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Large structures made of nowhere L^q functions

by

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Abstract. We say that a real-valued function f defined on a positive Borel measure space (X,μ) is nowhere q-integrable if, for each nonvoid open subset U of X, the restriction $f|_U$ is not in $L^q(U)$. When (X,μ) has some natural properties, we show that certain sets of functions defined in X which are p-integrable for some p's but nowhere q-integrable for some other q's $(0 < p,q < \infty)$ admit a variety of large linear and algebraic structures within them. The presented results answer a question of Bernal-González, improve and complement recent spaceability and algebrability results of several authors and motivate new research directions in the field of spaceability.

1. Introduction. This work is a contribution to the study of large linear and algebraic structures within essentially nonlinear sets of functions with special properties; the presence of such structures is often described using the terms *lineable*, *algebrable* and *spaceable*. This subject gained impulse especially during the last decade, with important contributions from several authors, see for example [1], [4], [6], [12] and [23]. For an up-to-date survey on this topic, see [10].

Recall that a subset S of a topological vector space V is said to be *lineable* (respectively, *spaceable*) if $S \cup \{0\}$ contains an infinite-dimensional vector subspace (respectively, a *closed* infinite-dimensional vector subspace) of V. Though results in this field date back to the sixties ([1]), this terminology was introduced only recently: it first appeared in unpublished notes by Enflo and Gurariy and was first published in [2]. We should mention that Enflo and Gurariy's notes have been completed in collaboration with Seoane-Sepúlveda and will finally be published in [14]. It is also common to say that S is *dense-lineable* if $S \cup \{0\}$ contains a *dense* infinite-dimensional vector subspace of V.

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([1]) In [18], Gurariy showed that there exists in $C([0,1])$ a closed infinite-dimensional subspace consisting, except for the null function, only of nowhere differentiable functions; see also [19] for a version in English.

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The qualifier *maximal* is often added to *dense-lineable* or *spaceable* when the corresponding space contained in $S \cup \{0\}$ has the same dimension of V. There are, though, natural and more restrictive concepts of spaceability than maximal spaceability, as we shall discuss in Section 3.

The term *algebrability* was introduced later, in [3]; if V is a linear algebra, S is said to be *κ-algebrable* if $S \cup \{0\}$ contains an infinitely generated algebra, with a *minimal* set of generators of cardinality κ (see [3] for details). We shall work with a strengthened notion of κ-algebrability, named *strong κ-algebrability*:

Definition 1.1. We say that a subset S of an algebra $\mathcal{A}$ is *strongly κ-algebrable*, where κ is a cardinal number, if there exists a κ-generated free algebra $\mathcal{B}$ contained in $S \cup \{0\}$.

We recall that, for a cardinal number κ, to say that an algebra $\mathcal{A}$ is a *κ-generated free algebra* means that there exists a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset \mathcal{A}$ such that every function f from Z into any algebra $\mathcal{A}'$ can be uniquely extended to a homomorphism from $\mathcal{A}$ into $\mathcal{A}'$. The set Z is called a *set of free generators* of $\mathcal{A}$. If Z is a set of free generators of some subalgebra $\mathcal{B} \subset \mathcal{A}$, we say that Z is a set of free generators *in* $\mathcal{A}$. If $\mathcal{A}$ is *commutative*, a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset \mathcal{A}$ is a set of free generators in $\mathcal{A}$ if for each polynomial P and for any $z_{\alpha_1}, \ldots, z_{\alpha_n} \in Z$ we have

$$P(z_{\alpha_1}, \ldots, z_{\alpha_n}) = 0 \quad \text{if and only if} \quad P = 0.$$

The definition of strong κ-algebrability was introduced in [5], though in several papers, sets which are shown to be algebrable are in fact strongly algebrable, as can be easily seen from the proofs. See e.g. [3], [7] and [15]. Strong algebrability is indeed a stronger condition than algebrability: for example, c_{00} is ω-algebrable in c_0 but it is not strongly 1-algebrable (see [5]).

1.1. Results on large structures of nonintegrable functions: recent and new. Our object of study will be the quasi-Banach spaces $L^p(X, \mathcal{M}, \mu)$. For brevity, when there cannot be any confusion or ambiguity, we shall write L^p, $L^p(X, \mu)$ or $L^p(X)$ instead of $L^p(X, \mathcal{M}, \mu)$. Our main focus will be on functions which are p-integrable but not q-integrable, for some $0 < p, q \leq \infty$, and specially on functions which are p-integrable but *nowhere q-integrable*. The notion of nowhere-q-integrability we consider is connected with open sets:

Definition 1.2. Let $0 < q \leq \infty$. A scalar-valued function f defined on a Borel measure space X is said to be *nowhere q-integrable* (or *nowhere L^q*) if, for each nonvoid open subset U of X, the restriction $f|_U$ is not in $L^q(U)$.

In our context it would be pointless to substitute “for each nonvoid open subset U of X” by “for each Borel subset U of positive measure of X” in the definition above; the reason is that, if $0 < p, q \le \infty$ and $f \in L^p(X)$, there is always a Borel subset of X of positive measure on which f is q-integrable. This follows from a simple argument (see e.g. the final remarks in [8]). Of course, not all Borel measure spaces (X, μ) admit L^p-nowhere-L^q functions, but there is a large class of such spaces which admit plenty of such functions, as we will see.

Let us start by mentioning some recent results and open questions on large structures within sets of functions which are p-integrable but not q-integrable. For a survey on the results in this direction, we recommend [11].

THEOREM 1.3 (Bernal-González and Ordóñez Cabrera [9]). *Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider the conditions*

(α) $\inf\{\mu(A) : A \in \mathcal{M},\ \mu(A) > 0\} = 0$, *and*
(β) $\sup\{\mu(A) : A \in \mathcal{M},\ \mu(A) < \infty\} = \infty$.

Then the following assertions hold:

(1) *if $1 \le p < \infty$, then $L^p \setminus \bigcup_{q>p} L^q$ is spaceable if and only if (α) holds;*
(2) *if $1 < p \le \infty$, then $L^p \setminus \bigcup_{q<p} L^q$ is spaceable if and only if (β) holds;*
(3) *if $1 < p < \infty$, then $L^p \setminus \bigcup_{q\neq p} L^q$ is spaceable if and only if both (α) and (β) hold;*
(4) *if $1 < p < \infty$ and L^p is separable, then $L^p \setminus \bigcup_{q<p} L^q$ is maximal dense-lineable if and only if (β) holds.*

Note that there are measure spaces X that satisfy (α) and (β), but for which there are no nowhere q-integrable functions in $L^p(X)$: it suffices that X has an open singleton of positive measure, since in that case all functions will be q-integrable on $\{x\}$. Bernal-González et al. use the convenient terminology '(*left, right*) *strict order integrability*' when a function is p-integrable but not q-integrable for $q \neq p$ (resp. $q < p$, $q > p$). We refer to [11] for improvements on Theorem 1.3(2) above. And in [13] there is a version of that same item which includes quasi-Banach spaces:

THEOREM 1.4 (Botelho, Fávaro, Pellegrino and Seoane-Sepúlveda [13]). *The set $L^p[0,1] \setminus \bigcup_{q>p} L^q[0,1]$ is spaceable for every $p > 0$.*

When it comes to nowhere integrable functions, Bernal-González gave the first initial result:

THEOREM 1.5 (Bernal-González [8]). *Let $(X, \mathcal{M}, \mu)$ be a measure space such that X is a Hausdorff first-countable separable locally compact perfect topological space and that μ is a positive Borel measure which is continuous, regular and has full support. Let $1 \le p < \infty$. Then the set*

$$\{f \in L^p : f \text{ is nowhere } q\text{-integrable, for each } q > p\} \tag{1.1}$$

is dense in L^p.

It is clear that μ having full support (that is, $\mu(U) > 0$ for every nonvoid open subset $U \subset X$) is a necessary condition for the existence of nowhere q-integrable functions. Based on the above result, Bernal-González rose the following question:

Problem 1. *Is the set* (1.1) *lineable/maximal lineable/dense-lineable?*

It is quite natural to seek for other large structures within (1.1).

