. Theorem 1.2 follows from formula (3.3) together with [19, Corollary 3.5] and formula [19, (3.1)] with $n = 1$.

4. Estimate for the evolution on N.

In this section we estimate the evolution kernel on $N = M \rtimes V$. Denote

$$P_{t,s}^\sigma(m,v) := P_{t,s}^\sigma(0,0;m,v).$$

The main result of this section is the following estimate where k_o is as in (1.16).

THEOREM 4.1. *There are positive constants C, D such that*

$$\begin{aligned}&A_{M,\Pi}^\sigma(0,t)^{1/2} A_{V,\Pi}^\sigma(0,t)^{1/2} P_{t,0}^\sigma(m,v)\\ &\le C(\|m\|^{1/(2k_o)}+1)\exp\Big(-\frac{D\|v\|^2}{A_{V,\Sigma}^\sigma(0,t)} - \frac{D\|m\|^2}{(\|m\|^{1/(2k_o)}+\|v\|+2)^{2k_o} A_{M,\Sigma}^\sigma(0,t)}\Big)\\ &\quad + CA_{V,\Sigma}^\sigma(0,t)^{1/2}\exp\Big(-D\frac{\|m\|^{1/k_o}+\|v\|^2}{A_{V,\Sigma}^\sigma(0,t)}\Big).\end{aligned}$$

REMARK. One can also consider $P_{t,s}^\sigma$ in Theorem 4.1. Then in the estimate all exponential functionals $A_{\cdot,\cdot}^\sigma(0,t)$ must be replaced by $A_{\cdot,\cdot}^\sigma(s,t)$. The proof remains essentially the same.

Proof of Theorem 4.1. We allow the constants C and D to change from line to line. By Lemmas 3.1 and 3.3,

$$\begin{aligned}P_{t,s}^{M,\sigma,\eta}(m,m') &= \mathcal{D}(A_M^{\sigma,\eta}(s,t))e^{-\mathcal{B}(A_M^{\sigma,\eta}(s,t))(m-m')}\\ &\le CA_{M,\Pi}^\sigma(s,t)^{-1/2} e^{-\frac{\|m-m'\|^2}{2\|A_M^{\sigma,\eta}(s,t)\|}}.\end{aligned} \tag{4.1}$$

From Theorem 1.2, for $m, m' \in M$ and $v, v' \in V$,

$$\int_V P^{\sigma}_{t,0}(m, v; m', v')\psi(v')\,dv' = \int P^{M,\sigma,\eta}_{t,0}(m, m')\psi(\eta(t))\,d\mathbf{W}^{V,\sigma}_v(\eta)$$

$$\le C A^{\sigma}_{M,\Pi}(0,t)^{-1/2} \int \psi(\eta(t)) e^{-\frac{\|m-m'\|^2}{2\|A^{\sigma,\eta}_M(0,t)\|}}\,d\mathbf{W}^{V,\sigma}_v(\eta).$$

Then, by Lemma 3.4,

$$A^{\sigma}_{M,\Pi}(0,t)^{1/2} \int_V P^{\sigma}_{t,0}(m, v)\psi(v)\,dv \le C \int \exp\left(-\frac{D\|m\|^2}{(1 + \Lambda^{\eta}(0,t)^{2k_o})A^{\sigma}_{M,\Sigma}(0,t)}\right)\psi(\eta(t))\,d\mathbf{W}^{V,\sigma}_0(\eta). \tag{4.2}$$

For $v \in \mathbb{R}^n$ given and $\varepsilon > 0$, let

$$\psi_\varepsilon(\cdot) = \varepsilon^{-n}\mathbf{1}_{B_\varepsilon(v)}(\cdot),$$

where

$$B_\varepsilon(v) = \prod_{j=1}^{n} B^1_\varepsilon(v_j) \quad \text{and} \quad B^1_\varepsilon(v_j) = [v_j - \varepsilon/2, v_j + \varepsilon/2].$$

We will estimate (4.2) with ψ_ε in place of ψ as ε tends to zero.

Let $\mathbf{E}^{\eta}_v$ denote expectation with respect to $d\mathbf{W}^{V,\sigma}_v(\eta)$. For $k = 1, 2, \ldots,$ define the sets of paths in V,

$$\mathcal{A}_k(t) = \Big\{\eta : k - 1 \le \Lambda^{\eta}(0,t) = \sup_{0\le u\le t} \|\eta(u)\|_\infty < k\Big\},$$

where by $\|\cdot\|_\infty$ we denote the maximum norm $\|x\|_\infty = \max_{1\le i\le n} |x_i|$. The integral on the right in (4.2) can be written as an infinite sum and estimated as follows:

$$\sum_{k=1}^{\infty} \mathbf{E}^{\eta}_0 \exp\left(-\frac{D\|m\|^2}{(1 + \Lambda^{\eta}(0,t)^{2k_o})A^{\sigma}_{M,\Sigma}(0,t)}\right)\psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k(t)}(\eta) \le \sum_{k=1}^{\infty} \exp\left(-\frac{D\|m\|^2}{k^{2k_o} A^{\sigma}_{M,\Sigma}(0,t)}\right)\mathbf{E}^{\eta}_0\psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k(t)}(\eta). \tag{4.3}$$