The authors of this work have also presented some results on large structures of nowhere integrable functions, including the following:

Theorem 1.6 (Głąb, Kaufmann and Pellegrini [17]). *The set of nowhere essentially bounded functions in $L^1[0,1]$ is*

(1) *spaceable, and*
(2) *strongly $\mathfrak{c}$-algebrable.*

In this landscape, we present a few new results which solve Problem 1 and, under quite mild conditions on the measure space where our functions are defined, complement/generalize the results mentioned above. We summarize these results in Theorem 1.7 below.

Theorem 1.7. *Suppose that X is a topological space admitting a countable π-base (that is, a family $(U_n)_n$ of nonvoid open subsets of X such that, for each nonvoid open subset A of X, $U_j \subset A$ for some j) and that μ is a positive Borel measure on X. Let $0 < p < \infty$ and set*

$$\begin{aligned} S_p(X) \doteq S_p &\doteq \{f \in L^p : f \text{ is nowhere } L^q, \text{ for each } q \in (p,\infty]\},\\ S'_p &\doteq S_p \setminus \bigcup_{0<q<p} L^q,\\ \mathcal{G} &\doteq \Big\{ f \in \bigcap_{0<q<\infty} L^q : f \text{ is nowhere } L^\infty \Big\}. \end{aligned}$$

Then we have the following:

(a) *if μ is atomless, outer regular and has full support, then $S_p \cup \{0\}$ contains an ℓ_p-isometric subspace of L^p, which is in addition complemented if $p \geq 1$;*
(b) *if μ is infinite and σ-finite, then $L^p \setminus \bigcup_{0<q<p} L^q$ contains an ℓ_p-isometric subspace of L^p, which is in addition complemented if $p \geq 1$;*
(c) *if μ is atomless, infinite, outer regular and has full support, then $S'_p \cup \{0\}$ contains an ℓ_p-isometric subspace of L^p, which is in addition complemented if $p \geq 1$;*
(d) *if μ is atomless, outer regular and has full support, then S_p is maximal dense-lineable;*
(e) *if μ is atomless, outer regular and has full support, then $\mathcal{G}$ is strongly $\mathfrak{c}$-algebrable.*

See Section 6 for comments on working with π-bases instead of the more usual bases of open sets. In addition to Theorem 1.7 we also prove that, for special classes of positive Borel measure spaces, S_p contains an isomorphic copy of ℓ_2 (see Theorems 3.2 and 3.3, and Corollary 3.4). This motivates a new investigation direction concerning spaceability (see Section 3).

REMARK 1.8. Referring to items (a)–(c), it is worth recalling that for $p < 1$, L^p contains no complemented copy of ℓ_p. This is easily seen if one recalls that, for $p < 1$, ℓ_p admits nontrivial continuous linear forms (e.g. the evaluation functionals), while every nontrivial linear form on L^p is discontinuous.

REMARK 1.9. In any measurable space which admits a set of strictly positive finite measure (in particular for (X, μ) under the conditions in (e)) and $0 < p < q < \infty$, the set of L^p functions which are not L^q is *not* algebrable; to see this, just note that if f is p-integrable but not q-integrable on some set of finite measure U, then f^n is not p-integrable if we choose a large enough power n. There is therefore no hope of finding algebraic structures of strict-order integrable functions in many cases. One exception is given by:

THEOREM 1.10 (García-Pacheco, Pérez-Eslava, Seoane-Sepúlveda [16]). *If $(X, \mathcal{M}, \mu)$ is a measure space in which there exists an infinite family of pairwise disjoint measurable sets A_n satisfying $\mu(A_n) \geq \epsilon$ for some $\epsilon > 0$, then*

$$L^\infty \setminus \bigcap_{p=1}^{\infty} L^p$$

is spaceable in L^∞ and algebrable.

Note that Theorem 1.7(e) complements, in some sense, the algebrability part of Theorem 1.10.

REMARK 1.11. Theorem 1.7 relates to the previous results as follows:

- (a) generalizes Theorem 1.4, Theorem 1.6(1) and, under our assumptions, also Theorem 1.3(1).
- It is not hard to adapt Theorem 1.3(2) for $p < 1$ and to see that the space guaranteeing the spaceability can be isometric to ℓ_p and complemented in case $p \geq 1$; since condition (β) from Theorem 1.3 is milder that the conditions in (b), it follows that (b) does not really add much. But the construction in the proof we present is used to prove also (c), thus we include (b) for completeness and clarity.
- Under our assumptions, (c) improves Theorem 1.3(3).
- (d) improves Theorem 1.5 and gives a positive answer to Bernal-González's Problem 1.
- (e) improves Theorem 1.6(2).

The remaining sections are organized as follows. In Section 2 we will prove Theorem 1.7(a)–(c), that is, its spaceability part. In Section 3 we discuss some cases where we have a copy of ℓ_2 in $S_p \cup \{0\}$, and motivate a new research direction on spaceability, touching more qualitative aspects. Section 4 contains the proof of the dense-lineability result (Theorem 1.7(d)), and Section 5 concerns the algebrability result (Theorem 1.7(e)). In Section 6 we briefly discuss conditions on positive Borel measure spaces under which there exist, or not, functions p-nowhere-q integrable in the corresponding L^p spaces. We include related open problems throughout the text.

2. Spaceability: proof of Theorem 1.7(a)–(c). Recall the following standard result from functional analysis on Banach spaces:

THEOREM 2.1. *Suppose that (X,μ) is a Borel measure space. Let $1 \leq p < \infty$, and suppose that (f_n) is a sequence of norm-one, disjointly supported functions in $L^p(\mu)$. Then (f_n) is a complemented basic sequence isometrically equivalent to the canonical basis of ℓ_p.*

It is not hard to see that the same holds for $0 < p < 1$, though complementability is lost, as we previously pointed out. Our strategy to prove Theorem 1.7(a)–(c) will be to find sequences of norm-one, disjointly supported functions in S_p, $L^p \setminus \bigcup_{0<q<p} L^q$ and S'_p, under the corresponding assumptions.

LEMMA 2.2. *Let X be a topological space with a countable π-base. Suppose that μ is an atomless and outer-regular positive Borel measure on X with full support. Let U be an open set such that $\mu(U) > 0$ and let $\varepsilon \in (0,1)$. Then there is a nowhere-dense Borel subset N of U such that $\mu(N) > \mu(U)\varepsilon$.*

Proof. Let (U_n) be a π-base of U. Since μ is atomless, there are Borel sets $B_n \subset U_n$ such that $\mu(B_n) < \varepsilon\mu(U)/2^n$. Since μ is outer-regular, there are open sets $V'_n \supset B_n$ with $\mu(V'_n) < \varepsilon\mu(U)/2^n$. Let $V_n = V'_n \cap U$ and put $V = \bigcup_n V_n$. Then $\mu(V) < \varepsilon\mu(U)$ and V is a dense open subset of U. Therefore $N = U \setminus V$ is a nowhere dense subset of U with measure greater than $\mu(U)\varepsilon$. ∎

LEMMA 2.3. *Suppose that μ is an atomless positive Borel measure on X with full support. Let A be a measurable set in X such that $\mu(A) > 0$ and let (a_n) be a sequence in $(0,\infty)$. Then there is a sequence (A_n) of pairwise disjoint measurable subsets of A such that $0 < \mu(A_n) < \infty$ and $\mu(A_{n+1}) \leq a_n\mu(A_n)$.*

Proof. We may assume that $a_n \leq 1/2$ for all n. Since μ is atomless, there is a Borel set $A_1 \subset A$ such that

$$0 < \mu(A_1) < \tfrac{1}{2}\mu(A).$$

Likewise, there is a Borel set $A_2 \subset A \setminus A_1$ such that

$$0 < \mu(A_2) < a_1\mu(A_1) \le \tfrac{1}{2}\mu(A_1).$$

Proceeding this way, we can find inductively $A_n \subset A \setminus \bigcup_{k<n} A_k$ such that

$$0 < \mu(A_n) < a_{n-1}\mu(A_{n-1}) \le \tfrac{1}{2}\mu(A_{n-1});$$

this is possible since $\mu(A \setminus \bigcup_{k<n} A_k) > 0$. ∎

LEMMA 2.4. *Suppose that μ is an atomless positive Borel measure on X with full support. Then for any given Borel set A in X such that $\mu(A) > 0$ there is a norm-one A-supported function h_A in $L^p \setminus \bigcup_{q>p} L^q$.*