To simplify notation we introduce

$$c_k = \exp\left(-\frac{D\|m\|^2}{k^{2k_o} A^{\sigma}_{M,\Sigma}(0,t)}\right),$$

$$\mathcal{E}_k(\varepsilon) = \mathbf{E}^{\eta}_0\psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k(t)}(\eta) = \varepsilon^{-n}\mathbf{W}^{V,\sigma}_0\big(\eta \in \mathcal{A}_k(t) \text{ and } \eta(t) \in B_\varepsilon(v)\big).$$

Let $v \ne 0$ and choose $\varepsilon/2 < \|v\|_\infty$. If $\eta \in \mathcal{A}_k(t)$ and $\eta(t) \in B_\varepsilon(v)$ then $\|\eta(t)\|_\infty \ge \|v\|_\infty - \varepsilon/2$. Hence,

$$\mathcal{E}_k = 0 \quad \text{for } k < \|v\|_\infty - \varepsilon/2. \tag{4.4}$$

Let, for $k = 1, 2, \ldots$,

$$\Lambda^{\eta_j}(0,t) = \sup_{0\le u\le t} |\eta_j(u)| \quad \text{and} \quad \mathcal{A}_k^j(t) = \{\eta : k-1 \le \Lambda^{\eta_j}(0,t) < k\}.$$

Since the coordinates $\eta_j(t)$ of $\eta(t)$ are independent, we can estimate

$$\begin{aligned}
(4.5)\qquad \mathcal{E}_k(\varepsilon) &= \varepsilon^{-n}\mathbf{W}_0^{V,\sigma}\big(\eta \in \mathcal{A}_k(t) \wedge \eta(t) \in B_\varepsilon(v)\big) \\
&\le \varepsilon^{-n}\sum_{j=1}^n \mathbf{W}_0^{V,\sigma}\big(\eta \in \mathcal{A}_k^j(t) \wedge \eta(t) \in B_\varepsilon(v)\big) \\
&= \varepsilon^{-n}\sum_{j=1}^n \mathbf{W}_0^{V,\sigma}\big(\eta \in \mathcal{A}_k^j(t) \wedge \eta_j(t) \in B_\varepsilon^1(v_j)\big)\mathbf{W}_0^{V,\sigma}\big(\forall i \ne j, \eta_i(t) \in B_\varepsilon^1(v_i)\big) \\
&= \sum_{j=1}^n \varepsilon^{-1}\mathbf{W}_0\big(\eta \in \mathcal{A}_k^j(A_{V,j}^\sigma(0,t)) \wedge \eta_j(A_{1,j}^\sigma(0,t)) \in B_\varepsilon^1(v_j)\big) \\
&\quad \times \prod_{i\ne j}\big(\varepsilon^{-1}\mathbf{W}_0\big(\eta_i(A_{V,i}^\sigma(0,t)) \in B_\varepsilon^1(v_i)\big)\big).
\end{aligned}$$

LEMMA 4.2. *Assume that $a > \|v\|_\infty + \delta$, $\delta > 0$, and $0 < \varepsilon/2 < \delta$. Then*

$$\begin{aligned}
&\varepsilon^{-n}\mathbf{W}_0^{V,\sigma}\Big(\sup_{u\in[0,t]} \|\eta(u)\|_\infty \ge a \text{ and } \eta(t) \in B_\varepsilon(v)\Big) \\
&\qquad \le A_{V,\Pi}^\sigma(0,t)^{-1/2}\sum_{j=1}^n\big(e^{-(2a-v_j)^2/(4A_{V,j}^\sigma(0,t))} + e^{-(2a+v_j-\varepsilon/2)^2/(4A_{V,j}^\sigma(0,t))}\big) \\
&\qquad \le A_{V,\Pi}^\sigma(0,t)^{-1/2}\sum_{j=1}^n\big(e^{-(2a-v_j)^2/(4A_{V,j}^\sigma(0,t))} + e^{-(2a+v_j-1/2)^2/(4A_{V,j}^\sigma(0,t))}\big).
\end{aligned}$$

Proof. Reasoning as in (4.5) we see that the left side of the above inequality is bounded by

$$\begin{aligned}
&C\sum_{j=1}^n\Big(\prod_{i\ne j} A_{V,i}^\sigma(0,t)\Big)^{-1/2} \\
&\qquad \times \varepsilon^{-1}\mathbf{W}_0\Big(\sup_{u\in[0,A_{V,j}^\sigma(0,t)]} |\eta_j(u)| \ge a \text{ and } \eta_j(A_{V,j}^\sigma(0,t)) \in B_\varepsilon^1(v_j)\Big).
\end{aligned}$$

By our assumption it follows that for every j, $a > |v_j| + \delta$. Hence, the result follows by Corollary 2.7. ■

LEMMA 4.3. *We have*

$$A_{M,\Pi}^\sigma(0,t)^{1/2}P_{t,0}^\sigma(m,v) \le CI, \quad \text{where} \quad I = \limsup_{\varepsilon\to 0^+}\sum_{k\ge\|v\|_\infty} c_k\mathcal{E}_k(\varepsilon).$$

Furthermore, the sum converges uniformly in ε.

Proof. The inequality follows by letting ε tend to 0 in (4.3). The uniform convergence follows from Lemma 4.2. ∎

Let n_o be the smallest natural number such that $n_o \geq \|v\|_\infty$.

LEMMA 4.4. *We have the following estimates:*

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_{n_o}(\varepsilon) \leq CA^\sigma_{V,\Pi}(0,t)^{-1/2} e^{-\|v\|^2_\infty/(4A^\sigma_{V,\Sigma}(0,t))},$$

while for $k \geq n_o + 1$,

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_k(\varepsilon) \leq CA^\sigma_{V,\Pi}(0,t)^{-1/2} \exp\biggl(-\frac{(2(k-1)-\|v\|_\infty)^2}{4A^\sigma_{V,\Sigma}(0,t)}\biggr).$$

Proof. Consider $\mathcal{E}_{n_o}$. Let $j \in \{1,\ldots,n\}$ be fixed. Suppose first that $|v_j| < n_o - 1$. Then, using Corollary 2.8, the jth term in (4.5) (with $k = n_o$) is dominated by a multiple of

$$A^\sigma_{V,\Pi}(0,t)^{-1/2} e^{-\frac{(2(n_o-1)-|v_j|)^2}{4A^\sigma_{V,j}(0,t)}} \prod_{i\neq j} e^{-\frac{|v_i|^2}{4A^\sigma_{V,i}(0,t)}}.$$

Notice that $|v_j|$ cannot be equal to $\|v\|_\infty$. Thus we are done in this case. Now suppose that $|v_j| \geq n_o - 1$. Then, using Corollary 2.8 again, we dominate the jth term in (4.5) by a multiple of

$$A^\sigma_{V,\Pi}(0,t)^{-1/2} e^{-\frac{|v_j|^2}{4A^\sigma_{V,j}(0,t)}} \prod_{i\neq j} e^{-\frac{|v_i|^2}{4A^\sigma_{V,i}(0,t)}}.$$

The result for $\mathcal{E}_{n_o}$ follows.

Now we consider $\mathcal{E}_k$. Since $k \geq n_o + 1$ it follows that $k - 1 \geq |v_j|$ for every j. Therefore, by Corollary 2.8 the jth term in (4.5) is estimated by

$$CA^\sigma_{V,\Pi}(0,t)^{-1/2} e^{-\frac{(2(k-1)-|v_j|)^2}{4A^\sigma_{V,j}(0,t)}} \prod_{i\neq j} e^{-\frac{|v_i|^2}{4A^\sigma_{V,i}(0,t)}}. \ ∎$$

Next we estimate $I = \limsup_{\varepsilon\to 0^+} \sum_{k\geq\|v\|_\infty} c_k\mathcal{E}_k(\varepsilon)$. From Lemma 4.4,

$$\begin{aligned}(4.6)\qquad A^\sigma_{V,\Pi}(0,t)^{1/2} I &= A^\sigma_{V,\Pi}(0,t)^{1/2} \limsup_{\varepsilon\to 0^+}\Bigl(c_{n_o}\mathcal{E}_{n_o}(\varepsilon) + \sum_{k\geq n_o+1} c_k\mathcal{E}_k(\varepsilon)\Bigr)\\ &\leq C\exp\biggl(-\frac{\|v\|^2_\infty}{2A^\sigma_{V,\Sigma}(0,t)} - \frac{D\|m\|^2}{n_o^{2k_o}A^\sigma_{M,\Sigma}(0,t)}\biggr)\\ &\quad + \sum_{k=n_o+1}^\infty \exp\biggl(-\frac{D\|m\|^2}{k^{2k_o}A^\sigma_{M,\Sigma}(0,t)} - \frac{(2(k-1)-\|v\|_\infty)^2}{2A^\sigma_{V,\Sigma}(0,t)}\biggr).\end{aligned}$$

For a, b non-negative, $a+b \geq \sqrt{a^2+b^2}$. Also, for $k \geq n_o+1$,

$$(k-1)+(k-1)-\|v\|_\infty \geq n_o+(k-1-\|v\|_\infty),$$
$$k-1-\|v\|_\infty \geq n_o-\|v\|_\infty \geq 0.$$