Proof. Let $A \subset X$ be measurable and $\mu(A) > 0$; by Lemma 2.3 there exists a family $\{A_{n,m} : n, m \in \mathbb{N}\}$ of pairwise disjoint subsets of A of positive measure such that $\mu(A_{n,m+1}) \le \frac{1}{2}\mu(A_{n,m})$. Let (r_n) be a strictly decreasing sequence of real numbers tending to p. Put

$$h_n = \sum_{m=1}^{\infty} a_{n,m}\chi_{A_{n,m}},$$

where $a_{n,m}^{r_n}\mu(A_{n,m}) = 1/m$. Then $\|h_n\|_{r_n} = \infty$ and

$$\int_X |h_n|^p \, d\mu = \sum_{m=1}^{\infty} a_{n,m}^p \mu(A_{n,m}) = \sum_{m=1}^{\infty} \frac{1}{a_{n,m}^{r_n-p} m}.$$

Since

$$\limsup_{m\to\infty} \frac{\frac{1}{a_{n,m+1}^{r_n-p}(m+1)}}{\frac{1}{a_{n,m}^{r_n-p} m}} = \limsup_{m\to\infty} \left(\frac{\mu(A_{n,m+1})}{\mu(A_{n,m})}\right)^{(r_n-p)/r_n} \le \left(\frac{1}{2}\right)^{(r_n-p)/r_n} < 1,$$

by the ratio test for series we find that $h_n \in L^p$. Put

$$h_A = \sum_{n=1}^{\infty} \frac{h_n}{\|h_n\| 2^n}.$$

Then $h_A \in L^p \setminus \bigcup_{q>p} L^q$ and $\|h_A\| = 1$. ∎

Proof of Theorem 1.7(a). Let (U_n) be a π-base of X. Since μ is atomless and outer-regular, we may assume that $\mu(U_n) < \infty$ for each n. (Indeed, suppose that $\mu(U_n) = \infty$. Hence $U_n \neq \emptyset$; select $x \in U_n$. Since μ is atomless, $\mu(\{x\}) = 0$. By the outer-regularity of μ, there is an open neighborhood V of x with arbitrarily small μ-measure. Since μ does not vanish on open sets, we have $0 < \mu(V \cap U_n) < \infty$, and we may replace $U_n \cap V$ with U_n.)

By Lemma 2.2, there is a nowhere dense Borel set $N_1 \subset U_1$ with $0 < \mu(N_1) < 1/2$. Since N_1 is nowhere dense we can find a nonempty open set $U \subset U_2 \setminus N_1$, and again by Lemma 2.2 there is a nowhere dense Borel set $N_2 \subset U \subset U_2$ with $0 < \mu(N_2) < 1/2^2$. We can then inductively define a pairwise disjoint sequence of nowhere dense Borel sets (N_n) such that

$N_n \subset U_n$ and $0 < \mu(N_n) < 1/2^n$. Decompose each N_n into μ-positive and pairwise disjoint Borel sets $N_{n,m}$. For each n, m there exists, by Lemma 2.4, a norm-one $N_{n,m}$-supported function $h_{N_{n,m}}$ in $L^p \setminus \bigcup_{q>p} L^q$. If we put

$$f_m = \sum_{n=1}^{\infty} \frac{h_{N_{n,m}}}{2^n},$$

then (f_m) will form a norm-one basic sequence of elements from S_p with pairwise disjoint supports, and by Theorem 2.1 our proof is complete. ∎

LEMMA 2.5. *Suppose that μ is an infinite and σ-finite positive Borel measure on X. Then for any given Borel set $B \subset X$ of infinite measure, there exists a function $g_B \in L^p \setminus \bigcup_{q<p} L^q$ which is zero outside of B.*

Proof. Let $B \subset X$ be Borel of infinite measure, and let $\{B_{n,m} : n, m \in \mathbb{N}\}$ be a family of pairwise disjoint subsets of B of positive finite measure such that $2\mu(B_{n,m}) \le \mu(B_{n,m+1})$. Let (r_n) be a strictly increasing sequence of (strictly positive) real numbers tending to p. Put

$$g_n = \sum_{m=1}^{\infty} b_{n,m} \chi_{B_{n,m}},$$

where $b_{n,m}^{r_n} \mu(B_{n,m}) = 1/m$. Then

$$\int_X |g_n|^p \, d\mu = \sum_{m=1}^{\infty} b_{n,m}^p \mu(B_{n,m}) = \sum_{m=1}^{\infty} \frac{b_{n,m}^{p-r_n}}{m},$$

and since

$$\limsup_{m\to\infty} \frac{\frac{b_{n,m+1}^{p-r_n}}{m+1}}{\frac{b_{n,m}^{p-r_n}}{m}} = \limsup_{m\to\infty} \left(\frac{\mu(B_{n,m})}{\mu(B_{n,m+1})} \right)^{(p-r_n)/r_n} \le \left(\frac{1}{2}\right)^{(p-r_n)/r_n} < 1,$$

by the ratio test for series we find that $g_n \in L^p$. Letting

$$g_B \doteq \sum_{n=1}^{\infty} \frac{g_n}{\|g_n\| 2^n},$$

we deduce that $g_B \in L^p$. It suffices to show now that $g_B \notin L_q$ for any $q < p$. Fix such a q; then $r_n > q$ for large enough n, and so

$$\begin{aligned}
(\|g_n\|2^n)^q \int |g_B|^q \ge \int |g_n|^q &= \sum_m \left(\frac{1}{m\mu(B_{n,m})} \right)^{q/r_n} \mu(B_{n,m}) \\
&= \sum_m \left(\frac{1}{m}\right)^{q/r_n} \mu(B_{n,m})^{(r_n-q)/r_n} \\
&= \mu(B_{n,1})^{(r_n-q)/r_n} \sum_m \left(\frac{1}{m}\right)^{q/r_n} = \infty. \ \blacksquare
\end{aligned}$$

Proof of Theorem 1.7(b). Since μ is infinite and σ-finite, each Borel set D of infinite measure can be written as an infinite disjoint union of Borel sets of infinite measure. To see this it is enough to verify that D *contains* an infinite disjoint union of Borel sets of infinite measure D_n. In effect, we can define inductively Borel sets $C_k \subset D$ such that $1 \leq \mu(C_k) < \infty$; and let (M_n) be a pairwise disjoint family of infinite subsets of $\mathbb{N}$. Then $D_n \doteq \bigcup_{k \in M_n} C_k$ is a family of Borel sets with the desired properties. The conclusion then follows from Lemma 2.5 and the same argument that was used in the proof of Theorem 1.7(a). ■

The proof of Theorem 1.7(c) is a combination of the constructions from the proofs of parts (a) and (b):

Proof of Theorem 1.7(c). Consider U_n, N_n and f_m as in the proof of Theorem 1.7(a). Since $\mu(X \setminus \bigcup_n N_n) = \infty$, $X \setminus \bigcup_n N_n$ can be written as a disjoint union of Borel sets D_m of infinite measure. Then by Lemma 2.5, for each m there is a norm-one function $g_{D_m} \in L^p \setminus \bigcup_{q<p} L^q$ which is zero outside of D_m. Then the norm-one functions $(f_m + g_{D_m})/2$, $m \in \mathbb{N}$, are in S'_p and have almost disjoint supports. ■