Hence the summation in the last line of (4.6) is bounded by

$$\begin{aligned}(4.7)\qquad &\sum_{k=n_o+1}^{\infty} \exp\Big(-\frac{D\|m\|^2}{k^{2k_o}A^\sigma_{M,\Sigma}(0,t)} - \frac{(n_o+(k-1)-\|v\|_\infty)^2}{4A^\sigma_{V,\Sigma}(0,t)}\Big) \\ &\leq e^{-\frac{n_o^2}{4A^\sigma_{V,\Sigma}(0,t)}} \sum_{k=n_o+1}^{\infty} \exp\Big(-\frac{D\|m\|^2}{k^{2k_o}A^\sigma_{M,\Sigma}(0,t)} - \frac{(k-1-\|v\|_\infty)^2}{4A^\sigma_{V,\Sigma}(0,t)}\Big).\end{aligned}$$

We split the sum in (4.7) into two parts: $n_o+1 \leq k \leq n_o+\|m\|^{1/(2k_o)}$ and $k > n_o+\|m\|^{1/(2k_o)}$, and estimate the corresponding parts by the following two terms:

$$\|m\|^{1/(2k_o)} e^{-\frac{n_o^2}{4A^\sigma_{V,\Sigma}(0,t)}} \exp\Big(-\frac{D\|m\|^2}{(\|m\|^{1/(2k_o)}+\|v\|_\infty+2)^{2k_o}A^\sigma_{M,\Sigma}(0,t)}\Big)$$

and

$$e^{-\frac{n_o^2}{4A^\sigma_{V,\Sigma}(0,t)}} \sum_{k\geq n_o+\|m\|^{1/(2k_o)}+1} \exp\Big(-\frac{D\|m\|^2}{k^2 A^\sigma_{M,\Sigma}(0,t)} - \frac{(k-1-n_o)^2}{4A^\sigma_{V,\Sigma}(0,t)}\Big).$$

The above expression is bounded by

$$e^{-\frac{n_o^2}{4A^\sigma_{V,\Sigma}(0,t)}} \int_{\|m\|^{1/(2k_o)}}^{\infty} e^{-\frac{r^2}{4A^\sigma_{V,\Sigma}(0,t)}} dr \leq 2A^\sigma_{V,\Sigma}(0,t)^{1/2} e^{-\frac{\|v\|_\infty^2}{4A^\sigma_{V,\Sigma}(0,t)} - \frac{\|m\|^{1/k_o}}{4A^\sigma_{V,\Sigma}(0,t)}}.$$

Hence, by Lemmas 4.3 and 4.4, Theorem (4.1) follows. ∎

4.1. Proof of Theorem 1.3. To simplify our notation we set

$$(4.8)\qquad \begin{aligned} A_0 &= A^\sigma_{N,\Pi}(0,t)^{1/2} = A^\sigma_{M,\Pi}(0,t)^{1/2}A^\sigma_{V,\Pi}(0,t)^{1/2}, \\ A_1 &= A^\sigma_{V,\Sigma}(0,t), \\ A_2 &= A^\sigma_{M,\Sigma}(0,t), \\ A_3 &= A^\sigma_{N,\Sigma}(0,t). \end{aligned}$$

For $k \in \mathbb{N}$ and the ℓ^2-norm $\|\cdot\|$, we let

$$\phi_k(m) = \Big(\frac{\|m\|^{1/k}}{\|m\|^{1/k}+1}\Big)^k, \qquad m \in M.$$

It follows from Theorem 4.1 that there are positive constants C and D such that in the region $\|v\| \leq \|m\|^{1/(2k_o)}$,

$$(4.9)\qquad A_0 P^{\sigma}_{t,0}(m,v) \le C(\|m\|^{1/(2k_o)}+1)\exp\Big(-D\frac{\|v\|^2}{A_1}-D\frac{\|m\|}{A_2}\phi_{2k_o}(m)\Big) + CA_1^{1/2}\exp\Big(-D\frac{\|m\|^{1/k_o}+\|v\|^2}{A_1}\Big).$$

In fact, if $\|v\| \le \|m\|^{1/(2k_o)}$ then

$$\frac{\|m\|^2}{(\|m\|^{1/(2k_o)}+\|v\|+2)^{2k_o}} \ge \frac{\|m\|^2}{2^{2k_o}(\|m\|^{1/(2k_o)}+1)^{2k_o}} = \frac{1}{2^{2k_o}}\phi_{2k_o}(m)\|m\|,$$

and we get (4.9).