3. Qualitative spaceability. Before proceeding with our investigation of the set S_p, we briefly discuss some qualitative aspects of studying spaceability. The starting point of such studies is always a topological vector space V and an (often highly nonlinear) subset $S \subset V$. The first step is to find a closed linear structure W in $S \cup \{0\}$, and the second step is frequently to determine how big, in terms of dimension, W can be. But one might also wish to determine what other properties we can expect W to satisfy. For instance, Rodríguez-Piazza [24] proved that, in $C([0,1])$, the set of nowhere differentiable functions, together with the zero function, contains an isometric copy of every separable Banach space. This result was improved by Hencl [20], who showed the same for the set of nowhere approximatively differentiable and nowhere Hölder functions. It is worth noting that the property "$S \cup \{0\}$ contains a copy of the original space V" is more restrictive than "S is maximal spaceable in V"; several examples show that these properties are not equivalent in many cases (take for instance $V = L^1[0,1]$ and S a subspace isomorphic to ℓ_2). Another example of "$S \cup \{0\}$ contains a copy of V" is obtained as an application of 1.7(b):

Corollary 3.1. *$\ell_p \setminus \bigcup_{0<q<p} \ell_q$ contains an isometric (and, if $p \geq 1$, also complemented) copy of ℓ_p.*

These remarks naturally motivate new directions of investigation concerning spaceability. In our context of nowhere p-integrable functions, we can pose the following:

Problem 2. *Under appropriate assumptions, which subspaces of L^p have (isometric, complemented, etc.) copies in $S_p \cup \{0\}$ (or in $(L^p \setminus \bigcup_{0<q<p} L^q) \cup \{0\}$, or in $S'_p \cup \{0\}$)?*

The same could be asked when studying the spaceability of any other subset of a topological vector space. In the remainder of this section, we present some initial results towards solving this problem (Theorems 3.2 and 3.3, and Corollary 3.4):

Theorem 3.2. *Suppose that $1 \le p < \infty$, and that (X, μ) is a positive Borel measure space such that $S_p(X)$ is nonvoid. Then $S_p(X \times [0,1]) \cup \{0\}$ contains a copy of ℓ_2.*

Theorem 3.3. *$S_p([0,1]) \cup \{0\}$ contains a copy of ℓ_2 for each $1 \le p < \infty$.*

Corollary 3.4. *$S_p([0,1]^n) \cup \{0\}$ contains a copy of ℓ_2 for each $1 \le p < \infty$ and each $n \in \mathbb{N}$.*

Note that Corollary 3.4 follows easily by induction from Theorems 3.2 and 3.3. Note also that Theorem 3.3 is another improvement of Theorem 1.4 (for $p \ge 1$), in a different direction compared to the improvement provided by Theorem 1.7(a).

To prove Theorem 3.2, we shall need some auxiliary results. The first one is a corollary of the following:

Theorem 3.5 (Kitson and Timoney [21, Theorem 3.3]). *Let (E_n) be a sequence of Banach spaces and F be a Fréchet space. Let $T_n : E_n \to F$ be continuous linear operators and $W \doteq \operatorname{span}\{\bigcup_n T_n(E_n)\}$. If W is not closed in F, then $F \setminus W$ is spaceable.*

Corollary 3.6. *Let E be an infinite-dimensional Banach space and V be the linear span of a sequence of elements of E. Then $E \setminus V$ is spaceable. In particular, if E is a sequence space, then the set of elements $x = (x_n)$ of E such that $x_n \neq 0$ for infinitely many n is spaceable.*

Proof. Note that, for the first part, it suffices to show that $E \setminus V$ is spaceable when E is the closed linear span of (x_n) and V is the linear span of (x_n), for some linearly independent sequence (x_n) in E. But in this case, defining $T_n : \mathbb{R}^n \to E$ by $T_n(\lambda_1, \ldots, \lambda_n) \doteq \sum_{j=1}^n \lambda_j x_j$, we have $V = \operatorname{span}\{\bigcup_n T_n(\mathbb{R}^n)\}$. Since V is not closed in E, we can apply Theorem 3.5 to conclude the proof of the first part.

For the second part, just apply the first part to $V = \operatorname{span}\{e_n : \mathbb{N}\}$, where $\{e_n : n \in \mathbb{N}\}$ is the canonical basis of E. ∎

The sequence space we will be interested in will be ℓ_2. Recall that, if r_n are the Rademacher functions defined on $[0,1]$ by $r_n(t) \doteq \operatorname{sign}(\sin(2^n \pi t))$ and $0 < p < \infty$, then (r_n), as a sequence in $L^p[0,1]$, is equivalent to the canonical basis of ℓ_2.

LEMMA 3.7. *Let $(a_n) \in \ell_2$ have infinitely many nonzero entries. Then for all nonempty open $U \subset [0,1]$ we have $(\sum a_n r_n)|_U \not\equiv 0$, where $\sum a_n r_n$ is a series in $L^p[0,1]$ $(0 < p < \infty)$.*

Proof. Let (a_n) and U be as in the statement, and set

$$f \doteq \sum_{n=1}^{\infty} a_n r_n, \quad f_{<j} \doteq \sum_{n=1}^{j-1} a_n r_n, \quad f_{\geq j} \doteq \sum_{n=j}^{\infty} a_n r_n.$$

The set U contains an interval of the form $I = [k/2^N, (k+1)/2^N]$ for some $N \in \mathbb{N}$ and $k = 0, \ldots, N-1$. Note that $f_{<N}$ is constant in I, but since we have infinitely many nonzero a_n's, $f_{\geq N}$ is *not* constant in I. Thus f cannot be constant in I. ∎

Proof of Theorem 3.2. By Corollary 3.6, there is a closed infinite-dimensional subspace F of $\overline{\operatorname{span}}\{r_n : n \in \mathbb{N}\}$ such that, for each nonzero element $\sum_{n=1}^{\infty} a_n r_n$ from F, infinitely many a_n's are nonzero. By Lemma 3.7, for each nonzero element h of F and each nonvoid open subset $U \subset [0,1]$, we have $h|_U \not\equiv 0$. Note that F, being a closed infinite-dimensional subspace of a space isomorphic to ℓ_2, is isomorphic to ℓ_2.

Let f be a norm-one element of $S_p(X)$, and define $\Phi : F \to L^p(X \times [0,1])$ by

$$\Phi\Big(\sum_{n=1}^{\infty} a_n r_n\Big)(x,t) \doteq f(x) \sum_{n=1}^{\infty} a_n r_n(t).$$

Note that the support of $\Phi(h)$ is σ-finite for each $h \in F$, and Φ is clearly an isometric isomorphism onto its range. By Fubini's theorem,

$$\begin{aligned}\Big\|\Phi\Big(\sum_{n=1}^{\infty} a_n r_n\Big)\Big\|_p^p &= \int_X \int_0^1 \Big|f(x) \sum_{n=1}^{\infty} a_n r_n(t)\Big|^p \, dt \, dx \\ &= \int_0^1 \Big(\int_X |f(x)|^p \, dx\Big) \Big|\sum_{n=1}^{\infty} a_n r_n(t)\Big|^p \, dt \\ &= \int_0^1 \Big|\sum_{n=1}^{\infty} a_n r_n(t)\Big|^p \, dt = \Big\|\sum_{n=1}^{\infty} a_n r_n\Big\|_p^p\end{aligned}$$

for all $\sum_{n=1}^{\infty} a_n r_n \in F$. It follows that $\Phi(F)$ is an ℓ_2-isomorphic subspace of $L^p(X \times [0,1])$.

It remains to show that $\Phi(F) \subset S_p(X \times [0,1]) \cup \{0\}$. Let $\sum_{n=1}^{\infty} a_n r_n$ be a nonzero element of F, $U \times (a,b)$ be a nonvoid basic open subset of $X \times [0,1]$, and $p \leq q < \infty$. Then

$$\int_{U \times (a,b)} \Big|\Phi\Big(\sum_{n=1}^{\infty} a_n r_n\Big)\Big|^q = \int_a^b \Big|\sum_{n=1}^{\infty} a_n r_n(t)\Big|^q \, dt \int_U |f(x)|^q \, dx$$

converges if $q = p$ (by Khinchin's inequality and since f is p-integrable), and does not converge if $q > p$ (since the first factor is strictly positive by Lemma 3.7 and f is not q-integrable). This concludes our proof. ∎

Before proceeding to the proof of Theorem 3.3, we point out that it is not hard to prove, using Theorem 3.2, that $(L^p[0,1] \setminus \bigcup_{q>p} L^q[0,1]) \cup \{0\}$ contains a copy of ℓ_2, for $1 \leq p < \infty$. In effect, recall that there exists a measure-preserving Borel isomorphism ψ from $[0,1]^2$ onto $[0,1]$, which in turn induces an isometric isomorphism Ψ from $L^p([0,1]^2)$ onto $L^p[0,1]$, defined by $\Psi(f) \doteq f \circ \psi^{-1}$. It is easy to verify that, for each $f \in L^p([0,1]^2)$ and each $q > p$, f is q-integrable if and only if $\Psi(f)$ is q-integrable; in particular, *no* function in $\Psi(S_p([0,1]^2))$ is q-integrable for any $q > p$, and the claim follows. But the *nowhere* part is lost, since ψ is not a homeomorphism. We thus need to provide a finer construction.

Proof of Theorem 3.3. Let μ be the Lebesgue measure on $[0,1]$, that is, the unique Borel measure such that $\mu([k/2^n,(k+1)/2^n]) = 1/2^n$ for every $n \in \mathbb{N}$ and $k = 0,1,\ldots,2^n-1$. Denote by λ the Lebesgue measure on $\{0,1\}^{\mathbb{N}}$, that is, the unique Borel measure such that $\lambda(\langle s\rangle) = 1/2^{|s|}$ for every finite sequence s of zeros and ones, where $|s|$ is the length of s and $\langle s\rangle$ stands for the set of all $x \in \{0,1\}^{\mathbb{N}}$ such that $x(k) = s(k)$ for $k = 1,\ldots,|s|$. Let $g : \{0,1\}^{\mathbb{N}} \to [0,1]$ be given by $g(x) = \sum_{n=1}^{\infty} x(n)/2^n$. Note that $g^{-1}(k/2^n)$ consists of two elements x, y such that x is a binary representation of $k/2^n$ with $x(m) = 0$ for $m > n$, and y is a binary representation of $k/2^n$ with $x(m) = 1$ for $m > n$. Moreover $g^{-1}(t)$ is a singleton if t is not of the form $k/2^n$.

Claim 1. *$\mu(A) = \lambda(g^{-1}(A))$ for every Borel set A in $[0,1]$, and $\lambda(B) = \mu(g(B))$ for every Borel set B in $\{0,1\}^{\mathbb{N}}$.*

It is enough to show this for $A = [k/2^n,(k+1)/2^n]$ and $B = \langle s\rangle$. Let s be a finite sequence of zeros and ones which is the binary representation of the number $k/2^n$. Then $|s| = n$ and $g^{-1}(A) = \langle s\rangle$. Thus $\lambda(g^{-1}(A)) = \lambda(\langle s\rangle) = 1/2^n = \mu(A)$.

Let $n = |s|$ and define $k = s(n) + 2s(n-1) + 2^2 s(n-2) + \cdots + 2^{n-1}s(1)$. Then $g(B) = g(\langle s\rangle) = [k/2^n,(k+1)/2^n]$. Thus

$$\mu(g(B)) = \mu([k/2^n,(k+1)/2^n]) = 1/2^n = 1/2^{|s|} = \lambda(B).$$

Claim 1 is thus proved.

Claim 2. *$F : L^p[0,1] \to L^p(\{0,1\}^{\mathbb{N}})$ given by $F(f) \doteq f \circ g$ is an isometric isomorphism between $L^p[0,1]$ and $L^p(\{0,1\}^{\mathbb{N}})$, with inverse L :*

$L^p(\{0,1\}^{\mathbb{N}}) \to L^p[0,1]$ given by

$$L(h)(t) \doteq \begin{cases} h(g^{-1}(t)) & \text{if } t \in [0,1] \text{ is not of the form } k/2^n, \\ h(x) & \text{if } t = k/2^n \text{ and } x \text{ is the binary representation} \\ & \text{of } t \text{ with } x(m) = 0,\ m > n. \end{cases}$$

Moreover, $F|_{L^q[0,1]} = L^q(\{0,1\}^{\mathbb{N}})$ for each $q > p$.

If A is a Borel set in $[0,1]$ and $f = \chi_A$ is the characteristic function of a set A, then by Claim 1 we have

$$\int_{[0,1]} f\,d\mu = \mu(A) = \lambda(g^{-1}(A)) = \int_{\{0,1\}^{\mathbb{N}}} \chi_{g^{-1}(A)}\,d\lambda = \int_{\{0,1\}^{\mathbb{N}}} f \circ g\,d\lambda. \tag{3.1}$$

It is easily seen that (3.1) also holds if f is a step function, and it follows that

$$\int_{[0,1]} |f|\,d\mu = \int_{\{0,1\}^{\mathbb{N}}} |f \circ g|\,d\lambda$$

for each $f \in L^1[0,1]$. It follows easily that $\|f\|_p = \|f \circ g\|_p$ for $f \in L^p[0,1]$. This shows that F is norm-preserving and $F|_{L^q[0,1]} = L^q(\{0,1\}^{\mathbb{N}})$ for each $q > p$.

It is clear that L is a left inverse for F. Note that, for a given h in $L^p(\{0,1\}^{\mathbb{N}})$, $F(L(h))$ possibly differs from h on a countable set of elements $x \in \{0,1\}^{\mathbb{N}}$ with $x(m) = 1$ for almost every m. Since λ is a continuous measure, we have $\lambda(\{x \in \{0,1\}^{\mathbb{N}} : h(x) \neq F(L(h))(x)\}) = 0$. This means that h and $F(L(h))$ are the same element of $L^p(\{0,1\}^{\mathbb{N}})$. Thus Claim 2 is proved.

Claim 3. *$F(S_p([0,1])) = S_p(\{0,1\}^{\mathbb{N}})$ and $L(S_p(\{0,1\}^{\mathbb{N}})) = S_p([0,1])$.*

Let $f \in S_p([0,1])$ and fix a basic set $\langle s\rangle$ in $\{0,1\}^{\mathbb{N}}$. Note that there exists a positive integer k such that $g(\langle s\rangle) = [k/2^n, (k+1)/2^n]$, where $n = |s|$. Since

$$F(f)\chi_{\langle s\rangle} = (f \circ g)\chi_{\langle s\rangle} = (f\chi_{[k/2^n,(k+1)/2^n]}) \circ g = F(f\chi_{[k/2^n,(k+1)/2^n]})$$

and $f\chi_{[k/2^n,(k+1)/2^n} \notin L^q[0,1]$, it follows by the previous claim that $F(f)\chi_{\langle s\rangle}$ is not in $L^q(\{0,1\}^{\mathbb{N}})$. This means that $F(f)$ is nowhere L^q. We have thus proved that $F(S_p([0,1])) \subset S_p(\{0,1\}^{\mathbb{N}})$ and, in view of the previous claim, that $L(S_p(\{0,1\}^{\mathbb{N}})) \supset S_p([0,1])$.

Now let $h \in S_p(\{0,1\}^{\mathbb{N}})$ and fix a set $[k/2^n, (k+1)/2^n]$. Let s be a finite set which is the binary representation of $k/2^n$. Since $h\chi_{\langle s\rangle}$ is not in L^q for $q > p$, we have

$$L(h\chi_{\langle s\rangle}) = (h \circ g^{-1})\chi_{[k/2^n,(k+1)/2^n]} = L(h)\chi_{[k/2^n,(k+1)/2^n]}.$$

Thus $L(h)$ is nowhere L^q and the proof of Claim 3 is complete.

Given two positive Borel measure spaces X and Y, and $0 < p < \infty$, we shall say that a map $\varphi : L^p(X) \to L^p(Y)$ *preserves* S_p if $\varphi(S_p(X)) \subset S_p(Y)$. Claim 3 asserts in particular that both F and L preserve S_p.