Now we consider $\|v\| \ge \|m\|^{1/(2k_o)}$. Again, it follows from Theorem 4.1 that there are positive constants C, D such that

$$(4.10)\qquad A_0 P^{\sigma}_{t,0}(m,v) \le C(\|m\|^{1/(2k_o)}+1+A_1^{1/2})\exp\Big(-D\frac{\|m\|^{1/k_o}+\|v\|^2}{A_1}\Big).$$

In fact, it is easy to see that if $\|v\| \ge \|m\|^{1/(2k_o)}$ then there is a constant $D>0$ such that

$$\frac{\|v\|^2}{A_1}+\frac{\|m\|^2}{(\|m\|^{1/(2k_o)}+\|v\|+2)^{2k_o}A_2} \ge D\frac{\|m\|^{1/k_0}+\|v\|^2}{A_1}.$$

Hence, (4.10) follows from Theorem 4.1.

Since $0<\phi_{2k_o}(m)\le 1$ for all m, we note that there exists $c>0$ such that

$$\|m\|\phi_{2k_o}(m) \ge c\|m\|^{1/k_o} \quad \text{for } \|m\|\ge 1$$

and

$$\|m\|^{1/k_o} \ge \|m\|\phi_{2k_o}(m) \ge C\|m\|^2 \quad \text{for } \|m\|<1.$$

Since $A_1 \le A_3$ and $A_2 \le A_3$, Theorem 1.3 follows from (4.9) and (4.10).

5. Upper estimate for the Poisson kernel

5.1. Poisson kernel. Let μ_t be the semigroup of probability measures on $S = N \rtimes \mathbb{R}^k$ generated by $\mathcal{L}_\alpha$. It is known [5, 9] that

$$\lim_{t\to\infty}(\pi(\check{\mu}_t), f) = (\nu, f),$$

where π denotes the projection from S onto N and $(\check{\mu}, f) = (\mu, \check{f})$, $\check{f}(x) = f(x^{-1})$. Let $a \in \mathbb{R}^k$ and let μ be a measure on N. We define

$$(\mu^a, f) = (\mu, f\circ \mathrm{Ad}(a)).$$

For $a\in\mathbb{R}^k$ we have

$$(5.1)\qquad \nu^a(x) = \nu(a^{-1}xa)\chi(a)^{-1}, \qquad x\in N,$$

where χ is as in (1.18).

It is an easy calculation to check that

$$\lim_{t\to\infty}(\pi(\check{\mu}_t)^a, f) = (\nu^a, f). \tag{5.2}$$

We need the following

LEMMA 5.1. *We have*

$$(\pi(\check{\mu}_t)^a, f) = (\mathbf{E}_a \check{P}^{\sigma}_{t,0}, f).$$

Proof. This equality follows from formula (1.9). See [17, Lemma 4.1] for the details. ■

By (5.2) and Lemma 5.1 it follows that

$$(\nu^a, f) = \lim_{t\to\infty}(\pi(\check{\mu}_t)^a, f) = \lim_{t\to\infty}(\mathbf{E}_a \check{P}^{\sigma}_{t,0}, f). \tag{5.3}$$

Let $\nu(x) = \nu(m, v)$, $m \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, be the Poisson kernel on N for the operator $\mathcal{L}_\alpha$ in (1.10). Recall that we assume that

$$\lambda(\alpha) > 0 \quad \text{for all } \lambda \in \Lambda.$$

Hence α belongs to the positive Weyl chamber A^+. The operator Δ_α generates the Brownian motion with drift -2α,

$$\sigma(u) = b(u) - 2\alpha u,$$

where $b(u)$ is the k-dimensional standard Brownian motion normalized so that $\operatorname{Var} b_u = 2u$.

Let ν^a be as in (5.1). We also use the notation introduced in (1.19).

THEOREM 5.2. *For all compact subsets $K \not\ni e$ of $N = M \rtimes V$, all $\wp \in A^+$, and all $\varepsilon > 0$ there exists a constant $C = C(K, \wp, \varepsilon) > 0$ such that for all $s < 0$,*

$$\nu^{s\wp}(x) \le Ce^{-\rho_0(s\wp)} e^{(s/2)\gamma_\Theta(\wp)\overline{\gamma}_\Theta(\alpha)} e^{(s/2)\gamma_\Lambda(\wp)\overline{\gamma}_\Lambda(\alpha)} \quad \text{if } x = (m, v) \in K_1 = K \cap \{\|m\| \ge \varepsilon,\ \|v\| \ge \varepsilon\}, \tag{5.4}$$

$$\nu^{s\wp}(x) \le Ce^{-\rho_0(s\wp)} e^{s\gamma_\Theta(\wp)\overline{\gamma}_\Theta(\alpha)} \quad \text{if } x = (m, v) \in K_2 = K \cap \{\|m\| < \varepsilon,\ \|v\| \ge \varepsilon\}, \tag{5.5}$$

and

$$\nu^{s\wp}(x) \le Ce^{-\rho_0(s\wp)} e^{s\gamma_\Lambda(\wp)\overline{\gamma}_\Lambda(\alpha)} \quad \text{if } x = (m, v) \in K_3 = K \cap \{\|m\| \ge \varepsilon,\ \|v\| < \varepsilon\}. \tag{5.6}$$