Claim 4. *There is an isometric isomorphism G from $L^p(\{0,1\}^{\mathbb{N}})$ onto $L^p(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}})$ which preserves S_p.*

Let $\varphi : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be defined by

$$\varphi((x(1), x(2), \ldots), (y(1), y(2), \ldots)) \doteq (x(1), y(1), x(2), y(2), \ldots).$$

It is well known that φ is a homeomorphism of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$. Fix two finite sequences s and s' of zeros and ones. Note that

$$\frac{1}{2^{|s|}}\,\frac{1}{2^{|s'|}} = \lambda(\langle s\rangle)\lambda(\langle s'\rangle) = \lambda \times \lambda(\langle s\rangle \times \langle s'\rangle)$$

and

$$\lambda(\varphi(\langle s\rangle \times \langle s'\rangle)) = \frac{1}{2^{|s|+|s'|}}.$$

The last equality follows from the fact that $\varphi(\langle s\rangle \times \langle s'\rangle)$ is a subset of $\{0,1\}^{\mathbb{N}}$ with exactly $|s| + |s'|$ coordinates fixed.

Using this we infer that $\lambda \times \lambda(A) = \lambda(\varphi(A))$ for any Borel subset A of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Then $G : L^p(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}) \to L^p(\{0,1\}^{\mathbb{N}})$ defined by $G(f) \doteq f \circ \varphi$ has inverse given by $G^{-1}(h) = h \circ \varphi^{-1}$ and satisfies the desired condition. This completes the proof of Claim 4.

Mimicking the reasoning used to prove Claims 1–3, we can show that $T : L^p(\{0,1\}^{\mathbb{N}} \times [0,1]) \to L^p(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}})$ defined by

$$T(f)(x, y) \doteq f(x, g(y))$$

is an onto isometric isomorphism which preserves S_p. Hence, we have built the following chain of S_p-preserving isometric isomorphisms:

$$L^p(\{0,1\}^{\mathbb{N}} \times [0,1]) \xrightarrow{T} L^p(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}) \xrightarrow{G} L^p(\{0,1\}^{\mathbb{N}}) \xrightarrow{L} L^p[0,1].$$

Theorem 1.7 implies that $S_p(\{0,1\}^{\mathbb{N}})$ is nonempty, and thus an application of Theorem 3.2 gives us a copy of ℓ_2 in $S_p(\{0,1\}^{\mathbb{N}} \times [0,1]) \cup \{0\}$. The conclusion follows immediately. ∎

4. Dense-lineability: proof of Theorem 1.7(d). Before proceeding to the proof of Theorem 1.7(d), which will also be via Lemma 2.3, we establish some notation. First, recall that a family $\{A_i : i \in I\}$ of infinite subsets of $\mathbb{N}$ is said to be *almost disjoint* if $A_i \cap A_j$ is finite for any distinct $i, j \in I$. It is well known that there is a family of almost disjoint subsets of $\mathbb{N}$ of cardinality continuum; a way to see this is to take an enumeration $\{q_n\}$ of the rational numbers and to consider, for each $x \in \mathbb{R}$, a subsequence $(q_{n_k^x})$ of (q_n) converging to x; then the family $\{\{n_k^x\}_{k\in\mathbb{N}} : x \in \mathbb{R}\}$ has the desired property.

Let then $\{A'_\alpha : \alpha < \mathfrak{c}\}$ be a family of almost disjoint subsets of $\mathbb{N}$. Fix a sequence of integers $1 = n_0 < n_1 < \cdots$ such that

$$\sum_{i=n_k}^{n_{k+1}-1} \frac{1}{i} \geq 1,$$

and consider $M_k \doteq \{n_k, n_k+1, \ldots, n_{k+1}-1\}$. Define, for each $\alpha < \mathfrak{c}$, $A_\alpha \doteq \bigcup\{M_k : k \in A'_\alpha\}$. Note that the family $\{A_\alpha : \alpha < \mathfrak{c}\}$ is almost disjoint and

$$\sum_{i \in A_\alpha} \frac{1}{i} = \infty$$

for each $\alpha < \mathfrak{c}$. We fix $\{A_\alpha : \alpha < \mathfrak{c}\}$ in what follows.

Proof of Theorem 1.7(d). For a fixed Borel set A of positive finite measure and $\alpha < \mathfrak{c}$ we define a function h_A^α as follows. Let $\{A_{n,m} : n, m \in \mathbb{N}\}$ be a family of pairwise disjoint subsets of A of positive measure such that $\mu(A_{n,m}) \geq 2\mu(A_{n,m+1})$, and let (r_n) be a strictly decreasing sequence of reals tending to p. Put

$$h_n^\alpha \doteq \sum_{m \in A_\alpha} a_{m,n} \chi_{A_{n,m}}, \tag{4.1}$$

where $a_{m,n}^{r_n} \mu(A_{n,m}) = 1/m$. Then a similar argument to that used in Lemma 2.4 shows that the A-supported norm-one function

$$h_A^\alpha \doteq \sum_{n=1}^{\infty} \frac{h_n^\alpha}{\|h_n^\alpha\| 2^n} \tag{4.2}$$

is in $L^p \setminus \bigcup_{q>p} L^q$.

As in the proof of Theorem 2.2, fix a basis (U_n) for X, and let $N_n \subset U_n$ be a sequence of pairwise disjoint nowhere dense Borel sets satisfying $0 < \mu(N_n) < 1/2^n$. For each $\alpha < \mathfrak{c}$, by defining $h_{N_n}^\alpha$ as in (4.2) and putting

$$f^\alpha \doteq \sum_{n=1}^{\infty} \frac{h_{N_n}^\alpha}{2^n},$$

we obtain $f^\alpha \in S_p$ of norm one.

Note that any ordinal number $\alpha < \mathfrak{c}$ is of the form $\beta + n$, where β is a limit ordinal and $n = 0, 1, 2, \ldots$. Let $\{B_\beta : \beta < \mathfrak{c}\}$ be an indexation of all Borel subsets of X. Then the set $\{(B_\beta, n) : \beta < \mathfrak{c},\ n \in \mathbb{N}\}$ has cardinality $\mathfrak{c}$, thus there is a bijection $(B_\beta, n) \mapsto \alpha(\beta, n)$ onto all ordinals less than $\mathfrak{c}$.

For $\beta < \mathfrak{c}$ and $n \in \mathbb{N}$ consider the functions

$$g^{\beta,n} \doteq g^{\alpha(\beta,n)} \doteq \chi_{B_\beta} + \frac{1}{n} f^{\alpha(\beta,n)}. \tag{4.3}$$

By our construction, the linear span of $\{g^{\alpha(\beta,n)} : \beta < \mathfrak{c},\ n \in \mathbb{N}\}$ is dense in the set of all simple functions on X, and therefore it is also dense in L^p.

We will show that any nontrivial linear combination of functions of the form (4.3) is in S_p. Let $(\beta_1, n_1), \dots, (\beta_k, n_k)$ be distinct and consider $b_1, \dots, b_k \in \mathbb{R}$ which are not all zero, and write

$$\begin{aligned} g &\doteq b_1 g^{\beta_1, n_1} + \dots + b_k g^{\beta_k, n_k} \\ &= (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} f^{\alpha(\beta_1, n_1)} + \dots + \frac{b_k}{n_k} f^{\alpha(\beta_k, n_k)}. \end{aligned}$$

Consider $\alpha_i \doteq \alpha(\beta_i, n_i)$, and note that $\alpha_1, \dots, \alpha_k$ are distinct ordinals. We can then write

$$\begin{aligned} g &= (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} f^{\alpha_1} + \dots + \frac{b_k}{n_k} f^{\alpha_k} \\ &= (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} \sum_{n=1}^{\infty} \frac{h^{\alpha_1}_{N_n}}{2^n} + \dots + \frac{b_k}{n_k} \sum_{n=1}^{\infty} \frac{h^{\alpha_k}_{N_n}}{2^n}. \end{aligned}$$