Proof. First we consider elements $x = (m, v)$ from the set K_1. Let A_j be defined as in (4.8) but with $t = \infty$. By Theorem 1.3, we have

$$\begin{aligned}(5.7)\qquad \nu^{s\wp} &\le C\mathbf{E}_{s\wp}A_0^{-1}\exp\Big(-\frac{D}{A_1}-\frac{D}{A_3}\Big)\\ &\quad + C\mathbf{E}_{s\wp}A_0^{-1}A_1^{1/2}\exp\Big(-\frac{D}{A_1}-\frac{D}{A_3}\Big).\end{aligned}$$

We estimate the first expectation on the right:

$$\begin{aligned}(5.8)\qquad &\mathbf{E}_{s\wp}A_0^{-1}\exp\Big(-\frac{D}{A_1}-\frac{D}{A_3}\Big)\\ &\le (\mathbf{E}_{s\wp}(A_0^{-1})^2)^{1/2}\Big(\mathbf{E}_{s\wp}\exp\Big(-\frac{2D}{A_1}-\frac{2D}{A_3}\Big)\Big)^{1/2}\\ &\le (\mathbf{E}_{s\wp}(A_0^{-1})^2)^{1/2}\Big(\mathbf{E}_{s\wp}\exp\Big(-\frac{4D}{A_1}\Big)\Big)^{1/4}\Big(\mathbf{E}_{s\wp}\exp\Big(-\frac{4D}{A_3}\Big)\Big)^{1/4}.\end{aligned}$$

By the Cauchy–Schwarz inequality we get

$$\begin{aligned}(5.9)\qquad \mathbf{E}_{s\wp}(A_0^{-1})^2 &= \mathbf{E}_{s\wp}(A^\sigma_{M,\Pi})^{-1}(A^\sigma_{V,\Pi})^{-1}\\ &= e^{-2\rho_0(s\wp)}\mathbf{E}_0(A^\sigma_{M,\Pi})^{-1}(A^\sigma_{V,\Pi})^{-1}\\ &\le e^{-2\rho_0(s\wp)}(\mathbf{E}_0(A^\sigma_{M,\Pi})^{-2})^{1/2}(\mathbf{E}_0(A^\sigma_{V,\Pi})^{-2})^{1/2}.\end{aligned}$$

Since, by Lemma 2.3, the expected values $\mathbf{E}_0(A^\sigma_{M,j})^{-d}$, $j=1,\dots,m$, and $\mathbf{E}_0(A^\sigma_{V,i})^{-d}$, $i=1,\dots,n$, are finite for all $d>0$, we can apply the Cauchy–Schwarz inequality successively to each of the remaining expectations in (5.9) and conclude their finiteness.

Now we consider $\mathbf{E}_{s\wp}\exp(-4D_1/A_1)$ and $\mathbf{E}_{s\wp}\exp(-4D_2/A_3)$ from (5.8). Clearly,

$$(5.10)\qquad \mathbf{E}_{s\wp}\exp(-4D_1/A_1)\le \mathbf{E}_0\exp\big(-4D_1/(M(s\wp)A_1)\big),$$

where

$$M(s\wp)=\max_{\vartheta\in\Theta}e^{2\vartheta(s\wp)}=e^{2s\min_{\vartheta\in\Theta}\vartheta(\wp)}=e^{2s\gamma_\Theta(\wp)}.$$

Proceeding exactly in the same way as in the proof of [17, Lemma 6.2] we show that (5.10) is bounded by

$$(5.11)\qquad CM(s\wp)^{\overline{\gamma}_\Theta(\alpha)}=Ce^{2s\gamma_\Theta(\wp)\overline{\gamma}_\Theta(\alpha)}.$$

The expectation $\mathbf{E}_{s\wp}\exp(-4D_2/A_3)$ is similar. Again, in the same way as in the proof of [17, Lemma 6.2] we show that $\mathbf{E}_{s\wp}\exp(-4D_2/A_3)$ is bounded by

$$CM_1(s\wp)^{\overline{\gamma}_\Lambda(\alpha)},\quad\text{where}\quad M_1(s\wp)=\max_{\lambda\in\Lambda}e^{2\lambda(s\wp)}=e^{2s\gamma_\Lambda(\wp)}.$$

Hence,

$$\mathbf{E}_{s\wp}\exp(-4D_2/A_3)\le Ce^{2s\gamma_\Lambda(\wp)\overline{\gamma}_\Lambda(\alpha)}.$$