Consider the family $\{A_{l,m} : l, m \in \mathbb{N}\}$ of pairwise disjoint subsets of N_n of positive measure such that $\mu(A_{l,m}) \geq 2\mu(A_{l,m+1})$, and construct $h^{\alpha_i}_{N_n}$ as in (4.1), using these subsets and the corresponding $a_{l,m}$. Consider $N \in \mathbb{N}$ such that the sets $C_1 \doteq A_{\alpha_1} \setminus \{1, \dots, N\}$, $C_2 \doteq A_{\alpha_2} \setminus \{1, \dots, N\}, \dots, C_k \doteq A_{\alpha_k} \setminus \{1, \dots, N\}$ are disjoint; this is possible since $\{A_\alpha : \alpha < \mathfrak{c}\}$ is almost disjoint. Then we have

$$h^{\alpha_i}_l = \sum_{m \in A_{\alpha_i}} a_m \chi_{A_{l,m}} = \sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{m \in C_i} a_m \chi_{A_{l,m}},$$

and thus

$$\begin{aligned} h^{\alpha_i}_{N_n} &= \sum_{l=1}^{\infty} \frac{h^{\alpha_i}_l}{\|h^{\alpha_i}_l\| 2^l} = \sum_{l=1}^{\infty} \frac{1}{\|h^{\alpha_i}_l\| 2^l} \Big(\sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{m \in C_i} a_m \chi_{A_{l,m}} \Big) \\ &= \sum_{l=1}^{\infty} \frac{1}{\|h^{\alpha_i}_l\| 2^l} \sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{l=1}^{\infty} \frac{1}{\|h^{\alpha_i}_l\| 2^l} \sum_{m \in C_i} a_m \chi_{A_{l,m}}. \end{aligned}$$

Writing $w_i \doteq \sum_{l=1}^{\infty} \frac{1}{\|h^{\alpha_i}_l\| 2^l} \sum_{m \in C_i} a_m \chi_{A_{l,m}}$ for each $i = 1, \dots, k$, by our construction each w_i is in $L_p \setminus \bigcup_{q>p} L_q$ and $w_1, \dots, w_k$ have disjoint supports; more precisely, the support of each w_i is $N^i_n \doteq \bigcup_m \bigcup_{l \in C_i} A_{l,m}$. Note that $\operatorname{span}\{f^{\alpha_i} \chi_{\bigcup_n N^i_n}\} \subset S_p$. The fact that $g \in S_p$ follows then from the fact that adding a simple function to a function from S_p results in a function from S_p.

Since $L^p(X, \mu)$ is separable, it has dimension $\mathfrak{c}$, as does $\operatorname{span}\{g^{\alpha(\beta,n)} : \beta < \mathfrak{c},\ n \in \mathbb{N}\}$, which concludes our proof. ∎

5. Algebrability: proof of Theorem 1.7(e)

Proof of Theorem 1.7(e). Let (U_n) be a basis for X. Similarly to the construction at the beginning of the proof of Theorem 1.7(a), one can find pairwise disjoint nowhere dense Borel sets N_n such that $N_n \subset U_n$ and $0 < \mu(N_n) < 1/2^n$. Using Lemma 2.3, we can find for each n a pairwise disjoint family $(N_{n,j})_j$ of Borel subsets of N_n satisfying

$$\mu(N_{n,j+1}) \le \frac{1}{j+1}\mu(N_{n,j}).$$

Note that, for each n, j, we have $\mu(N_{n,j}) \le 1/(j!2^n)$. Let $B_j \doteq \bigcup_n N_{n,j}$. Then all nonvoid open subsets of X intersect each B_j in nonnull sets, and on the other hand $\mu(B_j) = \sum_n \mu(N_{n,j}) \le 1/j!$.

Let $\{\theta_\alpha : \alpha < \mathfrak{c}\}$ be a set of real numbers strictly greater than 1 such that the set $\{\ln\theta_\alpha : \alpha < \mathfrak{c}\}$ is linearly independent over the rational numbers. For each $\alpha < \mathfrak{c}$, define

$$g_\alpha \doteq \sum_{j=1}^{\infty} \theta_\alpha^j \chi_{B_j}.$$

For each α the series $\sum_j \theta_\alpha^{pj}/j!$ converges, thus $g_\alpha \in L^p$ for all $\alpha < \mathfrak{c}$ and $0 < p < \infty$.

Let us show that $\{g_\alpha : \alpha < \mathfrak{c}\}$ is a set of free generators, and the algebra generated by this set is contained in $\mathcal{G} \cup \{0\}$. It suffices to show that, for any positive integers m and n, for every matrix $(k_{il} : i = 1, \dots, m,\ l = 1, \dots, n)$ of nonnegative integers with nonzero and distinct rows, for all $\alpha_1, \dots, \alpha_n < \mathfrak{c}$ and for all $\beta_1, \dots, \beta_m \in \mathbb{R}$ which do not vanish simultaneously, the function

$$\begin{aligned} g &\doteq \beta_1 g_{\alpha_1}^{k_{11}} \cdots g_{\alpha_n}^{k_{1n}} + \cdots + \beta_m g_{\alpha_1}^{k_{m1}} \cdots g_{\alpha_n}^{k_{mn}} \\ &= \sum_{j=1}^{\infty} \big(\beta_1(\theta_{\alpha_1}^{k_{11}} \cdots \theta_{\alpha_n}^{k_{1n}})^j + \cdots + \beta_m(\theta_{\alpha_1}^{k_{m1}} \cdots \theta_{\alpha_n}^{k_{mn}})^j\big)\chi_{B_j} \end{aligned}$$

is in $\mathcal{G}$. First, let us show that it is in $\bigcap_{0<p<\infty} L^p$. Fix p and, for each $i = 1, \dots, m$, put $\theta_i \doteq \theta_{\alpha_1}^{k_{i1}} \cdots \theta_{\alpha_n}^{k_{in}}$. Then

$$\begin{aligned} \int |g|^p &\le \int \Big[\sum_{j=1}^{\infty}(|\beta_1|\theta_1^j + \cdots + |\beta_m|\theta_m^j)^p \chi_{B_j}\Big] \\ &\le \sum_{j=1}^{\infty} \frac{Q(\theta_1^j, \dots, \theta_m^j)}{j!}, \end{aligned} \tag{5.1}$$

where $Q : (x_1, \dots, x_m) \mapsto (|\beta_1|\theta_1^j + \cdots + |\beta_m|\theta_m^j)^p$. It is straightforward to find $C, b > 0$ such that $Q(\theta_1^j, \dots, \theta_m^j) < C + b^j$ for all j. Thus the sum on the right hand side of (5.1) converges, and $g \in L^p$.

Since $\ln\theta_i = \ln(\theta_{\alpha_1}^{k_{i1}} \cdots \theta_{\alpha_n}^{k_{in}}) = k_{i1}\ln\theta_{\alpha_1} + \cdots + k_{in}\ln\theta_{\alpha_n}$ and $\ln\theta_{\alpha_1}, \dots, \ln\theta_{\alpha_n}$ are $\mathbb{Q}$-linearly independent, the numbers $\ln\theta_1, \dots, \ln\theta_m$ are distinct. By the strict monotonicity of the logarithmic function we may assume that

$$\theta_1 > \cdots > \theta_m; \tag{5.2}$$

we may also assume $\beta_1 \neq 0$. Then we can write

$$g = \sum_{j=1}^{\infty} (\beta_1 \theta_1^j + \cdots + \beta_m \theta_m^j) \chi_{B_j}.$$

From (5.2) and since β_1 is assumed to be nonzero, we can find $j_0 \in \mathbb{N}$ such that

$$|\beta_2| \theta_2^j + \cdots + |\beta_m| \theta_m^j < \tfrac{1}{2} |\beta_1| \theta_1^j$$

for all $j \geq j_0$. Then for those j,

$$\begin{aligned} |\beta_1 \theta_1^j + \cdots + \beta_m \theta_m^j| &\geq |\beta_1| \theta_1^j - \left| \beta_2 \theta_2^j + \cdots + \beta_m \theta_m^j \right| \\ &\geq |\beta_1| \theta_1^j - (|\beta_2| \theta_2^j + \cdots + |\beta_m| \theta_m^j) > \tfrac{1}{2} |\beta_1| \theta_1^j. \end{aligned}$$

Since each nonvoid open subset of X intersects all B_j in nonnull sets, the inequality above shows that g is nowhere essentially bounded. ■

5.1. Comments and open problems. As a corollary of Theorem 1.7(e) we have the following:

COROLLARY 5.1. *If μ is an atomless and outer regular positive Borel measure on X with full support and $0 < p < \infty$, then*

$$\mathcal{G}_p \doteq \{f \in L^p(\mu) : f \text{ is nowhere } L^\infty(\mu)\}$$

is strongly $\mathfrak{c}$-algebrable.

It is a straightforward exercise for the reader to show, using a construction similar to the one used to prove Theorem 1.7(a), that $\mathcal{G}_p$ is spaceable in L^p. To finish this section we pose the following problem:

PROBLEM 3. *Does $\mathcal{G}_p \cup \{0\}$ admit* dense *or* closed *subalgebras of L^p?*

6. When are there nowhere q-integrable functions in L^p? We conclude this work with a couple of remarks and questions on necessary/sufficient conditions on a positive Borel measure space (X, μ), so that there exist nowhere q-integrable Borel functions in $L^p(X)$. An obvious necessary condition is that μ has full support, so we will always assume that. It is not hard to see that it is also necessary that X has the countable chain condition, as Proposition 6.1 below shows. Recall that X is said to have the *countable chain condition* (or ccc, in short) if any family consisting of open nonempty pairwise disjoint subsets of X is countable.

PROPOSITION 6.1. *Let X be a topological space without the ccc, assume that μ is a positive Borel measure on X with full support, fix $0 < q < \infty$ and let $f : X \to \mathbb{R}$ be a Borel function. If $f|_U$ is not in $L^q(U)$ for any nonvoid open set U, then f is not in $L^p(X)$ for any $0 < p < \infty$.*

Proof. Let $(U_s)_{s\in S}$ be an uncountable family of pairwise disjoint nonempty open sets. Since $f|_{U_s}$ is not in $L^q(U_s)$ for any U_s, f does not vanish on U_s. Thus for each $0 < p < \infty$ and each $s \in S$, $\|f|_{U_s}\|_p > 0$. Fix $0 < p < \infty$. Since S is uncountable, at least one of the sets

$$S_n \doteq \Big\{ s \in S : \int_{U_s} |f|^p \, d\mu \geq 1/n \Big\}$$

is uncountable. Hence $\int_X |f|^p \, d\mu = \infty$. ∎

The next natural step is to pose the following:

PROBLEM 4. *Suppose that X has the ccc, and that μ is a positive Borel measure on X with full support. Given $0 < p < \infty$, does there exist a Borel p-integrable function $f : X \to \mathbb{R}$ which is nowhere q-integrable for all $q > p$?*

We provide a partial answer to the problem above through a consistency result. Recall first that the product of two spaces with the ccc does not need to have the ccc, but this statement is independent of ZFC. Under Martin's axiom, the product of two ccc spaces does not have the ccc, but in some models of ZFC there exists a topological space called a *Suslin line*, which has the ccc but its square does not (cf. [22]).

THEOREM 6.2. *It is consistent with ZFC that there is a topological space X satisfying the ccc such that, for any positive Borel measure μ on X with full support and any $0 < p < \infty$, there is no Borel function $f : X \to \mathbb{R}$ in $L^p(\mu)$ but nowhere $L^q(\mu)$ for any $q > p$.*

Proof. It is consistent with ZFC that there exists a Suslin line X. Suppose that there is a Borel function $f : X \to \mathbb{R}$ in $L^p(\mu)$ but nowhere $L^q(\mu)$ for all $q > p$. Let $\tilde{f} : X^2 \to \mathbb{R}$ be defined by $\tilde{f}(x, y) = f(x)f(y)$. Clearly $\tilde{f}$ is Borel, and since $\operatorname{supp} f$ is σ-finite, so is $\operatorname{supp} \tilde{f} = (\operatorname{supp} f)^2$. Fubini's theorem then implies that

$$\int_{X^2} |\tilde{f}|^p \, d(\mu \times \mu) = \|f\|_p^{2p},$$

and for any two nonvoid open sets $U, V \subset X$ and $q > p$ we have

$$\int_{U\times V} |\tilde{f}|^q \, d(\mu \times \mu) = \int_U |f|^q \int_V |f|^q = \infty.$$

Hence $\tilde{f}$ is a Borel function in $L^p(\mu \times \mu)$ but nowhere $L^q(\mu \times \mu)$ for each $q > p$, and $\mu \times \mu$ is a positive Borel measure with full support. Since X^2 does not have the ccc, we get a contradiction. ∎

Finally we turn our attention to the presence of countable π-bases. First, note that there exist topological spaces X with countable π-bases but admitting no countable bases, and with positive Borel measures with full support defined on them: take for example the Sorgenfrey line (the set of real numbers with the topology generated by intervals of the form $[a,b)$) with the Lebesgue measure. The Sorgenfrey line $\mathbb{R}_S$ has a countable π-basis, so we can apply Theorem 1.7 to show that $S_p(\mathbb{R}_S)$ is ℓ_p-spaceable; but any basis of the Sorgenfrey line has cardinality $\mathfrak{c}$.

It turns out that the presence of a countable π-basis in X is also not necessary for the existence of nowhere q-integrable functions in $L^p(X)$. In fact, we have more.

Theorem 6.3. *Let X be a topological space with a countable π-basis. Suppose that μ is an atomless and outer-regular Borel probability measure on X with full support. Assume that κ is an uncountable cardinal number. Let $Y = X^\kappa$ be the Tychonoff product of κ many copies of X, and consider on Y the measure $\lambda = \mu^\kappa$, the product of κ many copies of μ. Then $S_p(Y)$ is spaceable in $L^p(Y)$.*

Proof. Let (U_n) be a countable π-base in X. By the construction used to prove Theorem 1.7(a) applied to X and μ, there is a norm-one basic sequence $f_1, f_2, \ldots$ of elements of $S_p(X)$ with pairwise disjoint supports, and with each f_i of the form $\sum_{k=1}^\infty a_k \chi_{A_k}$ where A_k are Borel subsets of X. For $(x_\alpha)_{\alpha<\kappa} \in X^\kappa$, put $\tilde{f}_i((x_\alpha)_{\alpha<\kappa}) \doteq f_i(x_0)$. Then $(\tilde{f}_i)$ is a norm-one basic sequence in $L^p(Y)$ with pairwise disjoint supports. We need to show that each $\tilde{f}_i$ is in $S_p(Y)$. Note that $\tilde{f}_i$ is of the form $\sum_{k=1}^\infty a_k \chi_{\tilde{A}_k}$, where $\tilde{A}_k \doteq A_k \times \prod_{1\le\alpha<\kappa} X$.

Let V be a nonempty open subset of Y. We may assume that V is of the form $\prod_{\alpha<\kappa} W_\alpha$ where W_α are nonempty open subsets of X, and there is a finite set $F \subset \kappa$ such that $W_\alpha = X$ if $\alpha \in \kappa \setminus F$. Let $F_0 = F \setminus \{0\}$. We have $V = W_0 \times \prod_{1\le\alpha<\kappa} W_\alpha$ and

$$\begin{aligned}
\int_V |\tilde{f}_i|^q \, d\lambda &= \int_V \sum_{k=1}^\infty |a_k|^q \chi_{\tilde{A}_k} d\lambda = \sum_{k=1}^\infty |a_k|^q \lambda(\tilde{A}_k \cap V) \\
&= \sum_{k=1}^\infty |a_k|^q \lambda\Big((A_k \cap W_0) \times \prod_{1\le\alpha<\kappa} W_\alpha\Big) \\
&= \sum_{k=1}^\infty |a_k|^q \mu(A_k \cap W_0) \prod_{\alpha\in F_0} \mu(W_\alpha) \\
&= \prod_{\alpha\in F_0} \mu(W_\alpha) \int_{W_0} |f_i|^q \, d\mu = \infty.
\end{aligned}$$

Similarly we get

$$\int_Y |\tilde{f}_i|^p \, d\lambda = \int_X |f|^p \, d\mu < \infty. \blacksquare$$

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