Now we estimate the second expectation on the right in (5.7):

$$\begin{aligned}
&\mathbf{E}_{s\wp} A_0^{-1} A_1^{1/2} \exp\Big(-\frac{D}{A_1} - \frac{D}{A_3}\Big) \\
&\quad \le \sum_{j=1}^{n} \mathbf{E}_{s\wp} A_0^{-1} A_{V,j}^{1/2} \exp\Big(-\frac{D}{A_1} - \frac{D}{A_3}\Big) \\
&\quad = \sum_{j=1}^{n} \mathbf{E}_{s\wp} A_{M,\Pi}^{-1/2} \prod_{k\ne j} A_{V,k}^{-1/2} \exp\Big(-\frac{D}{A_1} - \frac{D}{A_3}\Big) \\
&\quad = \sum_{j=1}^{n} e^{-\sum_{\xi\in\Xi}\xi(s\wp)} e^{-\sum_{\vartheta\ne\vartheta_j}\vartheta(s\wp)} \mathbf{E}_0 A_{M,\Pi}^{-1/2} \prod_{k\ne j} A_{V,k}^{-1/2} \exp\Big(-\frac{D}{A_1} - \frac{D}{A_3}\Big).
\end{aligned}$$

Since $s<0$,

$$e^{-\sum_{\xi\in\Xi}\xi(s\wp)} e^{-\sum_{\vartheta\ne\vartheta_j}\vartheta(s\wp)} \le e^{-\rho_0(s\wp)}.$$

To estimate

$$\mathbf{E}_0 A_{M,\Pi}^{-1/2} \prod_{k\ne j} A_{V,k}^{-1/2} \exp\Big(-\frac{D}{A_1} - \frac{D}{A_3}\Big)$$

we proceed as in (5.8) and (5.9) and get the same estimate. Hence, the estimate (5.4) holds on K_1.

Now we have to consider the set

$$K_2 = K \cap \{(m,v) : \|m\| < \varepsilon,\ \|v\| \ge \varepsilon\}.$$

On this set the estimate from Theorem 1.3 simplifies and we get

$$\nu^{s\wp}(x) \le C\mathbf{E}_{s\wp} A_0^{-1} \exp\Big(-\frac{D_1}{A_1}\Big) + C\mathbf{E}_{s\wp} A_0^{-1} A_1^{1/2} \exp\Big(-\frac{D}{A_1}\Big). \tag{5.12}$$

Using Lemma 2.3, (5.9), (5.10) and (5.11) as above, we get the estimate

$$\begin{aligned}
\mathbf{E}_{s\wp} A_0^{-1} \exp\Big(-\frac{D_1}{A_1}\Big) &\le (\mathbf{E}_{s\wp}(A_0^{-1})^2)^{1/2} \Big(\mathbf{E}_{s\wp} \exp\Big(-\frac{2D_1}{A_1}\Big)\Big)^{1/2} \\
&\le e^{-\rho_0(s\wp)} e^{s\gamma_\Theta(\wp)\overline{\gamma}_\Theta(\alpha)}.
\end{aligned}$$

As in the previous case the second expectation in (5.12) has the same estimate as the first one. Hence, the estimate (5.5) holds on K_2. Finally, we consider the set

$$K_3 = K \cap \{(m,v) : \|m\| \ge \varepsilon,\ \|v\| < \varepsilon\}.$$

On K_3,

$$\nu^{s\wp}(x) \le C\mathbf{E}_{s\wp} A_0^{-1} \exp\Big(-\frac{D_2}{A_3}\Big) + C\mathbf{E}_{s\wp} A_0^{-1} A_1^{1/2} \exp\Big(-\frac{D_2}{A_3}\Big). \tag{5.13}$$

We have

$$\mathbf{E}_{s\wp} A_0^{-1} \exp\left(-\frac{D_2}{A_3}\right) \le (\mathbf{E}_{s\wp}(A_0^{-1})^2)^{1/2} \left(\mathbf{E}_{s\wp} \exp\left(-\frac{2D_2}{A_3}\right)\right)^{1/2}$$
$$\le e^{-\rho_0(s\wp)} e^{s\gamma_\Lambda(\wp)\overline{\gamma}_\Lambda(\alpha)}.$$

Again, the second expectation in (5.13) has the same estimate as the first one. Thus (5.6) is proved. ∎

5.2. Proof of Theorem 1.5. Having Theorem 5.2, we use the standard homogeneity argument as follows.

Proof of Theorem 1.5. It is clear that for $x \in N$ with the norm $|x|_\wp \le 1$ we have $\nu(x) \le C_\wp$.

Let $\delta_t^\rho = \mathrm{Ad}((\log t)\wp)$. Then $|\delta_t^\wp x|_\wp = t|x|_\wp$. Let $x = \delta^\wp_{\exp(-s)} x_o$ with $|x_o|_\wp = 1$ and $s < 0$. Then $|x|_\wp = e^{-s} > 1$. Let $K = \{x_o : |x_o|_\wp = 1\}$. By definition (5.1) of $\nu^{s\wp}$, we get

$$\nu(x) = \nu(\delta^\rho_{\exp(-s)} x_o) = \nu\big((s\wp)^{-1} x_o (s\wp)\big) = e^{\rho_0(s\wp)} \nu^{s\wp}(x_o),$$

where $\rho_0 = \sum_j \vartheta_j + \sum_i \xi_i$, and the result follows from Theorem 5.2. ∎

6. Example. Consider $N = \mathcal{H}_n$, the $2n+1$-dimensional Heisenberg group, which we realize as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the Lie group multiplication given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 \cdot y_2).$$

The corresponding Lie algebra $\mathfrak{h}_n$ is then spanned by the left-invariant vector fields

$$X_j = \partial_{x_j}, \quad Y_j = \partial_{y_j} + x_j \partial_z, \quad Z = \partial_z,$$

where $1 \le j \le n$. Let $A = \mathbb{R}^k$ and let $\xi_{1,j}, \xi_{2,j}, \xi_3 \in (\mathbb{R}^k)^*$, $1 \le j \le n$, be such that

$$\xi_{1,j} + \xi_{2,j} = \xi_3$$

independently of j. For $x \in \mathbb{R}^n$, $a \in \mathbb{R}^k$, and $i = 1, 2$, we set

$$e^{\xi_i(a)} x = (e^{\xi_{i,1}(a)} x_1, \ldots, e^{\xi_{i,n}(a)} x_n).$$

We then define an A action on $\mathcal{H}_n$ by automorphisms of $\mathcal{H}_n$ by

$$a(x, y, z)a^{-1} = (e^{\xi_1(a)} x, e^{\xi_2(a)} y, e^{\xi_3(a)} z).$$

We then let $S = \mathcal{H}_n \rtimes A$.

Let $\overline{X}_j$, $\overline{Y}_j$, and $\overline{Z}$ be, respectively, X_j, Y_j, and Z considered as left-invariant vector fields on S. Then

$$\overline{X}_j = e^{\xi_{1,j}(a)} X_j, \quad \overline{Y}_j = e^{\xi_{2,j}(a)} Y_j, \quad \overline{Z} = e^{\xi_3(a)} Z.$$

We set

$$\mathcal{L}_\alpha = \sum_{j=1}^{n}(\overline{X}_j^2 + \overline{Y}_j^2) + \overline{Z}^2 + \Delta_\alpha = \sum_{j=1}^{n}(e^{2\xi_{1,j}(a)}X_j^2 + e^{2\xi_{2,j}(a)}Y_j^2) + e^{2\xi_3(a)}Z^2 + \Delta_\alpha, \tag{6.1}$$

where Δ_α is defined in (1.3).

EXAMPLE 6.1. Consider the operator $\mathcal{L}_\alpha$ defined in (6.1) on $\mathcal{H}_n \rtimes A$ with $A = \mathbb{R}^2$ and $\xi_{1,j} = (1,0)$, $\xi_{2,j} = (0,1)$. Theorem 1.2 of [17] gives

$$\nu(x,y,z) \le C(1 + |(x,y,z)|_\wp)^{-C_1\rho_0(\wp)\gamma(\alpha)/4},$$

where

$$\gamma(\alpha) = 2\min(\alpha_1, \alpha_2)$$

for some constant C_1 which depends on $\wp$ and can be computed. Take $\wp = (1,2)$. We have $\rho_0 = \sum_j \xi_{1,j} + \sum_j \xi_{2,j} + \xi_3$, where $\xi_{i,j}(a) = a_i$, $i = 1,2$, $j = 1,\dots,n$. To compute C_1 we proceed similarly as in [17, Example 1] and get

$$\nu(x,y,z) \le C(1 + |(x,y,z)|_{(1,2)})^{-\min(\alpha_1,\alpha_2)/2}.$$

whereas Theorem 1.5 gives, for $\|(x,z)\| > 1$ and $\|y\| > 1$,

$$\nu(x,y,z) \le C(1 + |(x,y,z)|_{(1,2)})^{-\alpha_1/2-\min(\alpha_1,\alpha_2)/2}.$$

Similarly, Theorem 1.1 of [18] gives, for every $q > 1$,

$$\nu(x,y,z) \le C_q(1 + |(x,y,z)|_\alpha)^{-\frac{2}{q}(\min(\alpha_1,\alpha_2))^2},$$

whereas Theorem 1.5 gives, for $\|(x,z)\| > 1$ and $\|y\| > 1$,

$$\nu(x,y,z) \le C(1 + |(x,y,z)|_\alpha)^{-\alpha_1^2/2-\alpha_2^2/2}$$

which is again a better estimate if we take for example an operator with $\alpha_1\sqrt{4/q-1} < \alpha_2$.

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